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STRONG CONVERGENCE THEOREMS FOR COMMUTATIVE LINEAR CONTRACTIVE MAPPINGS IN BANACH SPACES BASED ON NONLINEAR ANALYTIC METHODS

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Dedicated to Professor Hidetoshi Komiya
on his 65th birthday

Abstract. In this paper, we first discuss nonlinear analytic methods for studying linear contractive mappings in Banach spaces. Using these results, we obtain strong convergence theorems for commutative two linear contractive operators in Banach spaces. In theorems, the limit points are characterized by sunny generalized nonexpansive retractions in Banach spaces.

1. Introduction

Let E be a real Banach space and let C be a nonempty closed convex subset of E. For a mapping $T : C \to C$, we denoted by $F(T)$ the set of fixed points of T. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. In particular, a nonexpansive mapping $T : E \to E$ is called *contractive* if it is linear, that is, a linear contactive mapping $T : E \to E$ is a linear operator satisfying $\|T\| \le 1$. In 1932, von Neumann [37] proved the first mean convergence theorem for linear operators in a Hilbert space.

Theorem 1.1 ([37]). *Let T be a unitary operator in a Hilbert space H. Then, for any $x \in H$, the sequence*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges strongly to a point in H.

This theorem, in 1938, was extended to the following mean ergodic theorem for linear bounded operators by Yosida [39].

Theorem 1.2 ([39]). *Let E be a real Banach space and let T be a linear operator of E into itself such that there exists a constant C with $\|T^n\| \le C$ for $n \in \mathbb{N}$, and T is weakly completely continuous, i.e., T maps the closed unit ball of E into a weakly compact subset of E. Then, for each $x \in E$, the Cesàro means $S_n x$ converge strongly as $n \to \infty$ to a fixed point of T.*

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See Day [7] and Kido and Takahashi [22] for mean convergence theorems of semigroups of linear operators in a Banach space.

On the other hand, we know the first mean convergence theorem for nonexpansive mappings in a Hilbert space by Baillon [4].

Theorem 1.3 ([4])**.** *Let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is nonempty. Then for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

Such a theorem was extended to a noncommutative semigroup (called amenable) of nonexpanive mappings in a Hilbert space by Takahashi [31]; see also [32]. Baillon's theorem for nonexpansive mappings has been extended to Banach spaces by many authors; see, for example, [5, 6, 10, 11, 25].

From [34] we also know a weak convergence theorem by Mann's iteration [26] for nonexpansive mappings in a Hilbert space: Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ in C by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$. Then, $\{x_n\}$ converges weakly to an element z of $F(T)$, where $z = \lim_{n\to\infty} Px_n$ and P is the metric projection of H onto $F(T)$. By Reich [30], such a theorem was extended to a uniformly convex Banach space with a Fréchet differentiable norm. However, we have not known whether the limit point z is characterized under any projections in a Banach space. Using nonlinear analytic methods obtained by [15, 18, 19], Takahashi and Yao [35] solved such a problem for positively homogeneous nonexpansive mappings in a Banach space.

In this paper, we first discuss nonlinear analytic methods for studying linear contractive mappings in Banach spaces. Using these results, we obtain strong convergence theorems for commutative two linear contractive operators in Banach spaces. In theorems, the limit points are characterized by sunny generalized nonexpansive retractions in Banach spaces.

2. Preliminaries

Throughout this paper, we assume that a Banach space E with the dual space E^* is real. We denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of all positive integers and all real numbers, respectively. We also denote by $\langle x, x^* \rangle$ the dual pair of $x \in E$ and $x^* \in E^*$. A Banach space E is said to be strictly convex if $\|x + y\| < 2$ for $x, y \in E$ with

$\|x\| \le 1$, $\|y\| \le 1$ and $x \neq y$. Let E be a Banach space and let

$$\delta(\epsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| = \epsilon\right\}.$$

We call the function $\delta : [0,2] \to [0,1]$ the modulus of convexity. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive.

A Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$ with $\|x\| = \|y\| = 1$. Let E be a Banach space. With each $x \in E$, we associate the set

$$J(x) = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}.$$

The multivalued operator $J : E \to E^*$ is called the normalized duality mapping of E. From the Hahn-Banach theorem, $Jx \neq \emptyset$ for each $x \in E$. We know that E is smooth if and only if J is single-valued. If E is strictly convex, then J is one-to-one, i.e., $x \neq y \Rightarrow J(x) \cap J(y) = \emptyset$. If E is reflexive, then J is a mapping of E onto E^*. So, if E is reflexive, strictly convex and smooth, then J is single-valued, one-to-one and onto. In this case, the normalized duality mapping J_* from E^* into E is the inverse of J, that is, $J_* = J^{-1}$; see [33] for more details.

Lemma 2.1 ([33]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then $\langle x - y, Jx - Jy\rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy\rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the normalized duality mapping of E. We define the function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$$

for all $x, y \in E$. It is easy to see that $(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2$ for all $x, y \in E$. Thus, in particular, $\phi(x, y) \ge 0$ for all $x, y \in E$. We also know the following:

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle \tag{2.1}$$

for all $x, y, z \in E$. Further, we have

$$2\langle x - y, Jz - Jw\rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w) \tag{2.2}$$

for all $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \Leftrightarrow x = y. \tag{2.3}$$

The following lemma due to Kamimura and Takahashi [21] is well-known.

Lemma 2.2 ([21]). *Let E be a smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be sequences in E such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If* $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \phi(z, x) = \min_{y\in C} \phi(y, x)\}$$

is always a singleton. Let us define the mapping Π_C of E onto C by $z = \Pi_C x$ for every $x \in E$, i.e.,

$$\phi(\Pi_C x, x) = \min_{y\in C} \phi(y, x)$$

for every $x \in E$. Such Π_C is called the generalized projection of E onto C; see Alber [1]. The following lemma is due to Alber [1] and Kamimura and Takahashi [21].

Lemma 2.3 ([1, 21]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x, z) \in E \times C$. Then, the following hold:*

(a) $z = \Pi_C x$ *if and only if* $\langle y - z, Jx - Jz\rangle \le 0$ *for all* $y \in C$;
(b) $\phi(z, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(z, x)$.

Let D be a nonempty closed subset of a smooth Banach space E, let T be a mapping from D into itself and let $F(T)$ be the set of fixed points of T. Then, T is said to be generalized nonexpansive [16] if $F(T)$ is nonempty and $\phi(Tx, u) \le \phi(x, u)$ for all $x \in D$ and $u \in F(T)$. Let C be a nonempty subset of E and let R be a mapping from E onto C. Then R is said to be a retraction, or a projection if $Rx = x$ for all $x \in C$. It is known that if a mapping P of E into E satisfies $P^2 = P$, then P is a projection of E onto $\{Px : x \in E\}$. A mapping $T : E \to E$ with $F(T) \neq \emptyset$ is a retraction if and only if $F(T) = R(T)$, where $R(T)$ is the range of T. The mapping R is also said to be sunny if $R(Rx + t(x - Rx)) = Rx$ whenever $x \in E$ and $t \ge 0$. A nonempty subset C of a smooth Banach space E is said to be a generalized nonexpansive retract (resp. sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp. sunny generalized nonexpansive retraction) R from E onto C. The following lemmas were proved by Ibaraki and Takahashi [16].

Lemma 2.4 ([16]). *Let C be a nonempty closed sunny and generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then, the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 2.5 ([16]). *Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and let $(x,z) \in E \times C$. Then, the following hold:*

(a) *$z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;*
(b) *$\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.*

The following theorems were proved by Kohsaka and Takahashi [24].

Theorem 2.1 ([24]). *Let E be a smooth, strictly convex and reflexive Banach space, let C^* be a nonempty closed convex subset of E^* and let Π_{C^*} be the generalized projection of E^* onto C^*. Then the mapping R defined by $R = J^{-1}\Pi_{C^*}J$ is a sunny generalized nonexpansive retraction of E onto $J^{-1}C^*$.*

Theorem 2.2 ([24]). *Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty subset of E. Then, the following are equivalent.*

(1) *D is a sunny generalized nonexpansive retract of E;*
(2) *D is a generalized nonexpansive retract of E;*
(3) *JD is closed and convex.*

In this case, D is closed.

Let E be a smooth, strictly convex and reflexive Banach space, let J be the normalized duality mapping from E onto E^* and let C be a closed subset of E such that JC is closed and convex. Then, we can define a unique sunny generalized nonexpansive retraction R_C of E onto C as follows:

$$R_C = J^{-1}\Pi_{JC}J,$$

where Π_{JC} is the generalized projection from E^* onto JC.

Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. For an arbitrary point x of E, the set

$$\{z \in C : \|z - x\| = \min_{y \in C} \|y - x\|\}$$

is always nonempty and a singleton. Let us define the mapping P_C of E onto C by $z = P_C x$ for every $x \in E$, i.e.,

$$\|P_C x - x\| = \min_{y \in C} \|y - x\|$$

for every $x \in E$. Such P_C is called the metric projection of E onto C; see [33]. The following lemma is in [33].

Lemma 2.6 ([33]). *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let $(x,z) \in E \times C$. Then, $z = P_C x$ if and only if $\langle y - z, J(x - z) \rangle \leq 0$ for all $y \in C$.*

Let E be a Banach space and let K be a closed convex cone of E. Then, $T : K \to K$ is called a positively homogeneous mapping if $T(\alpha x) = \alpha T x$ for all $\alpha \geq 0$ and $x \in K$. Let M be a closed linear subspace of E. Then, $S : M \to M$ is called a homogeneous mapping if $T(\beta x) = \beta T x$ for all $\beta \in \mathbb{R}$ and $x \in M$.

Remark 2.1. *In L^p spaces, $1 \leq p \leq \infty$, we know examples of nonexpansive and positively homogeneous mappings; see, for instance, Wittmann* [38].

We know the following theorem from Takahashi, Yao and Honda [36].

Theorem 2.3 ([36]). *Let E be a smooth Banach space and let K be a closed convex cone of E. Then, a positively homogeneous mapping $T : K \to K$ is generalized nonexpansive if and only if for any $x \in K$ and $u \in F(T)$,*

$$\|Tx\| \leq \|x\| \text{ and } \langle x - Tx, Ju \rangle \leq 0.$$

Furthermore, let M be a closed linear subspace of E. Then, a homogeneous mapping $S : M \to M$ is generalized nonexpansive if and only if for any $x \in M$ and $v \in F(T)$,

$$\|Sx\| \leq \|x\| \text{ and } \langle x - Sx, Jv \rangle = 0.$$

We also know the follwing theorem from Takahashi and Yao [35]; see also Honda, Takahashi and Yao [15].

Theorem 2.4 ([35]). *Let E be a smooth Banach space and let K be a closed convex cone in E If $T : K \to K$ is a positively homogeneous nonexpansive mapping, then T is generalized nonexpansive. In particular, if $T : E \to E$ is a linear contractive mapping, then T is generalized nonexpansive.*

From Theorems 2.4 and 2.3, we have the following corollary.

Corollary 2.1. *Let E be a smooth Banach space and let K be a closed convex cone of E. If a mapping $T : K \to K$ is positively homogeneous nonexpansive, then for any $x \in K$ and $u \in F(T)$,*

$$\|Tx\| \leq \|x\| \text{ and } \langle x - Tx, Ju \rangle \leq 0.$$

Furthermore, let M be a closed linear subspace of E. If a mapping $S : M \to M$ is homogeneous nonexpansive, then for any $x \in M$ and $v \in F(T)$,

$$\|Sx\| \leq \|x\| \text{ and } \langle x - Sx, Jv \rangle = 0.$$

From Takahashi, Yao and Honda [36], we know the following concept.

Definition 2.1 ([36]). *Let E be a smooth Banch space, let $x \in E$ and let F be a nonempty subset of E. The set $R(x; F)$ between x and F is as follows:*

$$R(x; F) = \{z \in E : \langle x - z, Ju \rangle = 0 \text{ for all } u \in F \text{ and } \|z\| \leq \|x\|\}.$$

We have the following result from Takahashi, Yao and Honda [36].

Lemma 2.7 ([36]). *Let E be a strictly convex and smooth Banch space, let $x \in E$ and let F be a nonempty subset of E. Then $R(x; F)$ is nonempty, closed, convex and bounded, and $F \cap R(x; F)$ consists of at most one point.*

3. Strong convergence theorems

Let Y be a nonempty subset of a Banach space E and let Y^* be a nonempty subset of the dual space E^*. Then, we can define the annihilator $Y^*_\perp$ of Y^* and the annihilator $Y^\perp$ of Y as follows:

$$Y^*_\perp = \{x \in E : f(x) = 0 \text{ for all } f \in Y^*\}$$

and

$$Y^\perp = \{f \in E^* : f(x) = 0 \text{ for all } x \in Y\}.$$

We know the following result from Megginson [29].

Lemma 3.1 ([29])**.** *Let A be a nonempty subset of E. Then*

$$(A^\perp)_\perp = \overline{\mathrm{span}}A,$$

where $\overline{\mathrm{span}}A$ is the smallest closed linear subspace of E containing A.

Let $T : E \to E$ be a bounded linear operator. Then, the adjoint mapping $T^* : E^* \to E^*$ is defined as follows:

$$\langle x, T^*x^*\rangle = \langle Tx, x^*\rangle$$

for any $x \in E$ and $x^* \in E^*$. We know that T^* is also a bounded linear operator and $\|T\| = \|T^*\|$. If S and T are bounded linear operators form E into itself and $\alpha \in \mathbb{R}$, then $(S + T)^* = S^* + T^*$ and $(\alpha S)^* = \alpha S^*$. Let I be the identity operator on E. Then, I^* is the identity operator on E^*. Let $T^{**} : E^{**} \to E^{**}$ be the adjoint of T^*. Then we have $T^{**}(\pi(E)) \subset \pi(E)$ and $\pi^{-1}T^{**}\pi = T$, where π is the natural embedding from E into its second dual space E^{**}; see [29]. We know the following lemma from Takahashi, Yao and Honda [36].

Lemma 3.2 ([36])**.** *Let E be a smooth, strictly convex and reflexive Banach space, let T be a linear contractive operator of E into itself, i.e., $T : E \to E$ is a linear operator such that $\|T\| \leq 1$ and let $F(T)$ be the set of fixed points of T. Then $JF(T)$ is a closed linear subspace in E^* and $JF(T) = F(T^*) = \{z - Tz : z \in E\}^\perp$, where $J : E \to E^*$ is the normalized duality mapping and T^* is the adjoint operator of T.*

Using Lemma 3.2, we have the following result.

Lemma 3.3. *Let E be a smooth, strictly convex and reflexive Banach space, let S, T be linear contractive operators of E into itself. Then $J(F(S) \cap F(T))$ is a closed linear subspace in E^* and $J(F(S) \cap F(T)) = F(S^*) \cap F(T^*) = \{z - Sz, z - Tz : z \in E\}^\perp$, where $J : E \to E^*$ is the normalized duality mapping and S^*, T^* are the adjoint operators of S, T, respectively.*

Proof. Since E is a smooth, strictly convex and reflexive Banach space, J is single-valued, one-to-one and onto. Thus, we have that

$$J(F(S) \cap F(T)) = JF(S) \cap JF(T).$$

Using this equality and Lemma 3.2, we have that $J(F(S) \cap F(T))$ is a closed linear subspace in E^*. Furthermore, we have that

$$\begin{aligned} f \in J(F(S) \cap F(T)) &\Leftrightarrow f \in JF(S) \cap JF(T) \\ &\Leftrightarrow f \in F(S^*) \cap F(T^*) \\ &\Leftrightarrow f \in \{z - Sz : z \in E\}^{\perp} \cap \{z - Tz : z \in E\}^{\perp} \\ &\Leftrightarrow f \in \{z - Sz, z - Tz : z \in E\}^{\perp}. \end{aligned}$$

This completes the proof. □

Theorem 3.1. *Let E be a smooth, strictly convex and reflexive Banach space, let S, T be linear contractive operators on E and let $\{S_n : n \in \mathbb{N}\}$ be a sequence of contractive linear operators on E such that $F(S) \cap F(T) \subset F(S_n)$ for all $n \in \mathbb{N}$. Then, the following are equivalent:*

1. *$S_n x$ converges to an element of $F(S) \cap F(T)$ for each $x \in E$;*
2. *$S_n x$ converges to 0 for each $x \in (J(F(S) \cap F(T)))_{\perp}$;*
3. *$S_n x - S_n \circ Sx$ and $S_n x - S_n \circ Tx$ converge to 0 for each $x \in E$.*

Furthermore, if (1) holds, then $S_n x$ converges to $R_{F(S) \cap F(T)} x \in F(S) \cap F(T)$, where $R_{F(S) \cap F(T)} = J^{-1} \Pi_{J(F(S) \cap F(T))} J$ and $\Pi_{J(F(S) \cap F(T))}$ is the generalized projection of E^* onto $J(F(S) \cap F(T))$.

Proof. Suppose (1). Then, for any $x \in E$, $S_n x \in R(x; F(S_n)) \subset R(x; F(S) \cap F(T))$ for all $n \in \mathbb{N}$. We know from Lemma 2.7 that $R(x; F(S) \cap F(T)) \cap (F(S) \cap F(T))$ consists of at most one point. Since $R(x; F(S) \cap F(T))$ is closed and $S_n x$ converges strongly to an element z of $F(S) \cap F(T)$, we have

$$R(x; F(S) \cap F(T)) \cap (F(S) \cap F(T)) = \{z\}.$$

Let Rx be the unique element z of $R(x; F(S) \cap F(T)) \cap (F(S) \cap F(T))$. Then, a mapping $R : E \to F(S) \cap F(T)$ defined by $z = Rx$ is a retraction of E onto $F(S) \cap F(T)$. Furthermore, we know from Corollary 2.1 that $\langle x - S_n x, Ju \rangle = 0$ for all $u \in F(S_n)$ and $n \in \mathbb{N}$. Since $F(S) \cap F(T) \subset F(S_n)$ for all $n \in \mathbb{N}$, we have that $\langle x - S_n x, Ju \rangle = 0$ for all $u \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. So, we have that, for any $u \in F(S) \cap F(T)$,

$$\langle x - Rx, Ju \rangle = 0 \tag{3.1}$$

From $Rx \in F(S) \cap F(T)$, we also have $\langle x - Rx, JRx \rangle = 0$ and thus

$$\langle x - Rx, JRx - Ju \rangle = 0 \tag{3.2}$$

for all $u \in F(S) \cap F(T)$. We have from Lemmas 2.4 and 2.5 that R is the unique sunny generalized nonexpansive retraction of E onto $F(S) \cap F(T)$. Therefore, from Theorem 2.1, we have

$$R = R_{F(S) \cap F(T)} = J^{-1} \Pi_{J(F(S) \cap F(T))} J,$$

where $\Pi_{J(F(S) \cap F(T))}$ is the generalized projection of E^* onto $J(F(S) \cap F(T))$. If $x \in (J(F(S) \cap F(T)))_{\perp}$, then we have $\langle x, Ju \rangle = 0$ for all $u \in F(S) \cap F(T)$. We also have from (3.1) that $\langle x - Rx, Ju\rangle = 0$ for all $u \in F(S) \cap F(T)$. Then, we get $\langle Rx, Ju\rangle = 0$ for all $u \in F(S)\cap F(T)$. This implies $Rx \in (J\,(F(S) \cap F(T)))_\perp$. From $Rx \in F(S) \cap F(T) \cap (J\,(F(S) \cap F(T)))_\perp$ and $F(S) \cap F(T) \cap (J\,(F(S) \cap F(T)))_\perp = \{0\}$, we have that $S_n x \to R_{F(S)\cap F(T)}x = 0$ as $n \to \infty$. Therefore, we obtain (2).

Suppose (2). From Lemma 3.3, $J\,(F(S) \cap F(T)) = JF(S) \cap JF(T)$ is a closed linear subspace of E^*. Then, we have from [2, 3, 13, 14] that for any $x \in E$,

$$x = R_{F(S)\cap F(T)}x + P_{(J(F(S)\cap F(T)))_\perp}x,$$

where $P_{(J(F(S)\cap F(T)))_\perp}$ is the metric projection of E onto $(J\,(F(S) \cap F(T)))_\perp$. We have from (2) that

$$\begin{aligned} S_n x &= S_n(R_{F(S)\cap F(T)}x + P_{(J(F(S)\cap F(T)))_\perp}x) \\ &= S_n R_{F(S)\cap F(T)}x + S_n P_{(J(F(S)\cap F(T)))_\perp}x \\ &= R_{F(S)\cap F(T)}x + S_n P_{(J(F(S)\cap F(T)))_\perp}x \\ &\to R_{F(S)\cap F(T)}x \in F(S) \cap F(T), \end{aligned}$$

as $n \to \infty$. Then, we obtain (1). Furthermore, we know from Lemma 3.3 that $x - Sx, x - Tx \in (J\,(F(S) \cap F(T)))_\perp$ for all $x \in E$. Hence we have from (2) that $S_n(x - Sx) \to 0$ and $S_n(x - Tx) \to 0$ as $n \to \infty$. So, we obtain (3).

Suppose (3). We have that, for any $x \in E$,

$$S_n(x - Sx) \to 0 \text{ and } S_n(x - Tx) \to 0.$$

Then we have $S_n y$ converges to 0 for any $y \in \text{lin}\{x - Tx : x \in E\}$, where $\text{lin}A$ is the smallest linear subspace of E containing A. From Lemmas 3.2 and 3.1, we have

$$(J\,(F(S) \cap F(T)))_\perp = (\{z - Sz, z - Tz : z \in E\}^\perp)_\perp = \overline{\text{span}}\{z - Sz, x - Tx : x \in E\}.$$

Take $x \in (J\,(F(S) \cap F(T)))_\perp$. Then, for any $\epsilon > 0$, there exists an element $y \in \text{lin}\{x - Sx, x - Tx : x \in E\}$ such that $\|x - y\| < \epsilon$. Then we have

$$\begin{aligned} \|S_n x\| &= \|S_n y + (S_n x - S_n y)\| \\ &\le \|S_n y\| + \|S_n x - S_n y\| \\ &\le \|S_n y\| + \|x - y\| \\ &\le \|S_n y\| + \epsilon \end{aligned}$$

and hence

$$\limsup_{n\to\infty} \|S_n x\| \le \limsup_{n\to\infty}(\|S_n y\| + \epsilon) = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have that for any $x \in (J\,(F(S) \cap F(T)))_\perp$, $S_n x$ converges to 0. Then, we obtain (2).

Furthermore, if (1) holds, then we have from the proof of (1) that for any $x \in E$, $S_n x$ converges strongly to $R_{F(S)\cap F(T)}x \in F(S) \cap F(T)$. □

Using Theorem 3.1, we have the following useful result.

Theorem 3.2. *Let E be a smooth, strictly convex and reflexive Banach space, let S, T be linear contactive operators on E, let $\{T_i : i \in \mathbb{N}\}$ be a sequence of linear contractive operators on E such that $F(S) \cap F(T) \subset F(T_i)$ for all $i \in \mathbb{N}$ and let $S_n = T_n \circ T_{n-1} \circ \cdots \circ T_1$ for all $n \in \mathbb{N}$. Then, the following are equivalent:*

(1) *$S_n x$ converges to an element of $F(S) \cap F(T)$ for each $x \in E$;*
(2) *$S_n x$ converges to 0 for each $x \in (J(F(S) \cap F(T)))_\perp$;*
(3) *$S_n x - S_n \circ Sx \to 0$ and $S_n x - S_n \circ Tx \to 0$ for each $x \in E$.*

Furthermore, if (1) holds, then $S_n x$ converges to $R_{F(S)\cap F(T)}x \in F(S) \cap F(T)$, where $R_{F(S)\cap F(T)} = J^{-1}\Pi_{J(F(S)\cap F(T))}J$ and $\Pi_{J(F(S)\cap F(T))}$ is the generalized projection of E^* onto $J(F(S) \cap F(T))$.

Proof. For any $n \in \mathbb{N}$, $S_n = T_n \circ T_{n-1} \circ \cdots \circ T_1$ is a linear contractive operator on E and $F(S) \cap F(T) \subset F(S_n)$ for all $i \in \mathbb{N}$. Therefore, we have the desired result from Theorem 3.1

□

4. Applications

In this section, using Theorems 3.1 and 3.2, we obtain some strong convergence theorems for linear contractive mappings in a Banach space. Applying Theorem 3.2, we obtain a strong convergence theorem of Mann type for contractive linear mappings in a Banach space. The following lemma was proved by Eshita and Takahashi [8].

Lemma 4.1 ([8]). *Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$ and let $\{b_n\}$ and $\{\varepsilon_n\}$ be sequences in $[0, \infty)$ such that*

$$b_{n+1} \le \alpha_n b_n + (1 - \alpha_n)\varepsilon_n, \quad \forall n \in \mathbb{N}$$

and $\lim_{n\to\infty} \varepsilon_n = 0$. Then $\lim_{n\to\infty} b_n = 0$.

Using Lemma 4.1, we obtain the following theorem.

Theorem 4.1. *Let E be a smooth and uniformly convex Banach space and let S, T be commutative contractive linear operators on E. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\sum_{n=1}^{\infty}(1 - \alpha_n) = \infty$. Then a sequence $\{x_n\}$ generated by $x_1 = x \in E$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\frac{1}{(n+1)^2}\sum_{k=0}^{n}\sum_{l=0}^{n} S^k T^l x_n, \quad n \in \mathbb{N},$$

converges strongly to an element Rx of $F(S) \cap F(T)$, where $R = R_{F(S)\cap F(T)} = J^{-1}\Pi_{J(F(S)\cap F(T))}J$ and $\Pi_{J(F(S)\cap F(T))}$ is the generalized projection of E^* onto $J(F(S) \cap F(T))$.

Proof. Put $U_i = \frac{1}{(i+1)^2}\sum_{k=0}^{i}\sum_{l=0}^{i} S^k T^l$ and let $T_i = \alpha_i I + (1 - \alpha_i)U_i$ for all $i \in \mathbb{N}$, where I is the identity operator on E. Let $S_n = T_n \circ T_{n-1} \circ \cdots \circ T_1$ for all $n \in \mathbb{N}$. Then, we have that $x_{n+1} = S_n x$. Since S, T are linear cotractive operators,

$F(S) \cap F(T)$ is a closed linear subspace of E. For any $i \in \mathbb{N}$, we have $\|T_i\| \leq 1$ and $F(S) \cap F(T) \subset F(T_i)$. Using these results, we obtain that $\|S_n\| \leq 1$ and $F(S) \cap F(T) \subset F(S_n)$ for any $n \in \mathbb{N}$.

It is sufficient to show from Theorem 3.2 that, for all $x \in E$.

$$\|S_n x - S_n \circ Sx\| \to 0$$

and

$$\|S_n x - S_n \circ Tx\| \to 0.$$

Since S, T are commutative contractive linear operators on E, we can also show that $S \circ S_n = S_n \circ S$ and $T \circ S_n = S_n \circ T$. We may show from $x_{n+1} = S_n x$ that

$$\|x_{n+1} - Sx_{n+1}\| \to 0 \text{ and } \|x_{n+1} - Tx_{n+1}\| \to 0.$$

Since $x_{n+1} = \alpha_n x_n + (1 - \alpha_n)U_n x_n$ and S is linear, we have that

$$\|x_{n+1} - Sx_{n+1}\| \leq \alpha_n \|x_n - Sx_n\| + (1 - \alpha_n)\|U_n x_n - SU_n x_n\|. \tag{4.1}$$

Furthermore, we have

$$\begin{aligned}
\|x_{n+1}\| &= \|\alpha_n x_n + (1 - \alpha_n)U_n x_n\| \\
&\leq \alpha\|x_n\| + (1 - \alpha_n)\|U_n x_n\| \\
&\leq \alpha\|x_n\| + (1 - \alpha_n)\|x_n\| \\
&= \|x_n\|.
\end{aligned}$$

Then $\lim_{n\to\infty} \|x_n\|$ exists and hence $\{x_n\}$ is bounded. We have from $ST = TS$ that

$$\begin{aligned}
\|U_n x_n - SU_n x_n\| &= \left\| \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n - S \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x_n \right\| \\
&= \left\| \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} (S^k T^l x_n - S^{k+1} T^l x_n) \right\| \\
&= \left\| \frac{1}{(n+1)^2} \sum_{l=0}^{n} (ST^l x_n + T^l x_n - S^{n+2} T^l x_n + S^{n+1} T^l x_n) \right\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded, we have that $\|U_n x_n - SU_n x_n\| \to 0$. Using (4.1), Lemma 4.1 and $\|U_n x_n - SU_n x_n\| \to 0$, we have that

$$\|x_{n+1} - Sx_{n+1}\| \to 0. \tag{4.2}$$

Similarly, we have that

$$\|x_{n+1} - Tx_{n+1}\| \to 0. \tag{4.3}$$

By Theorem 3.2, $\{x_n\}$ converges strongly to the element Rx of $F(S) \cap F(T)$, where $R = R_{F(S)\cap F(T)} = J^{-1}\Pi_{J(F(S)\cap F(T))}J$ and $\Pi_{J(F(S)\cap F(T))}$ is the generalized projection of E^* onto $J(F(S) \cap F(T))$. This completes the proof. □

From Theorem 3.1, we can prove a mean strong convergence theorem for commutative contractive linear operators in a Banach space.

Theorem 4.2. *Let E be a smooth, strictly convex and reflexive Banach space and let S, T be commutative contractive linear operators on E. Then, for each $x \in E$,*

$$S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$$

converge strongly to the element Rx of $F(S) \cap F(T)$, where $R = R_{F(S)\cap F(T)} = J^{-1}\Pi_{J(F(S)\cap F(T))}J$ and $\Pi_{J(F(S)\cap F(T))}$ is the generalized projection of E^* onto $J(F(S) \cap F(T))$.

Proof. Put $S_n x = \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x$ for all $x \in E$ and $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, the operator $S_n : E \to E$ is a contractive linear operator. Furthermore, we have $F(S) \cap F(T) \subset F(S_n)$. To complete the proof, it is sufficient to show that $S_n x - S_n \circ Sx \to 0$ and $S_n x - S_n \circ Tx \to 0$ for each $x \in E$. We have

$$\begin{aligned} S_n x - S_n \circ Sx &= \left\| \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l x - \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} S^k T^l Sx \right\| \\ &= \left\| \frac{1}{(n+1)^2} \sum_{k=0}^{n} \sum_{l=0}^{n} (S^k T^l x - S^{k+1} T^l x) \right\| \\ &= \left\| \frac{1}{(n+1)^2} \sum_{l=0}^{n} (ST^l x + T^l x - S^{n+2} T^l x + S^{n+1} T^l x) \right\|. \end{aligned}$$

Then, for any $n \in \mathbb{N}$, we have

$$\|S_n x - S_n \circ Tx\| = \left\| \frac{1}{(n+1)^2} \sum_{l=0}^{n} (ST^l x + T^l x - S^{n+2} T^l x + S^{n+1} T^l x) \right\|.$$

Thus we obtain that $S_n x - S_n \circ Sx \to 0$ for each $x \in E$. Similarly, we have that $S_n x - S_n \circ Tx \to 0$ Using Theorem 3.1, $\{S_n x\}$ converges strongly to the element Rx of $F(S) \cap F(T)$, where $R = R_{F(S)\cap F(T)} = J^{-1}\Pi_{J(F(S)\cap F(T))}J$ and $\Pi_{J(F(S)\cap F(T))}$ is the generalized projection of E^* onto $J(F(S) \cap F(T))$. This completes the proof. □

Remark 4.1. *In Theorem 4.2, note that the point $z = \lim_{n\to\infty} S_n x$ is characterlized by the sunny generalized nonexpansive retraction $R = R_{F(S)\cap F(T)} = J^{-1}\Pi_{J(F(S)\cap F(T))}J$ of E onto $F(S) \cap F(T)$. Such a result is still new even if the operator T is linear.*

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