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THE WEAK AXIOM OF REVEALED PREFERENCE AND INVERSE PROBLEMS IN CONSUMER THEORY

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ABSTRACT. We consider the inverse problem of the consumer's optimization problem and derive a complete binary relation from a given candidate of demand that obeys the weak axiom of revealed preference. We show that the value of this candidate is in fact the unique maximal element with respect to the derived binary relation. Moreover, if this candidate satisfies several natural requirements, then this binary relation becomes transitive, and it is the unique complete, transitive, and upper semi-continuous binary relation corresponding to the given demand function.

1. INTRODUCTION

When considering a parameterized optimization problem, the solution is usually formulated as a multi-valued mapping on the space of parameters. If this optimization problem is a model of some phenomenon, then the objective function is often not observable, and only the solution mapping can be observed. In mathematics, the research area called "inverse problem" aims at reconstructing the original problem from the solution mapping.

There are two types of inverse problems: one where we only need to state the "existence" of the original problem, and another where we need additional information such as "form" or "uniqueness". The former is the case where it is only necessary to justify using an optimization problem to describe phenomena, or where a useful mathematical application can be given through an optimization problem. For example, Komiya [15] brought Berge's maximum theorem into the context of inverse problems and presented a method of constructing the original problem from what can be seen as a solution mapping, and he used this result to present a relatively easy proof of Kakutani's fixed point theorem. In another context, there is a research area that asks whether a model without stochastic fluctuations can explain chaotic behavior in the economy. In the context of dynamic programming, it has long been known that chaotic behavior arises when the policy function is a tent map. Boldrin and Montrucchio [1] showed that for a given candidate of a policy function, it is possible to build an economic model such that this candidate actually becomes the policy function. Both of these are typical examples of inverse problems where we only need the "existence".

Meanwhile, there are cases where the information in the objective function itself is important. For example, a Ramsey-Cass-Koopmans optimal capital accumulation

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model, introduced by Ramsey [21] and improved by Cass [3] and Koopmans [16], can be made into a "reduced form" that was treated by Boldrin and Montrucchio [1], but the converse is not true. Thus, Boldrin-Montrucchio's theorem does not imply the existence of a Ramsey-Cass-Koopmans model that produces chaos. Hosoya [8] presented a method of fully characterizing the class of policy functions that can be derived from a Ramsey-Cass-Koopmans model and calculating the objective function backwards from the policy function. This is an example of an inverse problem that requires a "form".

A more extreme example is the integrability problem in consumer theory. The purpose of this problem is to calculate the consumer's preference backwards from his/her behavior, and the "uniqueness" of the corresponding objective function is thus needed. Mas-Colell [17] gave a classical result for this uniqueness requirement, and this result has recently been extended by Hosoya [11]. The existence and uniqueness of the objective function have been treated simultaneously by Hosoya [7, 9].

The present paper deals with a branch of consumer theory called revealed preference theory. This is a research area that asks whether a candidate of the solution function for the consumer's behavior can actually be represented as a solution function for the consumer's optimization problem. The weak axiom of revealed preference proposed by Samuelson [23] is a straightforward necessary condition for a candidate of a solution function to be a solution function. However, Gale [5] showed that it is not sufficient. Houthakker [12] developed the strong axiom of revealed preference based on the idea of the Euler approximation of differential equations, and Richter [22] proved that this axiom is an appropriate necessary and sufficient condition for the present problem.

Meanwhile, consumer theory is a part of general equilibrium theory. In this regard, recent general equilibrium theory has evolved in the direction of dealing with preferences of a form that cannot be represented by some "objective function". This is because the "negative transitivity" of the strong order is not desirable for an axiom of the consumer's preference. For this reason, the theory of consumer behavior in which the preference is described by a binary relation that does not satisfy transitivity, and the theory of equilibrium that includes it, has developed. For example, Hildenbrand [6], Shafer and Sonnenschein [24], and Mas-Colell [18] are typical studies on this issue. The problem here is the inverse problem of that. What kind of demand behavior can be handled by such a model?

The strong axiom of revealed preference is undesirable as an assumption, because it is too strong and can only handle solution functions that can be expressed by an ordinary maximization of a transitive preference. Let us now turn our attention to the weak axiom of revealed preference. The problem of expressing a candidate of a solution function satisfying this axiom by a result of maximizing some (possibly non-transitive) binary relation was studied by Kim and Richter [13] and Quah [20]. However, the orders they created are unnatural. In particular, if we take the solution function from an ordinary maximization problem and calculate the binary relation that appears in their research, then we obtain binary relations that are not transitive. In short, their research focuses only on “existence”, and “uniqueness” is almost completely compromised.

Therefore, in this paper, we derive a natural binary relation from a candidate of the solution function that satisfies the weak axiom of revealed preference and show that this candidate is actually the solution function of the optimization problem of this binary relation (Theorem 2.4). In fact, under certain conditions, the binary relation that we have derived is the “unique” corresponding binary relation that satisfies several requirements (Proposition 2.8).

In Section 2.1, we explain standard concepts and terminology used in consumer theory. Section 2.2 is devoted to the proof of the main result and an explanation of related open problems.

2. Model and Results

2.1. **Preliminaries.** We fix $n \ge 2$. Let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \ge 0 \text{ for all } i\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n | x_i > 0 \text{ for all } i\}$. For vectors $x, y \in \mathbb{R}^n$, we write $x \ge y$ if $x - y \in \mathbb{R}^n_+$ and $x \gg y$ if $x - y \in \mathbb{R}^n_{++}$.

A set $\Omega \subset \mathbb{R}^n$ is called the **consumption set**, which denotes the set of all possible consumption plans. For a plan $x \in \Omega$, x_i denotes the amount of consumption of the i-th commodity. By this interpretation, $x_i \ge 0$ is usually assumed. Thus, throughout this paper, we assume that $\Omega \subset \mathbb{R}^n_+$. We also assume that Ω is nonempty and convex, and that $\Omega = \Omega + \mathbb{R}^n_+$: that is, $x \in \Omega$ and $y \ge x$ implies $y \in \Omega$.

Let a vector $p \gg 0$ denote the price system. Then, p_i denotes the price of the i-th commodity. The consumer's money is denoted by $m > 0$. Therefore, the consumer's behavior is represented by a function $f(p, m)$, where the domain P of f is a subset of $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$. Define

$$\Delta(p, m) = \{x \in \Omega | p \cdot x \le m\},$$

and $P = \mathbb{R}^n_{++} \times \mathbb{R}_{++} \setminus \Delta^{-1}(\emptyset)$. We call a function $f : P \to \Omega$ a **candidate of demand** (CoD) if $p \cdot f(p, m) \le m$ for all $(p, m) \in P$. We consider that this function f represents the consumer's choice behavior. If $p \cdot f(p, m) = m$ for all $(p, m) \in P$, then f is said to satisfy **Walras' law**. Meanwhile, if $f(p, m) = f(ap, am)$ for all $(p, m) \in P$ and $a > 0$, then f is said to be **homogeneous of degree zero**. We say that a CoD f is **income-Lipschitzian** if f is locally Lipschitz in m: that is, for every compact subset $C \subset P$, there exists $L > 0$ such that if $(p, m_1), (p, m_2) \in C$, then

$$\|f(p, m_1) - f(p, m_2)\| \le L|m_1 - m_2|.$$

Let $R(f)$ denote the range of f.

Next, consider a binary relation $\succsim$ on Ω: that is, $\succsim \subset \Omega^2$. We write $x \succsim y$ instead of $(x, y) \in \succsim$. Consequently, $x \not\succsim y$ means $(x, y) \notin \succsim$. We also write $x \succ y$ if $x \succsim y$ and $y \not\succsim x$, and $x \sim y$ if $x \succsim y$ and $y \succsim x$.

For each $(p, m) \in P$, define

$$f^{\succsim}(p, m) = \{x \in \Delta(p, m) | \forall y \in \Delta(p, m), y \not\succ x\}.$$

This multi-valued function $f^{\succsim} : P \twoheadrightarrow \Omega$ is called the **demand function** corresponds to $\succsim$. Note that every single-valued demand function is a CoD that is homogeneous of degree zero. We say that a CoD f corresponds to $\succsim$, or equivalently, $\succsim$ corresponds to f when $f = f^{\succsim}$.

Suppose that $u : \Omega \to \mathbb{R}$ satisfies the following relationship:

$$u(x) \ge u(y) \Leftrightarrow x \succsim y.$$

Then, we say that u represents $\succsim$, or equivalently, u is a **utility function** of this relation $\succsim$. If u represents $\succsim$, then we sometimes write f^u instead of $f^{\succsim}$, and say that this function f^u is the demand function corresponds to u. Note that, f^u is the solution function of the following optimization problem:

$$\begin{aligned} \max \quad & u(x) \\ \text{subject to.} \quad & x \in \Omega, \\ & p \cdot x \le m. \end{aligned} \tag{2.1}$$

In economics, this optimization problem is called the **utility maximization problem**.

We say that a binary relation $\succsim$ on Ω is

- **complete** if either $x \succsim y$ or $y \succsim x$ for all $(x, y) \in \Omega^2$,
- **transitive** if $x \succsim y$ and $y \succsim z$ imply $x \succsim z$,
- **p-transitive** if $\dim(\text{span}\{x, y, z\}) \le 2$, $x \succsim y$, and $y \succsim z$ imply $x \succsim z$,
- **asymmetric** if $x \succsim y$ implies $y \not\succsim x$,
- **monotone** if $x \gg y$ implies $x \succ y$,
- **convex** if $y \succsim x$ and $0 < t < 1$ imply $(1-t)x + ty \succsim x$,
- **strictly convex** if $y \neq x,\ y \succsim x$ and $0 < t < 1$ imply $(1-t)x + ty \succ x$,
- **upper semi-continuous** if the set $U(x) = \{y \in \Omega | y \succsim x\}$ is closed with respect to the relative topology of Ω for all x, and
- **continuous** if $\succsim$ itself is closed with respect to the relative topology of Ω^2.

Note that, if u represents $\succsim$, then $\succsim$ must be complete and transitive. Conversely, if $\succsim$ is complete, transitive, and upper semi-continuous, then there exists an upper semi-continuous function u that represents $\succsim$. If, in addition, $\succsim$ is continuous, then we can construct such u as a continuous function.[1] Consequently, if Ω is closed and $\succsim$ is complete, transitive, and upper semi-continuous, then $f^{\succsim}$ is nonempty-valued and homogeneous of degree zero. If, in addition, $\succsim$ is monotone, then we can easily show that $f^{\succsim}$ satisfies Walras' law. Furthermore, if $\succsim$ is strictly convex, then we can also easily show that $f^{\succsim}$ is single-valued, and thus it is a CoD that is homogeneous of degree zero. In conclusion, we verified that if Ω is closed and $\succsim$ is complete, transitive, upper semi-continuous, monotone, and strictly convex, then $f^{\succsim}$ is a CoD that satisfies Walras' law and homogeneity of degree zero.

Note that if $\succsim$ is transitive, then either $x \succsim y \succ z$ or $x \succ y \succsim z$ implies $x \succ z$. The proof of this fact is easy.

[1] See Debreu [4] or Bridges and Mehta [2]. Note that our Ω is convex, and thus it is connected.

Next, we introduce axioms of revealed preference. We say that a CoD f satisfies the **weak axiom of revealed preference** (WA) if for every $(p, m), (q, w) \in P$, if $p \cdot f(q, w) \le m$, then either $f(p, m) = f(q, w)$ or $q \cdot f(p, m) > w$. We also say that f satisfies the **strong axiom of revealed preference** (SA) if for every finite sequence $(p_1, m_1), ..., (p_N, m_N)$, if $p_i \cdot f(p_{i+1}, m_{i+1}) \le m_i$ for every $i \in \{1, ..., N-1\}$, then either $f(p_N, m_N) = f(p_1, m_1)$ or $p_N \cdot f(p_1, m_1) > m_N$. We can easily check that SA implies WA. Meanwhile, Gale [5] provided an example of a CoD that satisfies WA but violates SA. Therefore, WA does not imply SA.[2]

The interpretation of WA is as follows. Suppose that $x = f(p, m), y = f(q, w)$ and $x \neq y$. If $p \cdot y \le m$, then the consumer chooses x though he/she can choose y, and thus it is revealed that the consumer prefers x to y. Therefore, we can expect $q \cdot x > w$: if not, then $f(q, w)$ must not be y.

2.2. **Results.** The following result is known as Houthakker-Uzawa-Richter's theorem. The proof of this proposition is found in Theorem 3.J.1 of Mas-Colell et al. [19] or Theorem 3.2 of Hosoya [10].

Proposition 2.1. *A CoD f satisfies SA if and only if $f = f^{\succsim}$ for some complete and transitive binary relation $\succsim$.*

Using this result, we can obtain the following uniqueness result. Recall that $R(f)$ denotes the range of f. For the rigorous proof of this corollary, see Theorem 1 of Hosoya [11].

Corollary 2.2. *Suppose that $\Omega = \mathbb{R}^n_+$ and $P = \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. Let f be a continuous and income-Lipschitzian CoD that satisfies Walras' law, and $R(f)$ be open with respect to the relative topology of Ω and include $\mathbb{R}^n_{++}$. Then, f satisfies SA if and only if $f = f^u$ for some upper semi-continuous real-valued function u. Moreover, if $f = f^{\succsim}$ for some complete, transitive, and upper semi-continuous binary relation $\succsim$, then u represents $\succsim$.*

Remark 2.3. We can obtain u constructively as follows. First, fix $\bar{p} \in \mathbb{R}^n_{++}$. Choose any $x \in R(f)$, and let $x = f(p, m)$. Consider the following partial differential equation:

$$\nabla E(q) = f(q, E(q)), E(p) = m. \tag{2.2}$$

We can show that there uniquely exists a concave solution $E : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ of this equation (2.2). Define $u(x) = E(\bar{p})$. Next, choose $x \in \Omega \setminus R(f)$. Define

$$u(x) = \lim_{\delta \to 0} \sup\{u(y) | y \in R(f),\ \|x - y\| < \delta\}.$$

Then, we can show that u satisfies all requirements of the above corollary.

In fact, the solution of (2.2) can be described using a different feature. Because f satisfies SA, we have that $f = f^{\succsim}$ for some complete and transitive binary relation $\succsim$. Let

$$E^x(q) = \inf\{q \cdot y | y \succsim x\}.$$

[2] For a detailed arguments, see section 4.2 of Hosoya [10].

If $\succsim$ is represented by v, then it is the value function of a dual problem of (2.1):

$$\begin{aligned} \min\ & q\cdot y \\ \text{subject to. } & y\in\Omega, \\ & v(y)\ge v(x). \end{aligned} \tag{2.3}$$

This problem (2.3) is called the **expenditure minimization problem**, and the function E^x is called the **expenditure function**. Lemma 1 of Hosoya [11] showed that if $x=f(p,m)$, then E^x is a concave solution of (2.2). This result is called **Shephard's lemma** in economics.

We want to extend the above results to CoDs with WA. Before presenting the next result, we need several preparations. First, let $\rhd$ be any binary relation on Ω, and $\mathscr{P}$ denote the set of all p-transitive binary relations that includes $\rhd$. Because $\Omega^2\in\mathscr{P}$, we have that $\mathscr{P}$ is nonempty. It is easy to show that

$$\rhd^*=\cap_{\rhd'\in\mathscr{P}}\rhd'\in\mathscr{P}.$$

We call this $\rhd^*$ the **p-transitive closure** of $\rhd$.

We write $x\succ_r y$ if and only if there exists $(p,m)\in P$ such that $x=f(p,m)$, $y\in\Delta(p,m)$ and $x\neq y$. Then, WA is equivalent to the asymmetry of $\succ_r$. Let $\succ_{irp}$ be the p-transitive closure of $\succ_r$, and $x\succsim_f y$ if and only if $y\not\succ_{irp} x$. We now complete the preparation to present our main result.

Theorem 2.4. *If a CoD f satisfies Walras' law and WA, then $\succsim_f$ is complete and $f=f^{\succsim_f}$.*

Proof. First, we introduce a lemma.

Lemma 2.5. *Suppose that f is a CoD that satisfies Walras' law. Then, $\succ_r$ is asymmetric if and only if $\succ_{irp}$ is asymmetric.*[3]

Proof of Lemma 2.5. Clearly, if $\succ_{irp}$ is asymmetric, then $\succ_r$ is asymmetric because $\succ_r\subset\succ_{irp}$. Thus, it suffices to show the opposite direction.

First, suppose that $\succ_r$ is asymmetric, and there exists a sequence $x_1,...,x_k\in\Omega$ such that $\dim(\mathrm{span}\{x_1,...,x_k\})\le 2$ and $x_i\succ_r x_{i+1}$ for all $i\in\{1,...,k-1\}$. We will show that $x_k\not\succ_r x_1$. We use mathematical induction on k. If $k=2$, this claim is correct because $\succ_r$ is asymmetric.

Suppose that our claim holds if $k\le k^*$ for $k^*\ge 2$, and consider the case $k=k^*+1$. Choose any sequence $x_1,...,x_k$ such that $\dim(\mathrm{span}\{x_1,...,x_k\})\le 2$ and for all $i\in\{1,...,k-1\}$, $x_i\succ_r x_{i+1}$. Then, $x_i\neq x_{i+1}$ and there exists (p_i,m_i) such that $x_i=f(p_i,m_i)$ and $p_i\cdot x_{i+1}\le m_i$. Suppose that $x_k\succ_r x_1$. Then, $x_k\neq x_1$ and there exists (p_k,m_k) such that $x_k=f(p_k,m_k)$ and $p_k\cdot x_1\le m_k$. Define $V=\mathrm{span}\{x_1,...,x_k\}$. If $\dim V=1$, then $x_i=c_ix_1$ for $c_i\in]0,1]$, and $1=c_1>c_2>...>c_k$. Therefore,

$$p_k\cdot x_1>p_k\cdot c_kx_1=p_k\cdot x_k=m_k,$$

[3] If Ω is $\mathbb{R}^n_+$, then this result is just Theorem 4.1 of Hosoya [10]. However, the proof of this theorem in that paper has a slight error.

which is a contradiction. Thus, we have that $\dim V = 2$. Let P_V be the orthogonal projection mapping from $\mathbb{R}^n$ onto V. By definition of the orthogonal projection mapping, we have that $P_V p_i \cdot x_j = p_i \cdot x_j$ for $i, j \in \{1, ..., k\}$. Define $q_i = \frac{1}{p_i \cdot x_1} P_V p_i$. Then, we have $q_i \cdot x_1 = 1$ for all $i \in \{1, ..., k\}$, and thus all q_i are included in the line $\{q \in V | q \cdot x_1 = 1\}$.

We separate our proof into three cases.

Case 1. $q_1 \in [q_{k-1}, q_k]$. In this case, $q_1 = (1-t)q_{k-1} + tq_k$ for some $t \in [0,1]$. Therefore,

$$\begin{aligned} q_1 \cdot (x_1 - x_k) &= q_1 \cdot (x_1 - x_2) + q_1 \cdot (x_2 - x_k) \\ &= q_1 \cdot (x_1 - x_2) + (1-t)q_{k-1} \cdot (x_2 - x_k) + tq_k \cdot (x_2 - x_k) \\ &= q_1 \cdot (x_1 - x_2) + (1-t)q_{k-1} \cdot (x_2 - x_{k-1}) \\ &\quad + (1-t)q_{k-1} \cdot (x_{k-1} - x_k) + tq_k \cdot (x_2 - x_k) \geq 0, \end{aligned}$$

where the last inequality follows from the induction hypothesis. Therefore, we have that $x_1 \succ_r x_k$, which contradicts the asymmetry of $\succ_r$.

Case 2. $q_{k-1} \in [q_1, q_k]$. In this case, $q_{k-1} = (1-t)q_1 + tq_k$ for some $t \in [0,1]$. Therefore,

$$\begin{aligned} &(1-t)q_1 \cdot (x_1 - x_k) + tq_k \cdot (x_1 - x_k) \\ &= q_{k-1} \cdot (x_1 - x_k) \\ &= q_{k-1} \cdot (x_1 - x_{k-1}) + q_{k-1} \cdot (x_{k-1} - x_k) > 0. \end{aligned}$$

Because $q_k \cdot (x_1 - x_k) \leq 0$, we have $(1-t)q_1 \cdot (x_1 - x_k) > 0$. Therefore, we have $x_1 \succ_r x_k$, which is absurd.

Case 3. The other case. Define $v = q_1 - q_k$. The case in which $v = 0$ is included in Case 1, and thus we have $v \neq 0$ and $q_i = q_k + t_i v$ for some $t_i \in \mathbb{R}$. By definition, $t_1 = 1$. The case $t_{k-1} \geq 0$ is included in either Case 1 or Case 2, and thus we have $t_{k-1} < 0$. Therefore, we must have that there exists $i \in \{1, ..., k-2\}$ such that $t_i \geq 0$ and $t_{i+1} \leq 0$, and thus, $q_k \in [q_i, q_{i+1}]$. Then, $q_k = (1-t)q_i + tq_{i+1}$ for some $t \in [0,1]$, and

$$\begin{aligned} 0 &\geq q_k \cdot (x_1 - x_k) \\ &= q_k \cdot (x_1 - x_{i+1}) + q_k \cdot (x_{i+1} - x_k) \\ &= (1-t)q_i \cdot (x_1 - x_{i+1}) + tq_{i+1} \cdot (x_1 - x_{i+1}) + q_k \cdot (x_{i+1} - x_k) \\ &= (1-t)q_i \cdot (x_1 - x_i) + (1-t)q_i \cdot (x_i - x_{i+1}) \\ &\quad + tq_{i+1} \cdot (x_1 - x_{i+1}) + q_k \cdot (x_{i+1} - x_k) > 0, \end{aligned}$$

by the induction hypothesis, which is a contradiction.

Therefore, in all cases there is a contradiction. Thus, we conclude that $x_k \not\succ_r x_1$, and hence our claim is correct.

Second, suppose that $\succ_r$ is asymmetric. We will show that $x \succ_{irp} y$ if and only if either $x \in R(f)$ and $y = ax$ for $a \in [0, 1[$ or x is not proportional to y and there exists a finite sequence $x_1, ..., x_N$ such that $x_1 = x,\ x_N = y$, $\mathrm{span}\{x_1, ..., x_N\} = \mathrm{span}\{x, y\}$, and $x_i \succ_r x_{i+1}$ for every $i \in \{1, ..., N-1\}$. Let R be the set of all (x, y) such that 1) $x \in R(f)$ and $y = ax$ for some $a \in [0, 1[$, or 2) x is not proportional to y and there exists a finite sequence $x_1, ..., x_N$ such that $x = x_1, y = x_N$, $\mathrm{span}\{x_1, ..., x_N\} = \mathrm{span}\{x, y\}$, and $x_i \succ_r x_{i+1}$ for all $i \in \{1, ..., N-1\}$. By Walras' law, we have that $\succ_r \subset R$. Choose $x, y, z \in \Omega$ such that $(x, y), (y, z) \in R$ and $\dim(\mathrm{span}\{x, y, z\}) \le 2$. If $y = ax$ or $z = by$ for $a, b \in [0, 1[$, then we can easily show that $(x, z) \in R$. Suppose that x is not proportional to y and $z = ax$ for $a \ge 0$. Then, $x \in R(f)$. If $a \ge 1$, then we have that there exists a finite sequence $x_1, ..., x_N$ on $\mathrm{span}\{x, y\}$ such that $x_1 = x$, $x_i \succ_r x_{i+1}$ for all $i \in \{1, ..., N-1\}$, and $x_N \succ_r x_1$, which is a contradiction. Therefore, we have $a < 1$, and thus $(x, z) \in R$. Finally, if each pair of x, y, z is independent, then clearly $(x, z) \in R$. Therefore, R is p-transitive, and thus $\succ_{irp} \subset R$. Conversely, suppose that $(x, y) \in R$. If x is proportional to y, then $x \succ_r y$, and thus $x \succ_{irp} y$. If x is not proportional to y, then there exists a finite sequence $x_1, ..., x_N$ such that $x = x_1, y = x_N$, $\mathrm{span}\{x_1, ..., x_N\} = \mathrm{span}\{x, y\}$, and $x_i \succ_r x_{i+1}$ for all $i \in \{1, ..., N-1\}$. Because $\succ_r \subset \succ_{irp}$, we have that $x_i \succ_{irp} x_{i+1}$. By p-transitivity of $\succ_{irp}$, we have that $x = x_1 \succ_{irp} x_N = y$, which implies that $R \subset \succ_{irp}$. Thus, our claim is correct.

Now, suppose that $\succ_r$ is asymmetric, and $\succ_{irp}$ is not asymmetric. Then, there exists x, y such that $x \succ_{irp} y$ and $y \succ_{irp} x$. This implies that $x \succ_{irp} x$, which contradicts the arguments in the above paragraph. Therefore, we conclude that if $\succ_r$ is asymmetric, then $\succ_{irp}$ is also asymmetric. This completes the proof of this lemma. ☐

This lemma immediately implies that $\succsim_f$ is complete. Next, suppose that $x = f(p, m)$ and $p \cdot y \le m$. If $x = y$, then clearly $y \not\succ_{irp} x$, and thus $y \not\succ_f x$. If $x \ne y$, then $x \succ_r y$, and by the above lemma, we have that $y \not\succ_{irp} x$, which implies that $y \not\succ_f x$ and $x \succ_f y$. Therefore, we have that $x = f^{\succsim_f}(p, m)$ as desired. This completes the proof. ☐

We can obtain the following corollary.

Corollary 2.6. *Suppose that $\Omega = \mathbb{R}^n_{++}$ and $f = f^u$ for some C^2 strictly quasi-concave function u such that $\nabla u(x) \gg 0$ for all $x \in \Omega$. Then, u represents $\succsim_f$.*[4]

Proof. First, we show that $x \succ_f v$ implies that $u(x) > u(v)$. Suppose that $x \succ_f v$. As we have proved in the proof of Theorem 2.4, either $x = av$ for $a > 1$ or there exists $x_1, ..., x_N$ such that $x_1 = x, x_N = v$, and $x_i \succ_r x_{i+1}$ for all $i \in \{1, ..., N\}$. In the former case, clearly $u(x) > u(v)$. In the latter case, we have that $u(x_i) > u(x_{i+1})$, and thus $u(x) > u(v)$. Therefore, our claim is correct.

Second, we show the opposite direction: that is, we show that $u(x) > u(v)$ implies that $x \succ_f v$. Define $p(x) = \nabla u(x)$. By Lagrange's multiplier rule, we have that $x = f(p(x), p(x) \cdot x)$ for all $x \in \Omega$.

[4] Note that, by the requirement of u, f satisfies SA and Walras' law.

Suppose that $u(x) > u(v)$. If v is proportional to x, then $x \gg v$ and thus clearly $x \succ_f v$. Therefore, we assume that v is not proportional to x. Define

$$\xi(y) = (p(y) \cdot x)v - (p(y) \cdot v)x.$$

Then, $p(y) \cdot \xi(y) = 0$ for all $y \in \Omega$. Consider the following differential equation:

$$\dot{y}(t) = \xi(y),\ y(0) = x.$$

Let $y(t)$ be the nonextendable solution of the above equation. Hosoya [7] showed that there exists $t^* > 0$ such that $y(t^*)$ is proportional to v. Because

$$\frac{d}{dt}u(y(t)) = 0,$$

we have that $u(y(t^*)) = u(x)$, which implies that $y(t^*) \gg v$. Now, for small $h > 0$, define

$$x_0 = x,\ x_{i+1} = x_i + h\xi(x_i).$$

Then, the sequence (x_i) is a forward Euler approximated solution of the above differential equation, and thus for $i(h) = \max\{i \ge 0 | ih < t^*\}$, $x_{i(h)} \to y(t^*)$ as $h \to 0$. This implies that if h is sufficiently small, then there exists k such that $x_k \gg v$. Now, because $p(x_i) \cdot x_{i+1} = p(x_i) \cdot x_i$, we have that $x_i \succ_r x_{i+1}$. Moreover, $\mathrm{span}\{x_0, ..., x_k\} = \mathrm{span}\{x, v\}$. Therefore, we have that $x \succ_{irp} x_k$ and $x_k \succ_r v$, which implies that $x \succ_f v$. This completes the proof. □

To the best of our knowledge, there is no other previous research that has derived a binary relation with such properties in this context. For example, binary relations defined by Kim and Richter [13] and Quah [20] do not have this property. This is the main reason why we consider $\succsim_f$ is so natural.

Note that the converse relationship of Theorem 2.4 is open: that is, whether or not there is a demand function $f^{\succsim}$ such that $\succsim$ is complete but $f^{\succsim}$ violates WA is unknown.

We should mention another open problem related to Theorem 2.4. First, suppose that Ω is closed. As we have argued, if $f = f^{\succsim}$ for some complete, transitive, upper semi-continuous, monotone, and strictly convex binary relation, then f is a single-valued demand function that satisfies Walras' law, homogeneity of degree zero, and SA. Conversely, suppose that f is a continuous and income-Lipschitzian CoD such that $R(f)$ is relatively open subset of $\mathbb{R}^n_+$ that includes $\mathbb{R}^n_{++}$. If f satisfies Walras' law, homogeneity of degree zero, and SA, then by Corollary 2.2, we have that $f = f^{\succsim}$ for some complete, transitive, and upper semi-continuous binary relation $\succsim$. Therefore, we obtain an "almost" necessary and sufficient relationship between the existence of a good $\succsim$ corresponds to f and SA. Moreover, such a binary relation is unique. We want to extend this result for the case in which f satisfies only WA. However, there is a technical difficulty, and this problem is still open.

In fact, the following result was obtained by Sonnenschein [25].

Proposition 2.7. *Suppose that Ω is closed, and $\succsim$ is a complete, upper semi-continuous, and strictly convex binary relation on Ω such that for every $x \in \Omega$, the* *set*

$$P(x) = \{y \in \Omega | y \succ x\}$$

is convex. Then, $f^{\succsim}$ is a single-valued demand function defined on P.

In Theorem 2.4, we proved that under several conditions, a CoD f is $f^{\succsim}$ for some complete binary relation $\succsim$. We want to prove the reverse relationship. That is, we want to prove that if $\succsim$ is a complete, upper semi-continuous, and strictly convex binary relation, then $f^{\succsim}$ becomes a single-valued demand function. However, this problem is open. In fact, we cannot determine whether or not $P(x)$ in Proposition 2.7 is convex in this case. Because the proof of Proposition 2.7 uses the KKM theorem, the convexity of $P(x)$ is crucial.[5]

Suppose that f is surjective and $\succsim$ is a complete, upper semi-continuous, and strictly convex preference relation such that $f = f^{\succsim}$. Choose any $y \in P(x)$. Then, by strict convexity, we have that $(1-t)x + ty \in P(x)$ for every $t \in]0, 1[$.[6] In this case, is $P(x)$ convex? In other words, we are confronted with the following problem.

PROBLEM: Suppose that $C \subset \mathbb{R}^N$, and there exists $x \in \partial C$ such that if $y \in C$ and $0 < t < 1$, then $(1-t)x + ty \in C$. Is C convex?

Unfortunately, the answer to the above problem is NEGATIVE. Chaowen Yu presented the following counterexample. Let

$$C =]0, 1[\times \{(x_2, x_3) | |x_3| < |x_2|\}, \ x = (1, 0, 0).$$

This C and x satisfy all requirements of PROBLEM, and C is not convex. Therefore, our $P(x)$ might be not convex.

However, we may be able to extend Proposition 2.7 to our cases using techniques different from the KKM theorem, and thus this problem is still open.

We also mention a uniqueness result on $\succsim$. Suppose that a CoD f is continuously differentiable, and define

$$s_{ij}(p, m) = \frac{\partial f_i}{\partial p_j}(p, m) + \frac{\partial f_i}{\partial m}(p, m) f_j(p, m).$$

The $n \times n$ matrix-valued function $S_f(p, m) = (s_{ij}(p, m))_{i,j=1}^n$ is called the **Slutsky matrix**. We say that f satisfies (R) if the rank of $S_f(p, m)$ is always $n - 1$.

Suppose that $\Omega = \mathbb{R}^n_{++}$, and $f : P \to \Omega$ is a surjective and continuously differentiable CoD that satisfies Walras' law and WA. We can easily show that this f is homogeneous of degree zero. By Proposition 1 of Hosoya [7], f satisfies (R) if and only if there uniquely exists a function $g : \Omega \to \mathbb{R}^n_{++}$ such that the n-th coordinate of this function is always 1, and $f(g(x), g(x) \cdot x) = x$ for every $x \in \Omega$. Moreover, in this case g is continuously differentiable. Consider the following differential equation:

$$\dot{y}(t) = (g(y(t)) \cdot x)v - (g(y(t)) \cdot v)x, \ y(0; x, v) = x. \tag{2.4}$$

[5] Probably, Komiya's polytopal KKMS theorem is more useful for proving this theorem. See Komiya [14].

[6] In fact, if $\succsim = \succsim_f$, $R(f)$ is convex, and $x \in R(f)$, then the same result holds even if $\succsim$ is possibly not strictly convex. The proof is easy.

Let $w = (v \cdot x)v - (v \cdot v)x$. Define $t(x, v) = \inf\{t \geq 0 | y(t; x, v) \cdot w = 0\}$, where $y(t; x, v)$ is a nonextendable solution of (2.4). Let

$$u(x, v) = \frac{\|y(t(x, v); x, v)\|}{\|v\|}.$$

Using Hosoya's [7] results, we obtain the following result.

Proposition 2.8. *Suppose that $\Omega = \mathbb{R}^n_{++}$, and $f : P \to \Omega$ is a surjective and continuously differentiable CoD that satisfies Walras' law, WA, and (R). Then, $\succsim_f = u^{-1}([1, +\infty[)$, and $\succsim_f$ is the unique complete, p-transitive, and continuous binary relation that corresponds to f.*[7]

Proof. We mention several facts proved in Hosoya [7]. First, $t(x, v)$ is well-defined for every $(x, v) \in \Omega^2$. Second, $y(t(x, v); x, v)$ is proportional to v, and $y(t(x, v); x, v) = u(x, v)v$. Third, $u(x, v) \geq 1$ if and only if $u(v, x) \leq 1$. Fourth, if $x \succ_r v$, then $u(x, v) > 1$. Fifth, $a \mapsto u(ax, v)$ is increasing and $u(x, av) = a^{-1}u(x, v)$. Sixth, on $\mathrm{span}\{x, v\}$,

$$u(z_1, u(z_2, z_3)z_3) = u(z_1, z_2).$$

Now, choose $h > 0$ so small, and let (x^h_k) be a forward Euler approximated solution of (2.4): that is,

$$\varphi(y) = (g(y) \cdot x)v - (g(y) \cdot v)x,$$
$$x^h_0 = x,$$
$$x^h_{k+1} = x^h_k + h\varphi(x^h_k).$$

Since $\varphi(y) \in \mathrm{span}\{x, v\}$, we have that $x^h_k \in \mathrm{span}\{x, v\}$ for all k. Because $g(y) \cdot \varphi(y) = 0$, we have that for $k \geq 1$, either $x = x^h_k$ or

$$x \succ_{irp} x^h_k.$$

If $u(x, v) > 1$, then $y(t(x, v); x, v) \gg v$. Hence, for sufficiently small $h > 0$, there exists x^h_k such that $x^h_k \gg v$. This implies that $x^h_k \succ_r v$, and thus $x \succ_{irp} v$. Therefore, we have that if $u(x, v) > 1$, then $x \succ_{irp} v$.

Next, suppose that $x \succ_{irp} v$. If x is proportional to v, then $t(x, v) = 0$ and $u(x, v) = \frac{\|x\|}{\|v\|} > 1$. Hence, we assume without loss of generality that x is not proportional to v, and there exists a finite sequence $x_1, ..., x_N$ such that $x_1 = x, x_N = v$, $x_i \succ_r x_{i+1}$ for each i, and $\mathrm{span}\{x_1, ..., x_N\} = \mathrm{span}\{x, v\}$. Define $a_0 = 1$, and if a_{i-1} is already defined, then define $a_i = u(a_{i-1}x_i, x_{i+1})$. Because $x_i \succ_r x_{i+1}$, by mathematical induction, we have that

$$a_i = u(a_{i-1}x_i, x_{i+1}) \geq u(x_i, x_{i+1}) > 1$$

for all $i \in \{1, ..., N - 1\}$. Therefore,

$$1 = u(x_1, x_1) = u(x_1, a_{N-1}x_N) < u(x_1, x_N) = u(x, v),$$

[7]In the proof of this proposition, we frequently use knowledge of the proof of Theorem 1 in Hosoya [7].

which implies that $x \succ_{irp} v$ if and only if $u(x,v) > 1$. Because $u(x,v) \ge 1$ if and only if $u(v,x) \le 1$, we have that $x \succsim_f v$ if and only if $u(x,v) \ge 1$. The remainder of the proof was presented in Hosoya [7]. □

Hosoya [7] also showed that, if f satisfies SA, then $f = f^{u_v}$ for $u_v : x \mapsto u(x,v)$, and u_v is continuously differentiable. In this case, our Proposition 2.8 implies that u_v represents $\succsim_f$, and thus we immediately obtain the following corollary. This result also indicates a different feature of $\succsim_f$ from several known binary relations correspond to f in this context.

Corollary 2.9. *Suppose that $\Omega = \mathbb{R}^n_{++}$ and f is a surjective and continuously differentiable CoD that satisfies Walras' law, SA, and (R). Then, $f = f^u$ for some continuously differentiable function u, and u represents $\succsim_f$. In particular, $\succsim_f$ is transitive and continuous.*

Note that, if $\Omega = \mathbb{R}^n_{++}$, then $P = \mathbb{R}^n_{++} \times \mathbb{R}_{++}$. This restriction could be considered rather strong, because $\Delta(p,m)$ is not compact. In fact, Hosoya [7] showed that a similar result holds even if P is just an open cone in $\mathbb{R}^n_{++} \times \mathbb{R}_{++}$. However, the continuous differentiability assumption of f is strong, and thus we want to relax it. This is another open problem.

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References

[1] M. Boldrin and L. Montrucchio, *On the indeterminacy of capital accumulation paths*, J Econ Theory **40** (1986), 26–39.
[2] D. E. Bridges and G. B. Mehta, *Representations of Preference Orderings*, Springer, Berlin, 1995.
[3] D. Cass, *Optimum growth in an aggregative model of accumulation*, Rev Econ Stud **32** (1965), 233–240.
[4] G. Debreu, *Representation of a preference ordering by a numerical function*, in: Decision Processes, R. M. Thrall, C. H. Coombs, R. L. Davis (eds.), Wiley, 1954, pp.159–165.
[5] D. Gale, *A note on revealed preference*, Economica **27** (1960), 348–354.
[6] W. Hildenbrand, *Core and Equilibria of a Large Economy*, Princeton University Press, Princeton, 1974.
[7] Y. Hosoya, *Measuring utility from demand*, J. Math. Econ. **49** (2013), 82–96.
[8] Y. Hosoya, *Identification and testable implications of the Ramsey-Cass-Koopmans model*, J. Math. Econ. **50** (2014), 63–68.
[9] Y. Hosoya, *The relationship between revealed preference and Slutsky matrix*, J. Math. Econ. **70** (2017), 127–146.
[10] Y. Hosoya, *Revealed preference theory*, Appl. Anal. Optim. **3** (2019), 179–204.
[11] Y. Hosoya, *Recoverability revisited*, J. Math. Econ. **90** (2020), 31–41.
[12] H. S. Houthakker, *Revealed preference and the utility function*, Economica **17** (1950), 159–174.
[13] T. Kim and M. K. Richter, *Nontransitive-Nontotal consumer theory*, J. Econ. Theory **38** (1986), 324–363.

[14] H. Komiya, *A simple proof of K-K-M-S theorem*, Econ. Theory **4** (1994), 463–466.
[15] H. Komiya, *Inverse of the Berge maximum theorem*, Econ. Theory **9** (1997), 371–375.
[16] T. C. Koopmans, *On the concept of optimal economic growth*, Pontificiae Academiae Scientarum Varia **28** (1965), 225–300.
[17] A. Mas-Colell, *The recoverability of consumers' preferences from market demand behavior*, Econometrica **45** (1977), 1409–1430.
[18] A. Mas-Colell, *An equilibrium existence theorem without complete or transitive preferences*, in: General Equilibrium and Game Theory: Ten Papers, A. Mas-Colell (ed), 2016, pp.15–26.
[19] A. Mas-Colell, M. D. Whinston and J. R. Green, *Microeconomic Theory.* Oxford University Press, Oxford, 1995.
[20] J. K-H. Quah, *Weak axiomatic demand theory*, Econ. Theory **29** (2006), 677–699.
[21] F. Ramsey, *A mathematical theory of saving*, Econ. J **38** (1928), 543–559.
[22] M. K. Richter, *Revealed preference theory*, Econometrica **34** (1966), 635–645.
[23] P. A. Samuelson, *A note on the pure theory of consumer's behaviour*, Economica **5** (1938), 61–71.
[24] W. Shafer and H. F. Sonnenschein, *Some theorems on the existence of competitive equilibrium*, J. Econ. Theory **11** (1975), 83–93.
[25] H. F. Sonnenschein, *Demand theory without transitive preferences, with applications to the theory of competitive equilibrium*, in: Preferences, Utility and Demand, J. S. Chipman, L. Hurwicz, M. K. Richter, H. F. Sonnenschein (eds.), Harcourt Brace Jovanovich, Inc., New York, 1971, pp.215–223.

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