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SKOROHOD REPRESENTATION OF YOUNG MEASURES

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Dedicated to Professor Hidetoshi Komiya on his 65th birthday

Abstract. Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space. The Skorohod representation theorem holds for Young measures in which the conditional expectation is defined with respect to a subset of Ω. This paper generalizes, first, the Skorohod representation theorem for Young measures to the case in which the conditional expectation is defined with respect to a sub σ-algebra of $\mathcal{F}$. Secondly, we consider Skorohod representation theorem for sets of Young measures

1. Introduction

Let $(\mathbb{S}, d)$ be a complete separable metric (Polish) space, $\mathcal{B}(\mathbb{S})$ be its Borel σ-field, $\mathcal{M}_1^+(\mathbb{S})$ be the space of probability measures on $\mathbb{S}$. Skorohod [12] proved the following representation theorem. Let $\nu^n, \nu^0\,(n \in \mathbb{N})$ be probability measures on $\mathbb{S}$ such that $\nu^n \to \nu^0$ weakly* in $\mathcal{M}_1^+(\mathbb{S})$. Then there exist measurable functions, $\xi_n, \xi_0 : [0,1] \to \mathbb{S}$ such that

(a) $\xi_n(\lambda) = \nu^n, \xi_0(\lambda) = \nu^0$, and
(b) $\lambda(\{x \in [0,1] \mid \xi_n(x) \to \xi_0(x)\}) = 1$,

where λ is the Lebesgue measure on $[0,1]$. This theorem finds wide ranging applications in economic theory. In the realm of decision theory, Dumav and Stinchcombe [5] named the condition (a) as "domain equivalence" and called the condition (b) as "the continuity result". Since the seminal work of von Neumann and Morgenstern [10], decision theory has been providing theory of the axiomatizations of the (observable) preference relation $\succsim$ on a choice set and a utility representation of a preference relation $\succsim$. A preference relation $\succsim$ is a complete and transitive relation on a choice set. von Neumann and Morgenstern demonstrates that the independence and continuity axioms of a preference relation $\succsim$ on $\mathcal{M}_1^+(\mathbb{S})$ suffice to establish the (continuous) utility representation $u : \mathbb{S} \to \mathbb{R}$ which satisfies: $\kappa \succsim \tau$ iff

$$\int_{\mathbb{S}} u(s)\, d\kappa(s) \geq \int_{\mathbb{S}} u(s)\, d\tau(s).$$

On the other hand, Savage [12] introduced the space of the states of the world Ω and considers an "act" as a function ξ defined on Ω into $\mathbb{S}$. By assuming several axioms on a preference relation $\succsim$ on the space of "acts", Savage succeeds in establishing

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the existence of a (continuous) utility $u : \mathbb{S} \to \mathbb{R}$ and a probability measure P on some measurable space $(\Omega, \mathcal{F})$ such that $\xi_1 \succsim \xi_2$ iff

$$\int_\Omega u(\xi_1(\omega))\, dP(\omega) \geq \int_\Omega u(\xi_2(\omega))\, dP(\omega).$$

Remark that, since, for any non-atomic probability measure space $(\Omega, \mathcal{F}, P)$, there exists a measurable function $f : \Omega \to [0,1]$ such that $P \circ f^{-1} = \lambda$, if P is non-atomic, the space $([0,1], \mathcal{B}([0,1]), \lambda)$ in the Skorohod representation theorem can be replaced by the space $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}([0,1])$ stands for Borel σ-algebra on $[0,1]$. The condition (a) of Skorohod representation theorem states that the "domain equivalence" holds between utility representations of von Neumann and Morgenstern, and Savage. Furthermore, the condition (b) implies that the weak* convergence $\nu^n \to \nu^0$ in the utility representation of von Neumann and Morgenstern can be strengthened to be almost all convergence of acts: $\xi_n(\omega) \to \xi_0(\omega)$ for almost all $\omega \in \Omega$, in the utility representation of Savage.

Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space. In Tateishi [14], the author generalizes Skorohod representation theorem asserted on the probability space $\mathcal{M}_1^+(\mathbb{S})$ to the space of Young measures, that is, the space of probability measures on $(\Omega \times \mathbb{S}, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}))$ whose projection on Ω is equal to μ. The author established that the narrow topology on the space of Young measures admits Skorohod representation with respect to the conditional expectation of the Young measures. In the paper, we proved the Skorohod representation theorem for Young measures in which the conditional expectation is defined with respect to a measurable subset of Ω. The aim of this paper is twofold. First, we generalize the result of Tateishi [14] to the case in which the conditional expectation is defined with respect to a sub σ-algebra of $\mathcal{F}$. Secondly, we consider Skorohod representation theorem for sets of Young measures. The study of Skorohod representation for sets of probabilities is considered in Dumav and Stinchcombe [5]. This paper attempts to generalize their results to the case of Young measures.

Following Section 2, we first consider, in Section 3, the Skorohod representation theorem for Young measures in the case in which the conditional expectation is defined with respect to a measurable subset of Ω. We extends it, in Section 4, to the case in which the conditional expectation is defined with respect to a sub σ-algebra of $\mathcal{F}$. Section 5 is devoted to the extension of the Skorohod representation theorem to sets of Young measures.

2. Preliminaries

Let $(\mathbb{S}, d)$ be a complete separable metric (Polish) space, and let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space. The Borel σ-algebra on $\mathbb{S}$ is denoted by $\mathcal{B}(\mathbb{S})$, the set of bounded measures on $\mathcal{B}(\mathbb{S})$ by $\mathcal{M}^b(\mathbb{S})$, the set of nonnegative measures on $\mathcal{B}(\mathbb{S})$ by $\mathcal{M}^+(\mathbb{S})$, and the set of probability measures on $\mathcal{B}(\mathbb{S})$ by $\mathcal{M}_1^+(\mathbb{S})$. These spaces are endowed with the weak* topology, that is, the topology generated by the seminorms: $\nu \mapsto |\nu(\psi)|$, where ψ is a continuous and bounded function on $\mathbb{S}$. A probability measure on $(\Omega \times \mathbb{S}, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}))$ whose projection on Ω is equal to μ

is called a *Young measure*. By the theorem of disintegration of measures, for each Young measures ν there corresponds to a function $\nu^* : \Omega \times \mathcal{B}(\mathbb{S}) \to [0,1]$ called a *disintegration of measure* ν which satisfies the following conditions:

(i) $\nu^*(\omega, \cdot)$ is a probability measure on $\mathbb{S}$ for each $\omega \in \Omega$,
(ii) $\nu^*(\cdot, B)$ is measurable for each $B \in \mathcal{B}(\mathbb{S})$, and
(iii) $\nu(B) = \int_\Omega \left[\int_\mathbb{S} \mathbb{1}_B(\omega, s)\nu^*_\omega(ds)\right] d\mu(\omega)$ for each $B \in \mathcal{F} \otimes \mathcal{B}(\mathbb{S})$,

where $\mathbb{1}_B$ stands for the characteristic function of B. Remark that, each Young measure ν corresponds to many disintegrations, but there is a one-one correspondence between the space of Young measures and the space of the equivalence classes of μ-almost everywhere equal disintegrations. We shall identify, in this paper, each Young measure ν with its disintegration ν^*. We denote by $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ the space of all young measures on $(\Omega \times \mathbb{S}, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}))$.

A real-valued function ψ defined on $\Omega \times \mathbb{S}$ is called a *Carathéodory integrand* if the following conditions hold:

(i) ψ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{S})$-measurable,
(ii) for each $\omega \in \Omega$, the map $s \mapsto \psi(\omega, s)$ is continuous and bounded, and
(iii) the map $\omega \mapsto \|\psi(\omega, \cdot)\|_\infty$ is μ-integrable,

where $\|\cdot\|_\infty$ stands for the sup-norm. The set of Carathéodory integrands on $\Omega \times \mathbb{S}$ is denoted by $\mathcal{G}_C(\Omega, \mathcal{F}; \mathbb{S})$.

The space $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ of Young measures is endowed with the topology generated by the seminorms: $\nu \mapsto |\nu(\psi)|$, where ψ is a Carathéodory integrand on $\Omega \times \mathbb{S}$. The topology on $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ is called a *narrow topology*.

3. Skorohod representation theorem for Young measures

We start with the following simultaneous Skorohod representation theorem by Blackwell and Dubins.

Theorem 3.1 (Blackwell and Dubins [1])**.** *Let $\mathbb{S}$ be a Polish space. Then, there exists a joint measurable function $\xi : \mathcal{M}_1^+(\mathbb{S}) \times [0,1] \to \mathbb{S}$ such that*

(i) *the law of $\xi(\tau, \cdot)$ coincides with τ, and*
(ii) *the map $\tau \mapsto \xi(\tau, x)$ is continuous for almost all $x \in [0,1]$.*

The condition in Theorem 3.1(i) states that the domain equivalence holds for the two decision problems:

$$\sup_{\tau \in \mathcal{M}_1^+(\mathbb{S})} \int_\mathbb{S} u(s)\, d\tau(s), \text{ and}$$

$$\sup_{\tau \in \mathcal{M}_1^+(\mathbb{S})} \int_0^1 u(\xi(\tau, x))\, d\lambda(x).$$

The condition in Theorem 3.1(ii) implies that in the case where the space $\mathbb{S}$ is compact, and so the space $\mathcal{M}_1^+(\mathbb{S})$ is compact with respect to the weak* topology, the second problem also admits a maximum and has a strategy $\tau^\circ \in \mathcal{M}_1^+(\mathbb{S})$ such that the act $\xi(\tau^\circ, \cdot)$ maximizes utility.

The following theorem is some generalization of Skorohod representation theorem to Young measures.

Theorem 3.2 (Tateishi [14]). *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space, and let $\mathbb{S}$ be a metrizable Souslin space. Then, there exists a map $\xi : \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S}) \times \mathcal{F} \times [0,1] \to \mathbb{S}$ such that*

(i) *for each $\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ and $A \in \mathcal{F}$, the law of $\xi(\nu, A, \cdot)$ coincides with $\mathbb{E}(\nu \mid A)$,*
(ii) *for each $A \in \mathcal{F}$, the map $\nu \mapsto \xi(\nu, A, x)$ is continuous for almost all x in $[0,1]$,*

where

$$\mathbb{E}(\nu \mid A) = \frac{1}{\mu(A)} \int_A \nu_\omega \, d\mu(\omega)$$

stands for the conditional expectation of ν given A.

Remark 3.3. Tateishi [13, 14] *also studies another subject of interest of Young measures, namely, the open mapping property of Young measures. Let $\mathbb{S}, \mathbb{T}$ be metrizable Souslin spaces and let $\varphi : \mathbb{S} \to \mathbb{T}$ be a continuous, open, and surjective map. This map induces a map $\pi : \mathcal{M}_1^+(\mathbb{S}) \to \mathcal{M}_1^+(\mathbb{T})$ by the relation $\pi(\nu) = \nu \circ \varphi^{-1}$. We define a map $\Pi : \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S}) \to \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{T})$ by the relation: $\Pi(\nu)_\omega = \pi(\nu_\omega)$. Then the map Π can be seen to be a continuous, open and surjective map* (See Tateishi [14, Theorem 6]). *The property that Π is open implies, in particular, that the inverse map admits a continuous selection thanks to Michael's continuous selection theorem. It follows that if $\mathbb{S}$ has the Skorohod representation ξ which satisfies the conditions in* Theorem 3.2 (i),(ii)*, then $\mathbb{T}$ also has the Skorohod representation. The Skorohod representation property transmits across spaces in this way. Hence, the Skorohod representation property for the unit interval $\mathbb{S} = [0,1]$ suffices to show that a metrizable Souslin space $\mathbb{S}$ has the Skorohod representation property.*

If Ω is the space of the states of the world, the σ-algebra $\mathcal{F}$ represents the information structure the agent is faced with, and μ represents a prior of the agent, then the decision problem which confronts the agent who has an information $A \in \mathcal{F}$ is the following:

$$\sup_{\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})} \int_{\mathbb{S}} u(s) \, \mathbb{E}(\nu \mid A)(ds),$$

that is,

$$\sup_{\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})} \frac{1}{\mu(A)} \int_A \left[\int_{\mathbb{S}} u(s) \nu_\omega(ds) \right] d\mu(\omega).$$

The condition in Theorem 3.2(i) says that this problem can be reformulated, by introducing a joint measurable map $\xi : \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S}) \times \mathcal{F} \times [0,1] \to \mathbb{S}$, as follows:

$$\sup_{\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})} \int_0^1 u(\xi(\nu, A, x)) \, d\lambda(x).$$

Furthermore, since, in the case where $\mathbb{S}$ is compact, the space $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ is compact with respect to the narrow topology, the condition in Theorem 3.2(ii) implies that the problem admits a maximum and has a solution.

4. Skorohod representation and information structure

We consider, in this section, the generalization of the Skorohod representation for Young measures in the case in which the conditional expectation is defined with respect to a measurable subset of Ω to the case in which the conditional expectation is defined with respect to a sub σ-algebra of $\mathcal{F}$.

Let $\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ be given and let $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$. Since, by assumption, for each $B \in \mathcal{B}(\mathbb{S})$, the map $\omega \mapsto \nu_\omega(B)$ is measurable, it admits a conditional expectation $\mathbb{E}(\nu \mid \mathcal{G})_\omega(B)$ with respect to $\mathcal{G}$ (see, e.g., Dudley [4, Theorem 10.1.1]). Since, for all $A \in \mathcal{G}$,

$$\int_A \nu_\omega(B)\, d\mu(\omega) = \int_A \mathbb{E}(\nu \mid \mathcal{G})_\omega(B)\, d\mu(\omega) \quad \text{for all } B \in \mathcal{B}(\mathbb{S}), \tag{4.1}$$

we have

$$\frac{1}{\mu(A)} \int_A \mathbb{E}(\nu \mid \mathcal{G})(\omega)\, d\mu(\omega) \in \mathcal{M}_1^+(\mathbb{S}) \quad \text{for all } A \in \mathcal{G}. \tag{4.2}$$

We call $\mathbb{E}(\nu \mid \mathcal{G})$ defined in this way a *conditional expectation* of a Young measure $\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ with respect to the sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$.

Lemma 4.1. *Let $\nu^\alpha \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ be a net which converges to $\nu^0 \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ with respect to the narrow topology. Then, for all $A \in \mathcal{G}$,*

$$\int_A \mathbb{E}(\nu^\alpha \mid \mathcal{G})(\omega)\, d\mu(\omega) \to \int_A \mathbb{E}(\nu^0 \mid \mathcal{G})(\omega)\, d\mu(\omega)$$

weakly in $\mathcal{M}^+(\mathbb{S})$.*

Proof. Since the narrow convergence of a net $\{\nu^\alpha\}$ to ν^0 implies that, for each $A \in \mathcal{G}$, a net $\{\nu^\alpha(A \times \cdot)\}$ of elements of $\mathcal{M}^+(\mathbb{S})$ converges weakly* to $\nu^0(A \times \cdot)$ (see, Castaing, Raynaud de Fitte and Valadier [3, p.21]), the assertion of the lemma follows from (4.1). □

Theorem 4.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space, $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, and let $\mathbb{S}$ be a Polish space. Then there exists a joint measurable function $\xi : \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S}) \times \mathcal{G} \times [0,1] \to \mathbb{S}$ such that*

(i) *for each $\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$, each sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$, and each $A \in \mathcal{G}$, the law of $\xi(\nu, A, \cdot)$ coincides with*

$$\frac{1}{\mu(A)} \int_A \mathbb{E}(\nu \mid \mathcal{G})(\omega)\, d\mu(\omega),$$

(ii) *for each $A \in \mathcal{G}$, the map $\nu \mapsto \xi(\nu, A, x)$ is continuous for almost all $x \in [0,1]$.*

Proof. Let $\eta : \mathcal{M}_1^+(\mathbb{S}) \times [0,1] \to \mathbb{S}$ be a Skorohod representation of Blackwell and Dubins obtained in Theorem 3.1 and define $\xi : \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S}) \times \mathcal{G} \times [0,1] \to \mathbb{S}$ by

$$\xi(\nu, A, x) = \eta\left(\frac{1}{\mu(A)} \int_A \mathbb{E}(\nu \mid \mathcal{G})(\omega)\, d\mu(\omega), x\right)$$

Then, the condition (i) follows from the condition in Theorem 3.1(i) and (4.2). The condition (ii) also can be deduced from the condition Theorem 3.1(ii) and Lemma 4.1. □

The ex-ante utility of a decision maker who is confronted with the information structure $\mathcal{G}$ can be written as

$$\int_\Omega \left[\int_{\mathbb{S}} u(s)\,\mathbb{E}(\nu \mid \mathcal{G})_\omega\,(ds)\right] d\mu(\omega).$$

The ex-post utility of the decision maker, when he possesses an information $A \in \mathcal{G}$, can be calculated as

$$\frac{1}{\mu(A)}\int_A \left[\int_{\mathbb{S}} u(s)\,\mathbb{E}(\nu \mid \mathcal{G})_\omega\,(ds)\right] d\mu(\omega).$$

By Theorem 4.2, this problem encountered by the decision maker can be written, by some joint measurable function $\xi : \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S}) \times \mathcal{G} \times [0,1] \to \mathbb{S}$, as

$$\sup_{\nu \in \mathcal{Y}(\Omega,\mathcal{F};\mathbb{S})} \int_0^1 u(\xi(\nu, A, t))\,d\lambda(t).$$

If we interpret the space $([0,1], \lambda)$ as a type space à la Harsanyi [8], and consider $\xi(\nu, A, t)$ as an action which is selected by an agent of type t who has a strategy ν, and an information A, then the condition in Theorem 4.2(i) states that the decision maker's ex-post problem who possesses the restricted information structure $\mathcal{G}$ can be formulated only by introducing one dimensional type space $([0,1], \lambda)$ without having recourse to the information structure $\mathcal{G}$. Remark also that, when $\mathbb{S}$ is a compact metric space, the space $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ is compact with respect to the narrow topology. Thus, by the condition in Theorem 4.2(ii), the problem admits a maximum and has a solution.

5. Skorohod representation for sets of Young measures

In this section, we consider the Skorohod representation for sets of Young measures. The study in this section is inspired by Dumav and Stinchcombe [5].

Let $\mathfrak{K}(\mathcal{Y})$ be the set of all compact subsets of $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$. Let $\mathrm{BL}_1(\mathbb{S}, d)$ be the set of bounded Lipschitz function $f : \mathbb{S} \to [0,1]$ defined on the metric space $(\mathbb{S}, d)$ with Lipschitz modulus bounded by 1. Let $\mathcal{F}$ be essentially countably generated. Let $\{A_n\}$ be a sequence in $\mathcal{F}$ which generates $\mathcal{F}$ and let $\mathcal{C}^n, n \in \mathbb{N}$ be the partition of Ω generated by $A_0, \ldots, A_n$. We rearrange the elements of $\mathcal{C} = \cup_n \mathcal{C}^n$ so as to $\mathcal{C} = \{C_1, \ldots, C_n, \ldots\}$. If $\mathbb{S}$ is a Polish space and $\mathcal{F}$ is essentially countably generated, then the space $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ endowed with the narrow topology is metrizable by the metric:

$$\rho(\nu, \nu') := \sum_{n \geq 1} 2^{-n} \sup_{f \in \mathrm{BL}_1(\mathbb{S},d)} |(\nu - \nu')(\mathbb{1}_{C_n} \otimes f)|$$

(see Castaing, Raynaud de Fitte and Valadier [3, Proposition 2.3.1]).

We define the Hausdorff metric $h(K, K'), K, K' \in \mathfrak{K}(\mathcal{Y})$ by

$$h(K, K') = \max\Big\{\sup_{x \in K} \rho(x, K'), \sup_{y \in K'} \rho(y, K)\Big\}.$$

Remark that, since the Hausdorff metric h is defined on $\mathfrak{K}(\mathcal{Y})$, it is equivalent for any metric ρ compatible with the narrow topology on $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$. Remark also that, the compactness of $\mathbb{S}$ implies that the space $\mathfrak{K}(\mathcal{Y})$ is compact with respect to the

Hausdorff metric h. Let $\mathfrak{KC}(\mathcal{Y}) \subset \mathfrak{K}(\mathcal{Y})$ be a set of compact and convex subsets of $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$. We assume that:

Assumption 5.1. *$\mathfrak{KC}(\mathcal{Y}) \subset \mathfrak{K}(\mathcal{Y})$ is compact with respect to the Hausdorff metric h.*

The following theorem is some modification of Theorem 1(b) in Dumav and Stinchcombe [5]. Let $\Omega^\circ = [0,1] \times [0,1]$, $\mathcal{F}^\circ$ its usual Borel σ-algebra and $\Pi^\circ = \{\lambda_r : r \in [0,1]\}$ where for each $r \in [0,1]$, λ_r denotes the uniform distribution on $\{r\} \times [0,1]$.[1] We say that a σ-algebra $\mathcal{F}$ is *standard* if there exists a Polish topology $\mathcal{T}_\Omega$ on Ω such that $\mathcal{F}$ is generated by $\mathcal{T}_\Omega$.

Theorem 5.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a complete finite positive measure space and $\mathcal{F}$ be standard, $\mathcal{G}$ be a sub σ-algebra of $\mathcal{F}$, and let $\mathbb{S}$ be a Polish space. Then there exists a jointly measurable function $\xi : \mathfrak{KC}(\mathcal{Y}) \times \mathcal{G} \times \Omega^\circ \to \mathbb{S}$ such that*

(i) *for each $K \in \mathfrak{KC}(\mathcal{Y})$ and each $A \in \mathcal{G}$,*

$$\xi(K, A, \Pi^\circ) = \left\{ \frac{1}{\mu(A)} \int_A \mathbb{E}(\nu \mid \mathcal{G})(\omega)\, d\mu(\omega) : \nu \in K \right\}.$$

(ii) *for each $A \in \mathcal{G}$ and each K_n, K_0 in $\mathfrak{KC}(\mathcal{Y})$ with $h(K_n, K_0) \to 0$,*

$$P(\{(r,s) \in \Omega^\circ : \xi(K_n, A, (r,s)) \to \xi(K_0, A, (r,s))\}) = 1$$

for all $P \in \Pi^\circ$.

Proof. Let us define a function $\rho^\circ(\nu, \nu')$ as follows:

$$\rho^\circ(\nu, \nu') := \left(\sum_{n \geq 1} 2^{-n} \left(\sup_{f \in \mathrm{BL}_1(\mathbb{S}, d)} |(\nu - \nu')(\mathbb{1}_{C_n} \otimes f)| \right)^2 \right)^{\frac{1}{2}}.$$

Then ρ° is a metric on $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ which is compatible with ρ. Furthermore, for each $\nu \in \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ and each $K \in \mathfrak{KC}(\mathcal{Y})$, the problem $\inf_{\nu' \in K} \rho^\circ(\nu, \nu')$ has a unique solution.[2] We denote its solution by $f_K(\nu)$. Since $\mathcal{F}$ is standard and $\mathbb{S}$ is Polish, Proposition 2.3.3 in Castaing, Raynaud de Fitte and Valaider [3] implies that the space $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ is a Polish space. Hence, by the Borel isomorphism theorem [4, Theorem 13.1.1], there exists a measurable bijection with measurable inverse $\psi : [0,1] \to \mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$. Define $\xi : \mathfrak{KC}(\mathcal{Y}) \times \mathcal{G} \times \Omega^\circ \to \mathbb{S}$ as follows:

$$\xi(K, A, (r,s)) = \eta \left(\frac{1}{\mu(A)} \int_A \mathbb{E}(f_K(\psi(r)) \mid \mathcal{G})(\omega)\, d\mu(\omega), s \right).$$

It is clear that ξ is joint measurable. Furthermore, since

$$\xi(K, A, \lambda_r) = \frac{1}{\mu(A)} \int_A \mathbb{E}(f_K(\psi(r)) \mid \mathcal{G})(\omega)\, d\mu(\omega),$$

[1] Dumav and Stinchcombe [5] also treat more general case in which Π° is descriptively complete.

[2] The space $(\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S}), \rho^\circ)$ is a space of strictly convex metric space with convex round balls in the sense of Bula [2]. See [2, Lemma 3.2].

it follows that

$$\xi(K, A, \Pi^\circ) = \left\{ \frac{1}{\mu(A)} \int_A \mathbb{E}(\nu \mid \mathcal{G})(\omega)\, d\mu(\omega) : \nu \in K \right\}.$$

To prove (ii), let $A \in \mathcal{G}$ and let K_n, K_0 be in $\mathfrak{KC}(\mathcal{Y})$ such that $h(K_n, K_0) \to 0$. Since $f_K(\nu)$ is continuous in K, $f_{K_n}(\psi(r)) \to f_{K_0}(\psi(r))$. Hence Lemma 4.1 and the condition in Theorem 3.1(ii) imply the assertion of (ii). □

We consider an agent who assumes an attitude of ambiguity aversion in the sense of Ellsberg [6] in the set K of probabilities on $(\Omega \times \mathbb{S}, \mathcal{F} \otimes \mathcal{B}(\mathbb{S}))$ which belongs to $\mathfrak{KC}(\mathcal{Y})$ and who also assumes an attitude of usual utility maximizer over the set $\mathfrak{KC}(\mathcal{Y})$. Remind that Gilboa and Schmeidler [7] succeed in axiomatizing an ambiguity averse preference of Ellsberg [6] and derive the maxmin utility representation of the ambiguity averse preference. Gilboa and Schmeidler [7] consider the case in which preference relation is defined on the set $\mathcal{Y}(\Omega, \mathcal{F}; \mathbb{S})$ and establish the existence of some utility representation $u : \mathcal{M}_1^+(\mathbb{S}) \to \mathbb{R}$ and a set of probabilities C on the measurable set $(\Omega, \mathcal{F})$ which satisfies the following condition: $\nu \succsim \tau$ iff

$$\min_{P \in C} \int u(\nu)\, dP \geq \min_{P \in C} \int u(\tau)\, dP. \text{ [3]}$$

Let us now suppose that all the probabilities belonging to the set C are absolutely continuous with respect to μ. Then the above inequality can be written, by using Radon Nykodým derivative, as follows:

$$\min_{P \in C} \int_\Omega \left[\int_{\mathbb{S}} u(s) \frac{dP}{d\mu}(\omega) \nu_\omega(ds) \right] d\mu(\omega) \geq \min_{P \in C} \int_\Omega \left[\int_{\mathbb{S}} u(s) \frac{dP}{d\mu}(\omega) \tau_\omega(ds) \right] d\mu(\omega).$$

By setting $M = \{\frac{dP}{d\mu}\nu : P \in C\}$, $N = \{\frac{dP}{d\mu}\tau : P \in C\}$, this inequality can be rewritten by

$$\min_{\nu^* \in M} \int u(\nu^*)\, d\mu \geq \min_{\tau^* \in N} \int u(\tau^*)\, d\mu.$$

If we assume that the ambiguity averse agent follows the reasoning in the spirit of Gilboa and Schmeidler [7], then the problem the agent is faced with will be as follows:

$$\sup_{K \in \mathfrak{KC}(\mathcal{Y})} \inf_{\nu \in K} \int_\Omega \left[\int_{\mathbb{S}} u(s) \mathbb{E}(\nu \mid \mathcal{G})_\omega(ds) \right] d\mu(\omega),$$

that is, the decision problem is to maximize his utility over the set $\mathfrak{KC}(\mathcal{Y})$ which represents a menu of options the ambiguity averse agent is faced with. By deciding among these options, the agent optimizes his state of mind. This problem may also be considered as a problem of selecting *the state of confidence* of Keynes [9, Chapter 12, II]. The set $\mathfrak{KC}(\mathcal{Y})$ may be considered as a menu of low confidence of the agent. The problem the agent is faced with is that he selects from the menu his optimum state of mind with low confidence.

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[3] The setting of Gilboa and Schmeidler [7] is somewhat different from mine due to the specific constraint of decision theory.

The ex-post utility of the decision maker, when he possesses an information $A \in \mathcal{G}$, can be calculated as

$$\sup_{K \in \mathfrak{KC}(\mathcal{Y})} \inf_{\nu \in K} \frac{1}{\mu(A)} \int_A \left[\int_{\mathbb{S}} u(s) \mathbb{E}(\nu \mid \mathcal{G})_\omega(ds) \right] d\mu(\omega). \tag{5.1}$$

By Theorem 5.2, the problem (5.1) can be rewritten as follows:

$$\sup_{K \in \mathfrak{KC}(\mathcal{Y})} \inf_{Q \in \Pi^\circ} \int_0^1 u(\xi(K, A, x))\, dQ(x).$$

Remark that this problem is a usual maxmin utility representation of Gilboa and Schmeidler [7] with a parameter of the decision maker's state of mind. By the condition in Theorem 5.2(ii), the map $K \mapsto \inf_{Q \in \Pi^\circ} \int_0^1 u(\xi(K, A, x))\, dQ(x)$ is upper semi-continuous and hence, by Assumption 5.1, it admits a maximum. This implies that, if $\mathbb{S}$ is compact, the problem (5.1) has a solution and the decision maker has his optimal state of mind.

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