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THE KKM LEMMA AND THE FAN–BROWDER THEOREMS: EQUIVALENCES AND SOME CIRCULAR TOURS

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Abstract. Many theorems due to Fan and Browder are relevant to mathematical economics and game theory, and in this tribute to Hidetoshi Komiya, we provide their extension, as well as a comprehensive overview, under the rubric of equivalences, and generalized equivalences. We connect our theorems to recent results on discontinuous games, economies and variational relations, and briefly interrogate the way the word *equivalence* has been used in both the mathematics and the mathematical economics literature.

> At a conference in 1983, Fan listed various fields in mathematics which have applications of KKM maps, as follows: Potential theory, Pontrjagin spaces or Bochner spaces in inner product spaces, operator ideals, weak compactness of subsets of locally convex topological vector spaces, function algebras, harmonic analysis, variational inequalities, free boundary value problems, convex analysis, mathematical economics, game theory and mathematical statistics. Park (2012)

1. Introduction

The abstract of Komiya [35] reads as follows:

> We discuss fixed point properties of convex subsets of locally convex linear topological spaces. We derive equivalence among fixed point properties concerning several types of multivalued mappings.

It is a good introduction to his work, with the "equivalence of fixed point properties" playing a determining role in an *oeuvre* characterized not only by rigor and elegance, but also a deep commitment to exposition and to communication. Komiya published two papers in 1981, one on the "convexity of a topological space," and the other on the "closedness of cones." The first generalized a fixed point theorem of Browder, a result of Fan on inequalities of convex functions and Sion's minimax theorem; while the second furnished

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an extension of Farkas' lemma in normed spaces. All in all, Komiya published seven papers in the first five years of his scientific career, and his debut publications were underscored and supplemented by results on linear inequalities, majorization of functions and the Hahn-Banach theorem. The emphasis on relationships and equivalences, as well as his authoritative admiration of the work of Browder and Fan, rendered his theorems of especial interest to theorists of economics, games and optimization.

In this tribute to Hidetoshi Komiya, the authors present a relational overview, a circular tour or a network, so to speak, of work revolving around the following three seminal results of non-linear analysis, all pertaining to a topology that is not necessarily assumed to be locally convex.

Theorem 1.1. (Fan, 1961) *Let X be a nonempty convex, compact*[1] *set of a Hausdorff topological vector space Y and $F : X \twoheadrightarrow Y$ a correspondence satisfying:*

1. *$co\{x_1, .., x_n\} \subseteq \bigcup_{i=1}^n F(x_i)$ for all finite sets $\{x_1, \ldots, x_n\} \in X$.*
2. *$F(x)$ is closed for all $x \in X$.*

Then $\bigcap_{x\in X} F(x) \neq \emptyset$.

Theorem 1.2. (Fan, 1961) *Let X be a non-empty, compact and convex subset of a Hausdorff topological vector space. Suppose that $R \subseteq X \times X$ is a relation satisfying:*

1. *For each $x \in X, (x, x) \in R$.*
2. *The set $\{y \in X : (x, y) \notin R\}$ is convex for each $x \in X$.*
3. *The set $\{x \in X : (x, y) \in R\}$ is closed in X for each $y \in X$.*

Then there exists $\overline{x} \in X$ such that $(\overline{x}, y) \in R$ for all $y \in X$.

Theorem 1.3. (Browder, 1968) *Let X be a non-empty, compact and convex subset of a Hausdorff topological vector space, and $B : X \twoheadrightarrow X$ a correspondence such that*

1. *$x \notin B(x)$ for all $x \in X$.*
2. *$B(x)$ is convex for all $x \in X$.*
3. *The set $\{x \in X | y \in B(x)\}$ is open in X for all $y \in X$.*

Then there exists $\bar{x} \in X$ such that $B(\bar{x}) = \varnothing$.

Theorem 1.1 is a special case of Fan's [17] celebrated generalization of the equally celebrated Knaster–Kuratowski–Mazurkiewicz (KKM, hereafter) lemma, and is now generally known as the "FKKM theorem."[2] In the same

[1] Relative to Theorem 1.1 above, Fan's more general Lemma 1 in [17] assumes only that $F(x_0)$ is compact for some $x_0 \in X$.

[2] Reviews of these seven papers are available at *Mathscinet:* they were given favorable notice by senior mathematicians of the subject: Mahlon Day, Jean Paul Doignon, Wolfgang Lusky, Carslaw Scott and Ray Shiflet.

1961 paper, Fan uses his result to prove what we present, taking the mild generalization of Gwinner [22] into account, as our Theorem 1.2 concerning variational relations, a result that is lesser known. Finally, in separate seminal work, Browder [9] proved, what we present in a well known alternative equivalent form, as our Theorem 1.3: a fixed point/maximal-element theorem for correspondences with open lower sections. Our objective then is to clarify the relationship among these three results of Fan and Browder.[3] We show that they are not equivalent but we can identify conditions under which variants of these theorems are indeed equivalent. Furthermore, we can provide generalizations of these equivalences.[4] One may note here that one aspect of Komiya's multi-faceted work concerns maps that go beyond the Kakutani-map, and Komiya was deeply concerned about the equivalences of results for these several types of multi-valued maps as well as with the continuity assumption that is adduced on them.

In terms of a reader's guide to the paper, we present twelve equivalence theorems in three sections: each theorem presenting variants of Fan's two theorems and Browder's theorem recalled above. As such, we present a collection of "circular tours" each of which provides an equivalence between a set intersection theorem, a variational relation existence theorem and a fixed point theorem. Each result requires a "continuity" assumption, a "compactness" assumption and an assumption underscoring the overall "topological structure". Our main results are Theorems A, B and C, on the one hand, and Theorems D, E and F, on the other. All clarify the tradeoff between these three assumptions. They demonstrate, in particular, that as the continuity assumption is strengthened, the compactness and structure assumptions can be correspondingly weakened. Each can be used to prove the other two and in so doing, shed light on the underlying structure of all the three problems. Whereas the first revolves around Theorems 1.1, 1.2 and 1.3 presented above, and its three subsections distinguish between the Hausdorff and the locally convex Hausdorff case, the second focusses on theorems of Komiya, Fan and Peleg, catalogued as Theorems 4.1, 4.5, 4.7 and 4.9.

[3] In the antecedent literature, different combinations of these theorems have been variously presented as the Fan-Browder Theorem, and shown to have many applications in different fields, including mathematics, economics and statistics: the epigraph lists the manifold applications only in pure mathematics.

[4] Browder [10] also mentions the relationship and attributes his theorem to Fan. More to the point, Komiya has worked on generalizations and applications of FKKM theorem and fixed point theorems; see for example [34, 36, 35, 53], and note that our focus in this paper parallels his. In a different direction, L. Shapley provides a generalization of the KKM lemma which is especially useful in economics. Komiya has considerable amount of work in this direction; see for example [33] but we do not focus on this direction in this paper.

We conclude with the plan of the paper. Section 2 presents the basic conceptual and notational preliminaries, and it is followed by the two tours. Section 3 presents the first tour involving Theorems A, B and C, and Section 4 the second set of Theorems D, E and F. Section 5 takes up the opportunity to frame the issues pertaining to equivalence from a methodological and philosophic perspective drawing from the work of Dawson [13] and Thiele [64]. Finally, Section 6 concludes with a summary statement.

2. Maps, correspondences and relations

This section is devoted to laying out the related notation pertaining to a correspondence and its manifestation as a subset of the product set formed by the underlying domain and the codomain. The first two subsections introduce the basic notation, and the third provides specific definitions of KKM and Browder maps and the KF relation.

2.1. **Correspondences.** Let X, Y be two sets. A **correspondence (multi-function, multi-map, map)** from X into Y is a mapping $F : X \twoheadrightarrow Y$ such that $F(x) \subseteq Y$ for all $x \in X$. Given a correspondence $F : X \twoheadrightarrow Y$, define the correspondences $F^{-1} : Y \twoheadrightarrow X$ and $F^c : X \twoheadrightarrow Y$ as:

$$\begin{aligned} F^{-1}(y) =& \{x \in X |\ y \in F(x)\}, \\ F^c(x) =& \{y \in Y |\ y \notin F(x)\}. \end{aligned}$$

Note that for each $y \in X$,

$$(F^{-1})^c(y) = \{x \in X |\ y \notin F(x)\} = \{x \in X |\ y \in F^c(x)\} = (F^c)^{-1}(y).$$

If $F : X \twoheadrightarrow Y$ is correspondence, then the graph of F is the set

$$grF = \{(x, y) \in X \times Y | y \in F(x)\}.$$

For any set A in a vector space, let coA denote the convex hull of A. For any set A in a topological space, let clA denote the closure of A and $intA$ denote the interior of A.

Definition 2.1. Let X be a topological space and let Y be a topological vector space. A correspondence $F : X \twoheadrightarrow Y$ is **co-closed** if the convex hull correspondence $\mathrm{co}F : X \twoheadrightarrow Y$ defined as $(\mathrm{co}F)(x) = \mathrm{co}F(x)$ for all $x \in X$, has closed graph.

Definition 2.2. Let X and Y be topological spaces and let $F : X \twoheadrightarrow Y$ be a correspondence.

1. F has **closed graph** if $\mathrm{gr}F$ is closed in $X \times Y$ endowed with the product topology.
2. F is **closed valued** if $F(x)$ is closed in Y for all $x \in X$.

3. F is **transfer closed valued** if whenever $y \notin F(x)$, there exists an open set $U(y)$ containing y and $x^* \in X$ such that $y' \notin F(x^*)$ for all $y' \in U(y)$.
4. If X is a topological vector space, then F is **securely closed valued** if whenever $y \notin F(x)$, there exists an open set $U(y)$ containing y and a co-closed correspondence $d : U(y) \twoheadrightarrow X$ such that such that $y' \notin F(d(y'))$ for all $y' \in U(y)$.

Remark 2.3. The notion of transfer closed valued correspondence is due to Tian [65] and plays a role in many existence results. The notion of securely closed valued correspondence is new and is motivated by the various "security" conditions used to prove the existence of a Nash equilibrium in games with possibly discontinuous payoffs; see for example, Reny [55, 56, 57, 58], McLennan, Monteiro, and Tourky [44], Carmona and Podczeck [12], Barelli and Meneghel [5], Nessah and Tian [46] and Khan, McLean, and Uyanik [28].

Remark 2.4. It is well known that F is closed valued if F has closed graph but the converse is false. If $X = [0, 1]$ and $F : X \twoheadrightarrow X$ is defined as $F(x) = \{1\}$ if $x \neq 0$ and $F(0) = \{0\}$, then F is closed valued but grF is not closed. Furthermore, it is straightforward to show that F is transfer closed valued if F is closed valued but the converse is false. Let $F : X \twoheadrightarrow X$ be a correspondence with closed values. Pick $x, y \in X$ such that $x \in F^c(y)$. Since $F^c(y)$ is open there exists $U(x)$ containing x such that $x' \in F^c(y)$ for all $x' \in U(x)$. Hence, F is transfer closed valued. In order to see that transfer closed valuedness does not imply closed valuedness, let $X = [0, 1]$ and define $F : X \twoheadrightarrow X$ as $F(x) = (x, 1]$ for all $x < 1$ and $F(1) = \{1\}$. Then, F is transfer closed valued but $F(x)$ is not closed whenever $x < 1$.

If X is a Hausdorff topological vector space, then F is securely closed valued if F is transfer closed valued but the converse is false. If $F : X \twoheadrightarrow Y$ is a transfer closed valued correspondence, choose $x, y \in X$ such that $x \in F^c(y)$. Then there exists $U(x)$ containing x and $y^* \in X$ such that $x' \notin F(y^*)$ for all $x' \in U(x)$. Since a singleton set is closed in a Hausdorff space, the correspondence $d : U(x) \twoheadrightarrow X$ defined as $d(x') = \{y^*\}$ for each $x' \in U(x)$ is co-closed. Hence, F is securely closed valued. In order to see that securely closed valuedness does not imply transfer closed valuedness, let $X = [0, 1]$ and define $F : X \twoheadrightarrow X$ as $F(x) = X \backslash \{x\}$ for all $x \in X$. Then for all $x, y \in X$, $y \notin F(x)$ if and only if $y = x$. Pick $x \in X$. Define $U(x) = [0, 1]$ and $d(x) = \{x\}$ for all $x \in X$. Then d is co-closed. Therefore F is securely closed valued. It is easy to see that F is not transfer closed valued.

Definition 2.5. Let X and Y be topological spaces and let $F : X \twoheadrightarrow Y$ be a correspondence.

1. F has **open graph** if gr$F$ is open in $X \times Y$ endowed with the product topology.
2. F has **open lower sections** if $F^{-1}(y)$ is open for all $y \in Y$.
3. F has **the local intersection property** if for each $x \in X$ with $F(x) \neq \varnothing$, there exists an open set $U(x)$ containing x and $y^* \in Y$ such that $y^* \in F(x')$ for each $x' \in U(x)$.
4. If Y is a topological vector space, then F has **the continuous inclusion property** if for each $x \in X$ with $F(x) \neq \varnothing$, there exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \twoheadrightarrow Y$ such that $d(x') \subseteq F(x')$ for each $x' \in U(x)$.

Remark 2.6. The significance of correspondences with open lower sections in fixed point theory was made clear by Browder's 1968 result. Such correspondences have since been used to prove existence results for collective fixed point theorems and related problems in the theory of generalized games; see for example, Sonnenschein [60], Yannelis and Prabhakar [71], Deguire, Tan, and Yuan [14], Lin and Ansari [39] and Lin and Ansari [40]. Similar comments apply to correspondences with the local intersection property; see for example, [**?**] and Ansari and Yao [2]. The continuous inclusion property and its application to fixed point theory and generalized games appears in He and Yannelis [23] and Uyanik [67] and Khan and Uyanik [30]. □

Remark 2.7. From the definition of product topology, it is clear that F has open lower sections if F has open graph but the converse is false. If $X = [0, 1]$ and $F : X \twoheadrightarrow X$ is defined as $F(x) = [0, 1]\backslash\{x\}$ if $x < 1$ and $F(1) = [0, 1]$, then F has open lower sections but grF is not open.

Furthermore, F has the local intersection property if F has open lower sections but the converse is false**.** To see this , let $F : X \twoheadrightarrow X$ be a correspondence with open lower sections. Choose $x \in X$ such that there exists $y \in F(x)$. Since $F^{-1}(y)$ open, there exists an open set $U(x)$ containing x such that $y \in F(x')$ for each $x' \in F(x)$. Hence, F has the local intersection property. In order to see that local intersection property does not imply open lower sections, let $X = [0, 1]$ and define $F : X \twoheadrightarrow X$ as $F(x) = [0.5(x + 1), 1]$ for all $x < 1$ and $F(1) = \varnothing$. Then, F has the local intersection property since for all $x < 1$, selecting $U(x) = [0, 1)$ and $y^* = 1$ imply $y^* \in F(x')$ for all $x' \in U(x)$. However, $F^{-1}(\frac{1}{2}) = \{0\}$ is not open.

If Y is a Hausdorff TVS and F has the local intersection property then F has the continuous inclusion property but the converse is false. Let $F : X \twoheadrightarrow X$ be a correspondence with the local intersection property. Pick $x \in X$ such that $F(x) \neq \varnothing$. Then, there exists an open set $U(x)$ containing x and $y^* \in X$ such that $y^* \in F(x')$ for each $x' \in U(x)$. Since a singleton set is closed in a Hausdorff space, the correspondence $d : U(x) \twoheadrightarrow X$ defined as $d(x') = \{y^*\}$ for each $x' \in U(x)$ is co-closed. Hence B has the continuous inclusion property. In order to see that local intersection property does not imply the continuous inclusion property, let $X = [0,1]$ and define $F : X \twoheadrightarrow X$ as $F(x) = \{0.5(x+1)\}$ for all $x < 1$ and $F(1) = \emptyset$. It is clear that F does not have the local intersection property but it has the continuous inclusion property.

2.2. **Relations.** Let X, Y be two sets. A **relation** in $X \times Y$ is a subset R of $X \times Y$. For each $(x, y) \in X \times Y$, define the upper section $R(x)$, its complement $R^c(x)$ and the lower section $R^{-1}(y)$ as:

$$R(x) = \{y \in X | \ (x, y) \in R\}$$
$$R^c(x) = \{y \in X | \ (x, y) \notin R\}$$
$$R^{-1}(y) = \{x \in X | \ (x, y) \in R\}.$$

If $F : X \twoheadrightarrow Y$ is a correspondence and $R = grF$, then $R(x) = F(x), R^c(x) = F^c(x)$ and $R^{-1}(y) = F^{-1}(y)$ for each $(x, y) \in X \times Y$.

Definition 2.8. Let X and Y be topological spaces and let $R \subseteq X \times Y$ be a relation.

1. R is **closed** if R is closed in $X \times Y$ endowed with the product topology.
2. R has **closed lower sections** if $R^{-1}(y)$ is closed for all $y \in Y$.
3. R **transfer semi-continuous** if $x \in X$ and $(x, y) \notin R$ imply that there exists an open set $U(x)$ containing x and $y^* \in Y$ such that $(x', y^*) \notin R$ for all $x' \in U(x)$.
4. If Y is a topological vector space, then R is **correspondence secure** if whenever $x \in X$ and $(x, y) \notin R$, there exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \twoheadrightarrow Y$ such that $d(x') \subseteq \{y \in Y : (x', y) \notin R\}$ for all $x' \in U(x)$.

Remark 2.9. Transfer semi-continuous and correspondence secure relations appear in McLean [43]. □

Remark 2.10. A relation R has closed lower sections if R is closed, and R is transfer semi-continuous if R has closed lower sections but the converses are false and we omit the straightforward proofs. In addition, R is correspondence secure if R transfer semi-continuous but the convese is false and the proof follows from arguments similar to those of Remark 2.7. □

2.3. **KKM maps, KF relations and Browder maps.** We begin with three definitions.

Definition 2.11. Let X be a a non-empty and convex subset of a vector space.
A correspondence[5] $F : X \twoheadrightarrow X$ is a

[5] See Yuan [73], Fan [19].

1. **KKM map** if $\text{co}\{x_1, .., x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$ for every finite set $\{x_1, \ldots, x_n\} \subseteq X$.
2. **strong KKM map** if $(F^{-1})^c(y)$ is convex for each $y \in X$ and $x \in F(x)$ for each $x \in X$.

A relation[6] $R \subseteq X \times X$ is a

3. **KF relation** if the following holds: For each x, $x \notin coR^c(x)$, where $R^c(x) = \{y \in X : (x, y) \notin R\}$.
4. **strong KF relation** if $R^c(x)$ is convex for each $x \in X$ and $(x, x) \in R$ for each $x \in X$.

A correspondence[7] $B : X \twoheadrightarrow X$ is a

5. **Browder map** if the following holds: $x \notin coB(x)$ for each $x \in X$.
6. **strong Browder map** if for each $x \in X$, $B(x)$ is convex and $x \notin B(x)$.

The following lemma provides a charactization of KKM maps that plays a crucial role in many of the arguments of this paper.

Lemma 2.12. Let X be a a non-empty, convex subset of a vector space and $F : X \twoheadrightarrow X$ a correspondence. Then the following are equivalent:

1. F is a KKM map.
2. $x \notin co[(F^{-1})^c(x)]$ for each $x \in X$.

Proof of Lemma 2.12. Assume that for every $x \in X$, $x \notin \text{co}[X \backslash F^{-1}(x)]$ and that F is not a KKM map. Then there exists a set $\{x_1, .., x_n\}$ and $x \in \text{co}\{x_1, .., x_n\}$ such that $x \notin \cup_i F(x_i)$. Therefore, $x_i \notin F^{-1}(x)$ for each i implying that $x_i \in X \backslash F^{-1}(x)$ for each i. Therefore $x \in \text{co}[X \backslash F^{-1}(x)]$, a contradiction. Assume that F is a KKM map and that there exists an x such that $x \in \text{co}[X \backslash F^{-1}(x)]$. Then there exists a set $\{x_1, .., x_n\}$ such that $x_i \notin F^{-1}(x)$ for each i and $x \in \text{co}\{x_1, .., x_n\}$. Therefore $x \in \text{co}\{x_1, .., x_n\}$ but $x \notin \cup_i F(x_i)$. This contradicts the assumpton that F is a KKM map. □

Remark 2.13. From Lemma 2.12 and the definitions, we conclude that every strong KKM map (strong KF relation, strong Browder map) is a KKM map (KF relation, Browder map). □

In the results of this and the following sections, we will make frequent use of the following three Lemmas. In each result, indeed throughout the paper, we omit the "strong" version of our equivalence results since the proofs are identical in the strong case. The proofs of the Lemmas are not difficult; for

[6] See Yuan, Frenk, Kassay, and Kolumbán [20].

[7] In Yannelis and Prabhakar [71], a strong Browder map with open lower sections is called a KF correspondence. In Yannelis and Prabhakar [71], a Browder map with open lower sections is said to be of class $\mathcal{L}$. See also Tarafdar [61, 62], Khan and Uyanik [29], Park and Jeong [52].

the convenience of the reader, we provide the proof of the first one and omit the other two.

Lemma 2.14. Let X be a non-empty subset of a vector space and let $R \subseteq X \times X$ be a relation. Then the following are equivalent:

1. R is a (strong) KF relation.
2. $F : X \twoheadrightarrow X$ defined as $F(y) = R^{-1}(y)$ for each $y \in X$ is a (strong) KKM map.
3. $B : X \twoheadrightarrow X$ defined as $B(x) = R^c(x)$ for each $x \in X$ is a (strong) Browder map.

Proof of Lemma 2.14. $(1 \Rightarrow 2)$ Suppose that $R \subseteq X \times X$ is a KF relation. If $F(y) = R^{-1}(y)$ for each $y \in X$, then $(F^{-1})^c(y) = R^c(y)$ for each $y \in X$. Therefore $y \notin co[(F^{-1})^c](y)$ for each $y \in X$ and it follows that $F : X \twoheadrightarrow X$ is a KKM map.

$(2 \Rightarrow 3)$ Suppose that $F : X \twoheadrightarrow X$ defined as $F(y) = R^{-1}(y)$ for each $y \in X$ is a KKM map. If $B : X \twoheadrightarrow X$ is defined as $B(y) = R^c(y)$ for each $y \in X$, then $B(y) = ((R^c)^{-1})^{-1}(y) = ((R^{-1})^c)^{-1}(y) = (F^c)^{-1}(y)$ for each $y \in X$ implying that B is a Browder map.

$(3 \Rightarrow 1)$ This is an immediate consequence of the definitions. $\square$

Lemma 2.15. Let X be a non-empty subset of a vector space and let $F : X \twoheadrightarrow X$ be a correspondence. Then the following are equivent:

1. F is a (strong) KKM map.
2. $B : X \twoheadrightarrow X$ defined as $B(y) = (F^{-1})^c(y)$ for each $y \in X$ is a (strong) Browder map.
3. $R \subseteq X \times X$ defined as $R = [X \times X] \backslash gr(F^{-1})^c = grF^{-1}$ is a (strong) KF relation.

Lemma 2.16. Let X be a non-empty subset of a vector space and let $B : X \twoheadrightarrow X$ be a correspondence. Then the following are equivalent:

1. B is a (strong) Browder map.
2. $R = grB^c$ is a (strong) KF relation.
3. $F : X \twoheadrightarrow X$ defined as $F(y) = (B^{-1})^c(y)$ for each $y \in X$ is a (strong) KKM map.

3. Circular tours

The community to which this work is addressed sees little reason to interrogate the word *equivalence,* and takes its meaning for granted; see for example [22] and Borgersen [7]. However, we are using equivalence in our theorems is a nuancedly-different way, and it seems to us worthwhile to our results by elaborating the sense in which we use the term.[8] The point is that an object we work with can be portrayed in three different ways – a Fan-relation, a Browder-map, and a KKM-map – and what is true for one portrayal translates into a straightforward modification of the others. To put the matter in another way, once we obtain a theorem in one register, one has automatically proved a corresponding result in the two other registers. A "truth table" is helpful in illustrating this, and Lemmas 2.14 – 2.16 below, to take one example, show how the hypotheses of suitable versions of FKKM, Fan, and Browder Theorems are equivalent in the nuanced meaning that we are giving to the word. In Table 1 below, if a property is true (T) for a Browder map B, then it is true for a KF relation induced by grB^c as well as for a KKM map induced by $(B^{-1})^c$.

B is Browder Map with Open Graph	grB^c is KF Relation with Closed Graph	$(B^{-1})^c$ is KKM Map with Closed Graph
T	T	T
F	F	F

TABLE 1. Equivalence of Hypotheses

Truth-table 1 pertained to the hypotheses, but there are also conclusions to be considered, and we turn to them.

$B(x) = \emptyset,\ x \in X$	$\cap_{x\in X} grB^c(x) \neq \emptyset,\ x \in X$	$(B^{-1})^c(x) = \emptyset,\ x \in X$
T	T	T
F	F	F

TABLE 2. Equivalence of Conclusions

And so it is not only that two "if then" statements are equivalent in the colloquial sense of having the same truth values, it is the considerably stronger statement that that their hypotheses (their "ifs") as well as their conclusions (their "thens") have the same truth value.

3.1. **The Hausdorff locally convex case.** In our first circular tour, we make the weakest continuity assumption but the strongest compactness and topological structure assumptions. Again, in Theorem A and throughout this section, we omit the proofs for the strong version of each result.

Theorem A. *Let X be a non-empty and convex subset of a Hausdorff, locally convex topological vector space. Then the following are true and equivalent:*

[8]The uninterested reader can skip this opening and proceed directly to the results; in any case, Section 5 goes into the issues in some comprehensive detail.

1. *Suppose that* $F : X \twoheadrightarrow X$ *is a (strong) KKM map. If* F *is securely closed valued and* X *is compact, then* $\bigcap_{x\in X} F(x) \neq \varnothing$.
2. *Suppose that* $R \subseteq X \times X$ *is a (strong) KF relation. If* $R \subseteq X \times X$ *is correspondence secure and if* X *is compact, then* $R(\bar{x}) = X$ *for some* $\bar{x} \in X$.
3. *Suppose that* $B : X \twoheadrightarrow X$ *is a (strong) Browder map. If* B *has the continuous inclusion property and if* X *is compact, then* $B(\overline{x}) = \varnothing$ *for some* $\overline{x} \in X$.

Proof of Theorem A. The result is true[9] since Theorem A.3 is equivalent to Theorem 2 in He and Yannelis [24] so it remains to prove the equivalences.

($1 \Rightarrow 2$): Let R be a correspondence secure KF relation. Then the correspondence $F : X \twoheadrightarrow X$ defined as $F(y) = R^{-1}(y)$ is a securely closed KKM map. To show that F is securely closed valued, choose $x, y \in X$ such that $x \notin F(y)$. Then $(x, y) \notin \mathrm{gr}F^{-1}$ implies that $(x, y) \notin R$ since $\mathrm{gr}F^{-1} = R$. Then there exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \twoheadrightarrow X$ such that $d(x') \subseteq \{y \in X : (x', y) \notin R\}$ for all $x' \in U(x)$. Hence, $x' \notin F(d(x'))$ for all $x' \in U(x)$. Therefore, F is securely closed valued. Hence there exists $\overline{x} \in X$ such that $\overline{x} \in \bigcap_{y\in X} F(y)$ implying that $R(\overline{x}) = X$.

($2 \Rightarrow 3$): Let B be a Browder map with the continuous inclusion property. Defining $R = [X \times X] \backslash (\mathrm{gr}B) = \{(x, y) \in X \times X | y \notin B(x)\}$, it follows that R is a correspondence secure KF relation. To show that R is correspondence secure, choose $x, y \in X$ such that $(x, y) \notin R$. Then $y \in B(x)$. Hence there exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \twoheadrightarrow X$ such that $d(x') \subseteq B(x') = R^c(x')$ for each $x' \in U(x)$. Therefore, R is correspondence secure. Hence there exists $\overline{x} \in X$ such that $(\overline{x}, y) \in R$ for all $y \in X$. Therefore, $R^c(\bar{x}) = B(\bar{x}) = \varnothing$.

($3 \Rightarrow 1$): Let F be a securely closed valued KKM map. Define $B : X \twoheadrightarrow X$ as $B(x) = (F^{-1})^c(x)$ and note that B is a Browder map with the continuous inclusion property. To show that B has the continuous inclusion property, choose $x \in X$ with $B(x) \neq \emptyset$. Then there exists $y \in X$ such that $y \notin F^{-1}(x)$. Since $x \notin F(y)$, there exists an open set $U(x)$ containing x and a co-closed correspondence $d : U(x) \twoheadrightarrow X$ such that such that $x' \notin F(d(x'))$ for all $x' \in U(x)$. Therefore, $d(x') \subseteq (F^{-1})^c(x') = B(x')$ for all $x' \in U(x)$ implying that B has the continuous inclusion property. Hence there exists $\overline{x} \in X$ such that $(F^{-1})^c(\overline{x}) = B(\overline{x}) = \varnothing$ and it follows that $F^{-1}(\overline{x}) = X$ implying that $\overline{x} \in \bigcap_{y\in X} F(y)$. □

Remark 3.1. As mentioned in the proof, Theorem A.3 is exactly Theorem 2 in He and Yannelis [24] and Theorem A.2 is equivalent to Corollary 1 of the

[9] Note that the local convexity assumption is used only at this step.

same paper. Theorem A.2 is also a special case of Theorem 4 in McLean [43]. Theorem A.1 is new. Yannelis [70] proves a related equivalence result where it is shown that Theorem 1.3 is equivalent to Lemma 1 in Fan [17].

3.2. **The Hausdorff case: Strengthened continuity and weakened compactness.** In Theorem A, it was assumed that X is a compact subset of a Hausdorff, locally convex TVS. In this section, we present a result in which the continuity assumption in each part of Theorem A is strengthened but the local convexity assumption is dropped and the compactness assumption for X is significantly weakened. We begin by defining a condition that is found in a number of papers in the fixed point literature.

Definition 3.2. Let X be a a non-empty and convex subset of a topological vector space. A correspondence $G : X \twoheadrightarrow X$ satisfies the **compact intersection property** (CIP) if the following holds:

There exist non-empty and compact sets $K, M \subseteq X$, where M is convex, such that for each $x \in X \backslash K$ there exists an open set $U(x)$ containing x with

$$\left[\bigcap_{x' \in U(x)} G(x')\right] \cap M \neq \varnothing.$$

Theorem B. *Let X be a non-empty and convex subset of a Hausdorff topological vector space. Then the following are true and equivalent:*

1. *Suppose that $F : X \twoheadrightarrow X$ is a (strong) KKM map. If F is transfer closed valued and if $x \mapsto (F^{-1})^c(x)$ satisfies CIP, then $\bigcap_{x\in X} F(x) \neq \varnothing$.*
2. *Suppose that $R \subseteq X \times X$ is a (strong) KF relation. If $R \subseteq X \times X$ is transfer semi-continuous and if $x \mapsto R^c(x)$ satisfies CIP, then $R(\bar{x}) = X$ for some $\bar{x} \in X$.*
3. *Suppose that $B : X \twoheadrightarrow X$ is a (strong) Browder map. If B satisfies the local intersection property and if B satisfies CIP, then $B(\overline{x}) = \varnothing$ for some $\overline{x} \in X$.*

Proof of Theorem B. Theorem B.3 is true since it is a special case of Theorem 1 in Ansari and Yao [2] by applying Remark b.(ii)$''$ of their paper. It now remains to show that the claims are equivalent.

($1 \Rightarrow 2$) Let R be a transfer semi-continuous KF relation and suppose that $x \mapsto R^c(x)$ satisfies CIP. Then the correspondence $F : X \twoheadrightarrow X$ defined as as $F(y) = R^{-1}(y)$ is a transfer closed valed KKM map with closed graph. To see that F is transfer closed valued, choose $x, y \in X$ such that $x \notin F(y)$. Then $(x, y) \notin \mathrm{gr}F^{-1}$ implies that $(x, y) \notin R$ since $\mathrm{gr}F^{-1} = R$. Then there exists an open set $U(x)$ containing x and $y^* \in X$ such that $x' \notin R^{-1}(y^*) = F(y^*)$ for all $x' \in U(x)$. Therefore, F is transfer closed valued. Next, note that

$(F^{-1})^c(x) = X\backslash F^{-1}(x) = X\backslash R(x) = R^c(x)$ implying that F is a KKM map and that $x \mapsto (F^{-1})^c(x)$ satisfies CIP. Hence there exists $\overline{x} \in X$ such that $\overline{x} \in \bigcap_{y\in X} F(y)$ implying that $R(\overline{x}) = X$.

$(2 \Rightarrow 3)$ Let B be a Browder map with the local intersection property and suppose that B satisfies CIP. Defining $R = (X \times X)\backslash(\text{gr}B) = \{(x, y) \in X \times X | y \notin B(x)\}$, it follows that R is a transfer semi-continuous KF relation. To show that R is transfer semi-continuous, pick $x, y \in X$ such that $(x, y) \notin R$. Then $y \in B(x)$. Hence there exists an open set $U(x)$ containing x and $y^* \in X$ such that $y^* \in B(x') = R^c(x')$ for each $x' \in U(x)$. Therefore, R is transfer semi-continuous. Next, note that $R^c(x) = B(x)$ implying R is a KF relation and that $x \mapsto R^c(x)$ satisfies CIP. Hence there exists $\overline{x} \in X$ such that $(\overline{x}, y) \in R$ for all $y \in X$. Therefore, $R^c(\bar{x}) = B(\bar{x}) = \varnothing$.

$(3 \Rightarrow 1)$ Let F be a transfer closed KKM map and suppose that $x \mapsto (F^{-1})^c(x)$ satisfies CIP. Define $B : X \twoheadrightarrow X$ as $B(x) = (F^{-1})^c(x)$ and note that B is a Browder map with the local intersection property. To show that B has the local intersection property, pick $x \in X$ such that $B(x) \neq \emptyset$. Then there exists $y \in X$ such that $y \notin F^{-1}(x)$. Since $x \notin F(y)$, there exists $U(x)$ containing x and $y^* \in X$ such that $x' \notin F(y^*)$, i.e., $y^* \in B(x')$, for all $x' \in U(x)$. Therefore, B has the local intersection property. Furthermore, B satisfies CIP. Hence there exists $\overline{x} \in X$ such that $(F^{-1})^c(\overline{x}) = B(\overline{x}) = \varnothing$ and it follows that $F^{-1}(\overline{x}) = X$ implying that $\overline{x} \in \bigcap_{y\in X} F(y)$. □

Remark 3.3. As mentioned in the proof, Theorem B.3 is a special case of Theorem 1 in Ansari and Yao [2] and also Theorem 8 in Balaj and Muresan [4]. Theorem B.2 is a special case of Theorem 3 in McLean [43]. Theorem B.1 is new. Tian [65, Theorem 2] obtains the conclusion of Theorem B.1 for transfer closed KKM maps using an assumption different from CIP. □

Corollary 3.4. Let X be a non-empty, convex subset of a Hausdorff topological vector space. Then the following are true and equivalent:

1. Suppose that $F : X \twoheadrightarrow X$ is a (strong) KKM map. If F is transfer closed valued and if $clF(y_0)$ is compact for some $y_0 \in X$, then $\bigcap_{x\in X} F(x) \neq \varnothing$.
2. Suppose that $R \subseteq X \times X$ is a (strong) KF relation. If R is transfer semi-continuous and if $cl[R^{-1}(y_0)]$ is compact for some $y_0 \in X$, then $R(\bar{x}) = X$ for some $\bar{x} \in X$.
3. Suppose that $B : X \twoheadrightarrow X$ is a (strong) Browder map. If B has has the local intersection property and if $cl[X\backslash B^{-1}(y_0)]$ is compact for some $y_0 \in X$, then $B(\overline{x}) = \varnothing$ for some $\overline{x} \in X$.

Proof of Corollary 3.4. The equivalence of the three results follows from the same arguments as those of Theorem B. To show that each claim is true, it suffices to show that the hypotheses of Corollary 3.4.1 imply those of Theorem B.1. To see this, suppose that $clF(x_0)$ is compact for some $x_0 \in X$ and define $K = clF(x_0)$ and $M = \{x_0\}$. For each $x \in X\backslash K$, it follows that $U(x) = X\backslash K$ is open. Therefore,

$$x' \in U(x) \Rightarrow x' \in X\backslash K \Rightarrow x' \in X\backslash \mathrm{cl}F(x_0) \Rightarrow x' \notin F(x_0) \Rightarrow x_0 \notin F^{-1}(x') \cdots$$
$$\cdots \Rightarrow x_0 \in X\backslash F^{-1}(x').$$

It follows that

$$\left[\bigcap_{x' \in U(x)} (F^{-1})^c(x')\right] \cap M = \{x_0\}$$

implying that $x \mapsto (F^{-1})^c(x)$ satisfies CIP. □

Remark 3.5. Corollary 3.4.1 is a special case of Theorem 2.1 in Ansari, Lin, and Yao [1]. Corollary 3.4.2 is a special case of Theorem 3 in McLean [43] while Corollary 3.4.3 is new.

3.3. **The Hausdorff case: Further strengthening and weakening.** In this section, we present a result in which the continuity assumption of Theorem B is stregthened but the compact intersection property is weakened. We begin by defining a condition weaker than the compact intersection property that is also useful in a number of fixed point and maximal element existence results; see for example Deguire, Tan, and Yuan [14], Lin and Ansari [39, 40].

Definition 3.6. Let X be a non-empty and convex subset of a topological vector space. A correspondence $G : X \twoheadrightarrow X$ is satisfies the **weak compact intersection property** (WCIP) if the following holds: there exist non-empty and compact sets $K, M \subseteq X$, where M is convex, such that $G(x) \cap M \neq \varnothing$ for each $x \in X\backslash K$.

Remark 3.7. A correspondence $G : X \twoheadrightarrow X$ satisfies WCIP if G satisfies CIP but the converse is false. To see this, let $X = [0, 1[$ and define $G : X \twoheadrightarrow X$ as $G(x) = \{\frac{x}{2}\}$. Let $M = [0, \frac{1}{2}]$ and $K = \{0\}$. Then K and M are non-empty and compact with M convex such that $G(x) \cap M \neq \varnothing$ for each $x \in X\backslash K$. However, G does not satisfy CIP. Suppose that $K \subseteq [0, 1[$ is compact and nonempty. Then $x^* = \max\{x : x \in K\} < 1$. Choose $x^* < x < 1$. Then $x \in X\backslash K$ and if $U(x)$ is an open set containing x, it follows that $\bigcap_{x' \in U(x)} G(x') = \varnothing$.

Theorem C. *Let X be a non-empty, convex subset of a Hausdorff topological vector space. Then the following are true and equivalent:*

1. *Suppose that $F : X \twoheadrightarrow X$ is a (strong) KKM map. If F has closed values and if $x \mapsto (F^{-1})^c(x)$ satisfies WCIP, then $\bigcap_{x \in X} F(x) \neq \varnothing$.*

2. *Suppose that* $R \subseteq X \times X$ *is a (strong) KF relation. If* R *has closed lower sections and and if* $x \mapsto R^c(x)$ *satisfies WCIP, then* $R(\bar{x}) = X$ *for some* $\bar{x} \in X$.
3. *Suppose that* $B : X \twoheadrightarrow X$ *is a (strong) Browder map. If* B *has open lower sections and and if* B *satisfies WCIP, then* $B(\overline{x}) = \varnothing$ *for some* $\overline{x} \in X$.

Remark 3.8. Theorem C.1, Theorem C.2 and Theorem C.3 are, respectively, special cases of Theorem 3.2, 3.1 and 2.1 in Lin and Ansari [40]. The equivalence of Theorems C.1 and C.2 can also be deduced as a special case of the more general equivalence result stated as Remark 3.2 in Lin and Ansari. Using the argument in Lin and Ansari [40], one can use Theorem C.3 to prove Theorem C.2. Theorem C completes the circular tour by showing that the generalization of Browder's theorem given by Theorem C.3 is actually equivalent to Theorems C.1 and C.3.

Corollary 3.9. Let X be a non-empty, convex subset of a Hausdorff topological vector space. Then the following are true and equivalent:

1. Suppose that $F : X \twoheadrightarrow X$ is a (strong) KKM map. If F has closed values and if $F(y_0)$ is compact for some $y_0 \in X$, then $\bigcap_{x\in X} F(x) \neq \varnothing$.
2. Suppose that $R \subseteq X \times X$ is a (strong) KF relation. If R has closed lower sections and if $R^{-1}(y_0)$ is compact for some $y_0 \in X$, then $R(\bar{x}) = X$ for some $\bar{x} \in X$.
3. Suppose that $B : X \twoheadrightarrow X$ is a (strong) Browder map. If B has open lower sections and if $X \backslash B^{-1}(y_0)$ is compact for some $y_0 \in X$, then $B(\overline{x}) = \varnothing$ for some $\overline{x} \in X$.

Remark 3.10. If X is compact, then the equivalence of Theorems 1.1, 1.2 and 1.3 discussed in the introduction is an obvious consequence of Corollary 3.9. Corollary 3.9.2 is a generalization of Theorem 4′ in Gwinner [22].

Corollary 3.11. Let X be a non-empty, convex subset of a topological vector space. Then the following are true and equivalent:

1. Suppose that $F : X \twoheadrightarrow X$ is a (strong) KKM map. If F has closed graph and if $F(y_0)$ is compact for some $y_0 \in X$, then $\bigcap_{x\in X} F(x) \neq \varnothing$.
2. Suppose that $R \subseteq X \times X$ is a (strong) KF relation. If R is closed and if $R^{-1}(y_0)$ is compact for some $y_0 \in X$, then $R(\bar{x}) = X$ for some $\bar{x} \in X$.
3. Suppose that $B : X \twoheadrightarrow X$ is a (strong) Browder map. If B has open graph and if $X \backslash B^{-1}(y_0)$ is compact for some $y_0 \in X$, then $B(\overline{x}) = \varnothing$ for some $\overline{x} \in X$.

Remark 3.12. Under the additional assumption that X is compact, the strong version of Corollary 3.11.3 appears in Borglin and Keiding [8] who refer to a strong Browder map with open graph as a “KF correspondence.” See also Gale and Mas-Colell [21], Shafer and Sonnenschein [59] and Bergstrom, Parks, and Rader [6] for applications of open graph property in mathematical economics.

The main results of this section, Theorems A, B and C, are summarized in the table below and illustrate the trade-off between ”continuity”, compactness and the topological structure of the parent TVS containing X.

If X is a subset of a locally convex Hausdorff TVS, then the assumptions of Theorem A are

Result	KF map	KF relation	Browder map	domain X
Thm A	securely closed	correspondence secure	continuous inclusion	compact

If X is a subset of a Hausdorff TVS, then the assumptions of Theorem B and C and their relationships are

Result	KF map	KF relation	Browder map	domain X
Thm B	transfer closed	transfer semicontinuous	local intersection property	CIP
	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Downarrow$
Thm C	closed valued	closed lower sections	open lower sections	WCIP

If X is a subset of a locally, convex Hausdorff TVS, then a complete picture of the assumptions of Theorem A, B and C, and their relationships is

Result	KF map	KF relation	Browder map	domain X
Thm A	securely closed	correspondence secure	continuous inclusion	compact
	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Downarrow$
Thm B	transfer closed	transfer semicontinuous	local intersection property	CIP
	$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Downarrow$
Thm C	closed valued	closed lower sections	open lower sections	WCIP

4. Additional results

In this section we present three results based on the KKM lemma and provide additional circular tours.

4.1. **Komiya's coincidence theorem.**

Intersection theorems of FKKM type deal with self maps. If X and Y are different sets and if $F : X \twoheadrightarrow Y$ is a correspondence, then it is of interest to know when $\cap_{x\in X} F(x)$ is a non-empty subset of Y. Komiya [32] proves the following coincidence theorem that generalizes Browder's theorem.

Theorem 4.1. (Komiya, 1986) *Suppose that X is a non-empty, convex subset of a Hausdorff topological vector space and suppose that Y is a non-empty, convex, compact subset of a Hausdorff topological vector space. Let $A : X \twoheadrightarrow Y$ be a non-empty valued, closed valued, convex valued upper hemicontinuous correspondence and let $B : Y \twoheadrightarrow X$ be a non-empty valued, convex valued correspondence with open lower sections. Then there exists a $\overline{y} \in Y$ such that $B(\overline{y}) \cap A^{-1}(\overline{y}) \neq \varnothing$.*

In this section, we will apply Komiya's coincidence theorem to prove a result of FKKM type for a correspondence $F : X \twoheadrightarrow Y$. In fact, we prove an equivalence theorem that generalizes several results of Section 3 above.

Definition 4.2. Suppose that X is a non-empty subset of a vector space and Y is a non-empty set. Suppose that $A : X \twoheadrightarrow Y$ and $F : X \twoheadrightarrow Y$ are correspondences. Then F is a **KKM map with respect to** A if

$$A^{-1}(y) \cap \text{co}\{x_1, .., x_n\} \neq \varnothing \text{ implies } y \in \bigcup_{i=1}^{n} F(x_i)$$

for every $y \in Y$ and for every finite set $\{x_1, \ldots, x_n\} \subseteq X$.

Remark 4.3. If $X = Y$ and $A(x) = \{x\}$, then $F : X \twoheadrightarrow X$ is a KKM map with respect to A if and only if F is a KKM map in the sense of Definition 2.11. Similar definitions of KF relation with respect to A and Browder map with respect to A are straightforward.

The following result provides a characterization of a KKM map with respect to a correspondence; we provide its proof for the convenience of the reader.

Lemma 4.4. Suppose that X is a non-empty subset of a vector space and Y is a non-empty set. Suppose that $A : X \twoheadrightarrow Y$ and $F : X \twoheadrightarrow Y$ are correspondences. Then the following are equivalent:

1. F is a KKM map with respect to A
2. $co(F^{-1})^c(y) \cap A^{-1}(y) = \varnothing$ for all $y \in Y$

Proof of Lemma 4.4. There exists $\overline{y} \in Y$ such that $co(F^{-1})^c(\overline{y}) \cap A^{-1}(\bar{y}) \neq \varnothing$ iff there exists $\overline{y} \in Y$ and $\overline{x} \in A^{-1}(\overline{y})$ and a finite set $\{x_1, \ldots, x_n\} \subseteq X$ such that $\overline{x} \in \text{co}\{x_1, .., x_n\}$ and $\overline{y} \notin \cup_{i=1}^{n} F(x_i)$. □

We next reformulate Komiya's theorem in a manner analogous to the reformulation of Browder's theorem presented as Theorem 1.3.

Theorem 4.5. *Suppose that X is a non-empty, convex subset of a Hausdorff topological vector space and suppose that Y is a non-empty, convex, compact subset of a Hausdorff topological vector space. Let $A : X \twoheadrightarrow Y$ be a non-empty valued, closed valued, convex valued upper hemicontinuous correspondence and* *let $G : Y \twoheadrightarrow X$ be a correspondence with open lower sections. If $coG(y) \cap A^{-1}(y) = \varnothing$ for each $y \in Y$, then there exists a $\overline{y} \in Y$ such that $G(\overline{y}) = \varnothing$.*

Remark 4.6. Theorems 4.1 and 4.5 are equivalent. To see that Theorem 4.1 implies Theorem 4.5, suppose that X, Y, A and G satisfy the assumptions of Theorem 5 but $G(y) \neq \varnothing$ for each $y \in Y$. Define a correspondence $B : Y \twoheadrightarrow X$ as $B(y) = coG(y)$ and note that $B(y) \neq \varnothing$ for each $y \in Y$. Applying Lemma 5.1 in [71], it follows that B is convex valued with open lower sections. Applying Theorem 4.1, there exists a $\overline{y} \in Y$ such that $B(\overline{y}) \cap A^{-1}(\overline{y}) \neq \varnothing$, an impossibility.

To see that Theorem 4.5 implies Theorem 4.1, suppose that X, Y, A and B satisfy the assumptions of Theorem 4 but $B(y) \cap A^{-1}(y) = \varnothing$ for each $y \in Y$. Defining $G(y) = coB(y) = B(y)$, it follows that X, Y, A and G satisfy the assumptions of Theorem 5. Consequently, there exists a $\overline{y} \in Y$ such that $B(\overline{y}) = G(\overline{y}) = \varnothing$, an impossibility.

The next result provides a new result of FKKM type and new result for relations that are equivalent to Komiya's coincidence theorem.

Theorem D. *Suppose that X is a non-empty, convex subset of a Hausdorff topological vector space and suppose that Y is a non-empty, convex, compact subset of a Hausdorff topological vector space and let $A : X \twoheadrightarrow Y$ be a non-empty valued, closed valued, convex valued upper hemicontinuous correspondence. Then the following are true and equivalent.*

1. *Let $F : X \twoheadrightarrow Y$ be correspondence with closed values. If F is a KKM map with respect to A, i.e., $co(F^{-1})^c(y) \cap A^{-1}(y) = \varnothing$ for all $y \in Y$, then there exists a $\overline{y} \in Y$ such that $\overline{y} \in \cap_{x\in X} F(x)$, i.e., $\cap_{x\in X} F(x) \neq \varnothing$.*
2. *If $R \subseteq Y \times X$ is a relation with closed lower sections such that $coR^c(y) \cap A^{-1}(y) = \varnothing$ for all $y \in Y$, then there exists a $\overline{y} \in Y$ such that $(\overline{y}, x) \in R$ for each $x \in X$.*
3. *If $B : Y \twoheadrightarrow X$ is a convex valued correspondence with open lower sections such that $B(y) \cap A^{-1}(y) = \varnothing$ for each $y \in Y$, then there exists a $\overline{y} \in X$ such that $B(\overline{y}) = \varnothing$*

Proof of Theorem D. Theorem D.3 is exactly Komiya's coincidence theorem so we need only show that the three statements are equivalent.

($1 \Rightarrow 2$) Let $R \subseteq Y \times X$ be a relation with closed lower sections such that $R^c(y) \cap A^{-1}(y) = \varnothing$ for all $y \in Y$. Define a correspondence $F : X \twoheadrightarrow Y$ as $F(x) = R^{-1}(x)$ and note that F has closed values. Next, note that $(F^{-1})^c(y) = X \backslash F^{-1}(y) = X \backslash R(y) = R^c(y)$ implying that F is a KKM map and that $(F^{-1})^c(y) \cap A^{-1}(y) = R^c(y) \cap A^{-1}(y) = \varnothing$ for all $y \in Y$. Therefore, there exists a $\overline{y} \in Y$ such that $\overline{y} \in \cap_{x\in X} F(x)$, i.e., $\overline{y} \in \cap_{x\in X} R^{-1}(x)$ implying that $\overline{y} \in Y$ and $R(\overline{y}) = X$.

($2 \Rightarrow 3$) Let $B : Y \twoheadrightarrow X$ be a convex valued correspondence with open lower sections such that $B(y) \cap A^{-1}(y) = \varnothing$ for each $y \in Y$. Defining $R = (Y \times X)\backslash(\mathrm{gr}B) = \{(y, x) \in Y \times X | x \notin B(y)\}$, it follows that R is a relation with closed lower sections. Furthermore, $R^c(y) = B(y)$ and $B(y)$ convex imply that that $coR^c(y) \cap A^{-1}(y) = B(y) \cap A^{-1}(y) = \varnothing$ for all $y \in Y$. Then there exists a $\overline{y} \in Y$ such that $R(\overline{y}) = X$. That is, $B(\overline{y}) = R^c(\overline{y}) = \varnothing$.

($3 \Rightarrow 1$) Let $F : X \twoheadrightarrow Y$ be a correspondence with closed values such that $co(F^{-1})^c(y) \cap A^{-1}(y) = \varnothing$ for all $y \in Y$. Define a correspondence $G : Y \twoheadrightarrow X$ as $G(y) = (F^{-1})^c(y)$ and note that G has open lower sections since F is closed valued. Defining $B(y) = coG(y) = co[(F^{-1})^c(y)]$ and applying Lemma 5.1 in [71], it follows that the correspondence $B : Y \twoheadrightarrow X$ has open lower sections. Since $B(y) \cap A^{-1}(y) = \varnothing$ for each $y \in Y$, it follows that $B(\overline{y}) = co[(F^{-1})^c(\overline{y})] = \varnothing$ for some $\overline{y} \in Y$ implying that $(F^{-1})^c(\overline{y}) = \varnothing$. Therefore $F^{-1}(\overline{y}) = X$ implying that $\overline{y} \in \cap_{x \in X} F(x)$. □

4.2. **Fan's further generalization of the KKM lemma.** We next provide an equivalence result based on a further generalization of the classic KKM lemma he proved in Fan [18].

Theorem 4.7. (Fan, 1984) *Let Z be a non-empty, convex subset of a Hausdorff topological vector space and X a subset of Z. Suppose that (i) X_0 is a non-empty subset of X and (ii) $X_0 \subseteq K \subseteq Z$ where K is a compact, convex subset of Z. Suppose that*

(a) *$F : X \twoheadrightarrow Z$ is closed valued, i.e., $F(x)$ is closed in Z for each $x \in X$*
(b) *$co\{x_1, .., x_m\} \subseteq \cup_i F(x_i)$ whenever $\{x_1, .., x_m\} \subseteq X$.*
(c) *$\bigcap_{x \in X_0} F(x)$ is compact.*

Then there exists an $\overline{x} \in Z$ such that $\overline{x} \in \bigcap_{x \in X} F(x)$.

Remark 4.8. Letting $X = Z$ and $X_0 = K = \{x_0\}$ we recover the classic FKKM Theorem in Fan [17].

We can now prove the following equivalence result where Theorem E.1 is exactly Fan's theorem. Theorems E.2 and E.3 are new.

Theorem E. *Let Z be a non-empty, convex subset of a Hausdorff topological vector space and X a subset of Z. Suppose that (i) X_0 is a non-empty subset of X and (ii) $X_0 \subseteq K \subseteq Z$ where K is a non-empty compact, convex subset of Z. Then the following are true and equivalent:*

1. *If*
   (a) *$F : X \twoheadrightarrow Z$ is closed valued, i.e., $F(x)$ is closed in Z for each $x \in X$*
   (b) *$co\{x_1, .., x_m\} \subseteq \cup_i F(x_i)$ whenever $\{x_1, .., x_m\} \subseteq X$.*

(c) $\bigcap_{x\in X_0} F(x)$ *is compact.*
Then there exists an $\overline{x} \in Z$ *such that* $\overline{x} \in \bigcap_{x\in X} F(x)$.

2. *If*
   (a) $R \subseteq Z \times X$ *is a relation with closed lower sections, i.e., for each* $y \in X$, $R^{-1}(y) = \{x \in Z | (x, y) \in R\}$ *is closed in* Z.
   (b) $co\{x_1, .., x_m\} \subseteq \cup_i R^{-1}(x_i)$ *whenever* $\{x_1, .., x_m\} \subseteq X$.
   (c) $\bigcap_{x\in X_0} R^{-1}(x)$ *is compact.*
   Then there exists an $\overline{x} \in Z$ *such that* $(\overline{x}, y) \in R$ *for all* $y \in X$.
3. *If*
   (a) $B : Z \twoheadrightarrow X$ *has open lower sections, i.e.,* $B^{-1}(y)$ *is open in* Z *for each* $y \in X$.
   (b) $co\{x_1, .., x_m\} \subseteq \cup_i (B^{-1})^c(x_i)$ *whenever* $\{x_1, .., x_m\} \subseteq X$.
   (c) $\bigcap_{x\in X_0}[Z\backslash B^{-1}(x)] = \bigcap_{x\in X_0}[(B^{-1})^c(x)]$ *is compact.*
   Then there exists $\overline{x} \in Z$ *such that* $B(\overline{x}) = \varnothing$.

Proof of Theorem E. In light of Theorem 4.7 , we need only prove the equivalences.

(1 ⇒ 2) Suppose that a, b and c of E.1 are satisfied. Define $F : X \twoheadrightarrow Z$ as $F(y) = R^{-1}(y)$ for each $y \in Z$. Then $F(x)$ is closed in Z for each $x \in X$ and $co\{x_1, .., x_m\} \subseteq \cup_i F(y)$ whenever $\{x_1, .., x_m\} \subseteq X$. Since $F(y) = R^{-1}(y)$, it follows that $\bigcap_{x\in X_0} F(x)$ is compact and we conclude from Part 1 that there exists an $\overline{x} \in Z$ such that $\overline{x} \in \bigcap_{y\in X} F(y)$, i.e., there exists an $\overline{x} \in Z$ such that $\overline{x} \in R^{-1}(y)$ for each $y \in X$.

(2 ⇒ 3) Suppose that a, b and c of E.3 are satisfied. Define $R \subseteq Z \times X$ as $R = [Z \times X]\backslash grB$. If $y \in X$, then

$$R^{-1}(y) = \{x \in Z | (x, y) \in R\} = \{x \in Z | (x, y) \notin grB\} = Z\backslash B^{-1}(y)$$

is closed in Z and $co\{x_1, .., x_m\} \subseteq \cup_i R^{-1}(y_i)$ whenever $\{x_1, .., x_m\} \subseteq X$. Since $R^{-1}(y) = Z\backslash B^{-1}(y)$ for $y \in X$, it follows that $\bigcap_{x\in X_0} R^{-1}(x) = \bigcap_{x\in X_0}[Z\backslash B^{-1}(x)]$ is compact. Therefore, there exists $\overline{x} \in Z$ such that $(\overline{x}, y) \notin grB$ for all $y \in X$, i.e., there exists $\overline{x} \in Z$ such that $y \notin B(\overline{x})$ for all $y \in X$, i.e., there exists $\overline{x} \in Z$ such that $B(\overline{x}) = \varnothing$.

(3 ⇒ 1) Suppose that a, b and c of E.1 are satisfied. Define $B : Z \twoheadrightarrow X$ as $B(x) = X\backslash F^{-1}(x) = (F^{-1})^c(x)$ for each $x \in Z$. If $y \in X$, then

$$B^{-1}(y) = \{x \in Z | y \notin F^{-1}(x)\} = \{x \in Z | x \notin F(y)\} = Z\backslash F(y) = F^c(y)$$

implying that $F(y)$ is closed in Z for each $y \in X$. Since $F(x) = (B^{-1})^c(x)$ for each $x \in X$, it follows that $co\{x_1, .., x_m\} \subseteq \cup_i (B^{-1})^c(x_i)$ whenever $\{x_1, .., x_m\} \subseteq X$. Since $F(x) = (B^{-1})^c(x)$ for each $x \in X$, it follows that $\bigcap_{x\in X_0}[(B^{-1})^c(x)] = \bigcap_{x\in X_0} F(x)$ is compact implying that there exists $\overline{x} \in Z$

such that $B(\overline{x}) = \varnothing$, i.e., there exists $\overline{x} \in Z$ such that $X \backslash F^{-1}(\overline{x}) = \varnothing$, i.e., there exists $\overline{x} \in Z$ such that $F^{-1}(\overline{x}) = X$. □

4.3. **Peleg's generalization of KKM lemma.** In this section, we apply the arguments developed in our previous equivalence results to a trio of "collective" equilibrium problems. These problems are motivated by Peleg's [54] generalization of the classic finite dimensional KKM lemma and its infinite dimensional extension due to Lassonde and Schenkel [37].

For the remainder of this section, let $I = \{1, ..., n\}$. For each $i \in I$, let X_i be a a non-empty, convex , compact subset of a Hausdorff topological vector space and let $X = X_1 \times \cdots \times X_n$. If $i \in I$ and $S_i \subseteq X_i$, we define $(X_{-i}, S_i) = X_1 \times \cdots \times X_{i-1} \times S_i \times X_{i+1} \times \cdots \times X_n$. As a corollary of Theorem 5 in Lassonde and Schenkel [37], we have the following "collective" generalization of Browder's fixed point theorem[10].

Theorem 4.9. *For each $i \in I$, suppose that $B_i : X \twoheadrightarrow X_i$ is a correspondence satisfying:*

1. *$B_i(x)$ is convex for all $x \in X$.*
2. *The set $\{x \in X | y_i \in B(x)\}$ is open in X for each $y_i \in X_i$.*

Furthermore, suppose that for each $x \in X$, there exists $i \in I$ such that $B_i(x) \neq \varnothing$. Then there exists $\bar{x} \in X$ and $j \in I$ such that $\overline{x}_j \in B_j(\bar{x})$.

We next pose three propositions each of which provides a collective generalization of the corresponding Theorems 1.1 , 1.2 and 1.3 above.

Proposition 4.10. For each $i \in I$, suppose that $F_i : X_i \twoheadrightarrow X$ is a correspondence satisfying:

1. $(X_{-i}, coA_i) \subseteq \cup_{x_i \in A_i} F_i(x_i)$ for every finite subset $A_i \subseteq X_i$.
2. $F_i(x_i)$ is closed in X for all $x_i \in X_i$.

Then
$$\bigcap_{i=1}^{n} \left[\bigcap_{y_i \in X_i} F_i(y_i) \right] \neq \varnothing.$$
That is, there exists $(\overline{x}_1, .., \overline{x}_n) \in X$ such that $(\overline{x}_1, .., \overline{x}_n) \in \bigcap_{y_i \in X_i} F_i(y_i)$ for each $i \in I$.

Proposition 4.11. For each $i \in I$, suppose that $R_i \subseteq X \times X_i$. is a relation satisfying:

1. For each $x \in X, (x, x_i) \in R_i$.
2. The set $\{y_i \in X_i : (x, y_i) \notin R_i\}$ is convex for each $x \in X$.

[10]Lassonde and Schenkel [37] prove this result in the more general framework of abstract convex spaces. Every Hausdorff TVS is a convex space in their sense. See also, Marchi and Martínez-Legaz [42].

3. The set $R_i^{-1}(y_i) = \{x \in X | (x, y_i) \in R_i\}$ is closed in X for each $y_i \in X_i$.

Then there exists $(\overline{x}_1, .., \overline{x}_n) \in X$ such that $(\overline{x}_1, .., \overline{x}_n, y_i) \in R_i$ for each $i \in I$ and for each $y_i \in X_i$.

Proposition 4.12. For each $i \in I$, suppose that $B_i : X \twoheadrightarrow X_i$ is a correspondence satisfying:

1. $x_i \notin B_i(x)$ for all $x \in X$.
2. $B_i(x)$ is convex for all $x \in X$.
3. The set $B^{-1}(y_i) = \{x \in X | y_i \in B(x)\}$ is open in X for each $y_i \in X_i$.

Then there exists $\bar{x} \in X$ such that $B_i(\bar{x}) = \varnothing$ for each $i \in I$.

Remark 4.13. Proposition 4.10 is an infinite dimensional extension of the collective KKM Lemma in Peleg [54] and may be deduced as a corollary of Theorem 1 in Lassonde and Schenkel [37]. Proposition 4.12 is a collective maximal element result that is equivalent to Theorem 4.9 and we omit the simple argument. Proposition 4.11 is a corollary of Theorem 3.1 in Lin and Ansari [30]. Indeed, an alternative proof of the equivalence of Propositions 4.10 and 4.11 may be deduced from Theorems 3.1 and 3.2 in that paper.

Theorem F. *Propositions 4.10, 4.11 and 4.12 are true and equivalent.*

Proof of Theorem F. Each Proposition is true by Remark 4.13 so it only remains to verify the equivalences.

$(1 \Rightarrow 2)$ Suppose that for each $i \in I$, $R_i \subseteq X \times X_i$ is a relation satisfying the assumptions of Proposition 2. Then for each i, the correspondence $F_i : X_i \twoheadrightarrow X$ defined as $F_i(y_i) = R_i^{-1}(y_i)$ for each $y_i \in X_i$ is closed valued. To show that assumption 1 of Proposition 1 is satisfied, we argue by contradiction. Fix $i \in I$ and suppose that there exists a finite subset $A_i \subseteq X_i$ and $(z_1, .., z_n) \in (X_{-i}, coA_i)$ such that $(z_1, .., z_n) \notin \cup_{x_i \in A_i} F_i(x_i)$. Then $z_i \in coA_i$ but $x_i \in X_i \backslash F^{-1}(z_1, .., z_n)$ for each $x_i \in A_i$. Therefore, $x_i \in R_i^c(z_1, .., z_n)$ for each $x_i \in A_i$. Since assumption 2 of Proposition 2 implies that $R_i^c(z_1, .., z_n)$ is convex, it follows that $z_i \in R_i^c(z_1, .., z_n)$, contradicting assumption 1 of Proposition 2. Applying Proposition 1, it follows that there exists $(\overline{x}_1, .., \overline{x}_n) \in X$ such that $(\overline{x}_1, .., \overline{x}_n) \in \bigcap_{y_i \in X_i} F_i(y_i)$ for each $i \in I$. That is, $(\overline{x}_1, .., \overline{x}_n, y_i) \in R_i$ for each $i \in I$ and for each $y_i \in X_i$.

$(2 \Rightarrow 3)$ Suppose that for each $i \in I$, $B_i : X \twoheadrightarrow X_i$ is a correspondence satisfying the assumptions of Proposition 3. Let $R_i \subseteq X \times X_i$ be the relation defined as $R_i = (X \times X_i) \backslash (\mathrm{gr}B) = \{(x, y_i) \in X \times X_i | y_i \notin B(x)\}$. Since B_i has open lower sections and $R_i^{-1}(y_i) = (B_i^{-1})^c(y_i)$ for each $y_i \in X_i$, it follows $R_i^{-1}(y_i)$ is closed in X for each $y_i \in X_i$. Furthermore, $x_i \notin B_i(x)$ implies that $(x, x_i) \in R_i$ and $B_i(x)$ convex implies that $B_i(x) = R_i^c(x) = \{y_i \in X_i :$

$(x, y_i) \notin R_i\}$ is convex for each $x \in X$. Applying Proposition 2, it follows that there exists $(\overline{x}_1, .., \overline{x}_n) \in X$ such that $(\overline{x}_1, .., \overline{x}_n, y_i) \in R_i$ for each $i \in I$ and for each $y_i \in X_i$. That is, $(\overline{x}_1, .., \overline{x}_n, y_i) \notin grB_i$ for each $i \in I$ and for each $y_i \in X_i$ implying that $B_i(\bar{x}) = \varnothing$ for each $i \in I$.

$(3 \Rightarrow 1)$ Suppose that for each $i \in I$, $F_i : X_i \twoheadrightarrow X$ is a correspondence satisfying the assumptions of Proposition 1. Define for each $i \in I$ a correspondence $G_i : X \twoheadrightarrow X_i$ as $G_i(x) = (F_i^{-1})^c(x)$ for each $x \in X$. Since $G_i^{-1}(y_i) = F_i^c(y_i)$, it follows that $G^{-1}(y_i)$ is open in X for each $y_i \in X_i$. Applying Lemma 5.1 in [71], it follows that the correspondence $B_i : X \twoheadrightarrow X_i$ defined as $B_i(x) = coG_i(x)$ is convex valued with open lower sections. Using the same argument as that of Lemma 2.12, assumption 1 of Proposition 1 implies that $x_i \notin B_i(x)$ for all $x \in X$. Applying Proposition 3, we conclude that there exists $\bar{x} \in X$ such that $B_i(\bar{x}) = \varnothing$ for each $i \in I$. That is, there exists $\bar{x} \in X$ such that $F_i^{-1}(\bar{x}) = X_i$ for each $i \in I$ impling that $(\overline{x}_1, .., \overline{x}_n) \in \bigcap_{y_i \in X_i} F_i(y_i)$ for each $i \in I$. $\square$

5. An epistemological excursion

Gwinner [22] presents a circular tour of "fixed points and variational inequalities," and writes:

> The main object of this paper is to display relations and connections between some of the most fundamental results of modern nonlinear analysis: the existence theorem for pseudomonotone variational inequalities, Fan's minimax principle and its extension, the basic fixed point theorems for multivalued mappings, and the Gale-Nikaido-Debreu theorem in mathematical economics. Our tour starts in a now traditional way, but also ends with the classical Knaster-Kuratowski-Mazurkiewicz Theorem; thus all these results, which are the single-numbered theorems in this paper, are in some wide sense equivalent.

Given the expositional and referential motivation of this work, at least in part, we delve a little deeper into what Gwinner's phrase "in some wide sense" could possibly connote.[11]

Consider a model or mathematical object symbolically expressed as $\mathfrak{G}(x, g)$ with g indicating the parameters, the given datum, and x the variables that are to be explained. A result or a solution can then be expressed as $x = \varphi(g)$, and yet another as $x = \psi(g)$, and so a straightforward meaning of an equivalence theorem pertaining to the object $\mathfrak{G}$ is simply to say that under certain assumptions, say $g \in \mathfrak{A}$, $\varphi(g) = \psi(g)$. Indeed, we have an equivalence theorem

[11]This section could be viewed as a formalization, as well as a sociological excursion into mathematical economics, of the ideas informally expressed at the beginning of Section 3 above to introduce the results presented there and in the section subsequent to it.

which is to say that we have an *if-then* statement. Aumann's equivalence theorem of 1964 is a canonical example in mathematical economics.[12] However, one has to be careful even here in that the equivalence better pertain to the equivalence of like-objects or conditions. Dugundji presents an equivalence theorem that asserts the equivalence of Zermelo's theorem, Zorn's lemma and the axiom of choice.[13]

Furthermore, the difference between an *equivalence theorem* and an *equivalence of theorems* bears emphasis. The second goes beyond an if-then statement to come quite close to the words *applied* and *application.* When one says that Theorems A and B are equivalent, one can also be seen as saying each theorem, as an if-then statement, can be proved as a consequence of the other, again as an if-then statement. This can be expressed as $H(B) \Rightarrow H(A) \Rightarrow C(A) \Rightarrow C(B)$, and $H(A) \Rightarrow H(B) \Rightarrow C(B) \Rightarrow C(A)$, where H is an operator signifying a *hypothesis,* and C signifying a conclusion. A single one of these two directions could be referred to either as an *application,* or as a *corollary,* in the context of using one as the proof of the other.[14] It is precisely in this way that Borgerson (2004) presents the "equivalence of seven major theorems in combinatorics." However there is some dissonance here stemming from the results being used to execute the direction.[15] If these results are not elementary, then it is far from clear as to whose application or whose corollary the conclusions are. Or to argue from the other side, suppose one were not to appeal to Theorem A, but simply give a detailed *ab initio* argument for the proof of Theorem B that reproduces in full detail the proof of Theorem A, an alternative proof with no new ideas but also with no reference to Theorem A. The question then reduces to whether one can regard Theorem A as a corollary of Theorem B by virtue of only the statements of the two theorems.

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[12]See Aumann [3] that establishes the equivalence of core and competitive allocations of a measure-theoretic *economy* under the particular assumption that the measure is atomless. In this paper of [11], there are three equivalence theorems. But it calls the theorems involved "claims." A more recent example is [16] that establishes under certain parametric assumptions on a *game,* the equivalence of a Nash, Berk-Nash, self-confirming, ABEE (analogy-based expectational equilibrium) equilibria.

[13]See Dugundji [15, p. 30]. Also see Jech [25, p. 10] statement; and Jech [26].

[14]The word "corollary" has a complicated time line as far as English usage is concerned, with the OED listing four specific meanings further sub-classified into sub-items. Even in the mathematical register, one can ask whether a statement can be legitimately looked on as a corollary of another when the latter has been generalized by using its proof.

[15]The reader might want to keep the alleged equivalence of the Brouwer and Kakutani fixed point theorems as a way of concretizing the abstract statements to follow. To be sure, the theorems are equivalent in the sense that one can be proved from the other, but how useful is this? One direction is trivial, and the other requires us to know about barycentric subdivisions and retractions rather than their formalization as an additional theorem.

We now conclude this section by some observations of special interest for mathematical economics. The preoccupation with the word *equivalence* originates there as a defensive move in response to Marshall and Keynes: a suggestion of the former to use mathematics in economics as a back-of-the-envelope verification of economic insights, to be burnt once the insights have been had;[16] and comments of the latter that the theorems of the applied subject are really straightforward, if not trivial, exercises and applications of results in pure mathematics. In regard to the issue of the existence of competitive equilibrium, it then becomes a matter of some substantive and sociological consequence to workers in the field to show that the truth-value, colloquially put, of the theorem be of the same order as that of a fixed point theorem, and especially given the acknowledgement and appreciation of the fact that the Brouwer fixed point theorem is a theorem of some considerable depth in mathematics. This was formally established by Uzawa [68]. In his book, Majumdar [41] goes into the matter in some detail. He writes:

> We shall now prove the striking result of Uzawa [68] that links the Gale-Nikaido-Debreu lemma to the fixed point theorem of Brouwer. Assume the Gale-Nikaido-Debreu lemma. We wish to prove that any upper semicontinuous convex-valued correspondence from the interior of the simplex to itself has a fixed point.[17]

What is to be noted is Majumdar's studied avoidance of the word *equivalence* in preference to the word *links* so as to give a rigorous rendering to the assertion that "The existence of competitive equilibrium is "equivalent" to the fixed theorem, and therefore shows the two results to have equal mathematical depth." Perhaps one can follow Lassonde and Schenkel [37] who put the matter by referring to *interconnections*, a word that can be read as a metanomic stand-in for *equivalences* that has been so far our focus.[18]

6. Concluding remarks

The theorems that constitute our equivalence results have deep connections to Ky Fan's minimax inequality, variational inequalities, variational inclusions and many other related equilibrium problems. A comprehensive bibliography is beyond the scope of this paper but, in addition to the references cited below, we refer the reader to Tarafdar and Chowdhury [63] and the many references

[16] See Khan [27] for reaction and references.

[17] See pages 8 and 52. Majumdar credits [45] for the argument. Note also that Majumdar's theorem 2.3 is is labelled Uzawa's theorem and it asserts that "the Gale-Nikaido-Debreu lemma implies Brouwer's theorem." In this connection, also see [38].

[18] They authors write "The role of the topological assumptions in these results is emphasized. In particular, it is shown that the assumption of openness or closedness can be used indifferently. This observation definitely clarifies the interconnections between the KKM principle, the fixed point theorems, and the theorems on the existence of Nash equilibria."

therein. Our equivalence results could all be re-worked in the framework of abstract convex spaces studied in the very first paper Komiya wrote [31]. We will not pursue these generalizations here but refer the reader to the surveys in Park [47, 48, 49, 50, 51] and their references.

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