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HALPERN ITERATION FOR A FINITE FAMILY OF QUASINONEXPANSIVE MAPPINGS ON A COMPLETE GEODESIC SPACE WITH CURVATURE BOUNDED ABOVE BY ONE

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Dedicated to Professor Hidetoshi Komiya on his 65th birthday

Abstract. In this paper, we consider the Halpern iteration scheme for a finite family of quasinonexpansive mappings and then prove a strong convergence theorem to their common fixed point in a complete geodesic space with curvature bounded above by one.

1. Introduction

Let us begin with a historical explanation on Halpern schemes. In 1967, Halpern [5] considered an iterative method to find a fixed point of a nonexpansive mapping from the unit ball of a real Hilbert space into itself. In 1992, Wittmann [18] considered the following Halpern type iteration scheme in a real Hilbert space H: Let $C \subset H$ be a closed convex subset, and $u, x_1 \in C$ are given. The iteration scheme is

$$x_{n+1} := \alpha_n u + (1 - \alpha_n)Tx_n$$

for all $n \in \mathbb{N}$, where T is a nonexpansive mapping from C into itself such that the set $F(T)$ of its fixed points is nonempty, and where the real sequence $\{\alpha_n\}$ satisfies $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$. He showed that $\{x_n\}$ converges strongly to a fixed point which is nearest to u in $F(T)$.

In 1997, Shioji-Takahashi [16] extended Wittmann's result to the case where the Hilbert space H is replaced by a Banach space. In 1998, motivated by results of Ishikawa [6] and Das-Debata [3], Atsushiba-Takahashi [1] considered a variation of Halpern iteration using W-mappings $\{W_n\}$ (see Definition 2.1) in a Banach space: u, x_1 are given and

$$x_{n+1} := \beta_n u + (1 - \beta_n)W_n x_n$$

for all $n \in \mathbb{N}$.

A CAT(0) space is a generalization of Hilbert space in a direction different from that of a Banach space. In 2011, Saejung [14] considered the Halpern iteration using single nonexpansive mapping in a CAT(0) space. In 2011, Phuengrattana-Suantai [13] considered the same iteration scheme using W-mapping in a convex metric space. Remark that a CAT(0) space is a convex metric space, so that their result covers the case of CAT(0) space. In 2013, Kimura-Satô [12] considered the

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Halpern iteration using single strongly quasinonexpansive mapping in a CAT(1) space. Remark that a CAT(1) space is not necessarily a convex metric space.

In this paper, we consider the Halpern iteration with W-mapping generated by a finite family of quasinonexpansive mappings in a CAT(1) space, that is, we showed the following theorem under the similar condition in the result of Kimura-Satô:

Theorem 1.1. *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $T_1, T_2, \dots, T_r$ be a finite number of quasinonexpansive and Δ-demiclosed mappings of X into itself such that $F := \bigcap_{i=1}^{r} F(T_i) \neq \emptyset$, and let $\alpha_{n,1}, a_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$, where $0 < a < 1/2$. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. For a given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n) W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v, v') < \pi/2$;

(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_1, P_F u) < \pi/2$;

(c) $\sum_{n=1}^{\infty} \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_F u$.

The proof will be given in §3.

In §4, we give some applications of the main theorem. In Theorem 4.1, we give an approximation of a minimizer of convex functions on a complete CAT(1) space. A further application will be given in Theorem 4.2. We also give an example of quasinonexpansive mappings which is not strongly quasinonexpansive in Example 4.1.

2. Preliminaries

Let (X, d) be a metric space. For $x, y \in X$, a mapping $c : [0, l] \to X$ is a geodesic of $x, y \in X$ if $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, l]$. For $r > 0$, if a geodesic exists for every $x, y \in X$ with $d(x, y) < r$, then X is called an r-geodesic metric space. If a geodesic is unique for every $x, y \in X$, we define $[x, y] := c([0, l])$ and it is called a geodesic segment of $x, y \in X$. In what follows, a metric space X is always assumed to be π-geodesic and every geodesic is unique. For $x, y \in X$, let $c : [0, l] \to X$ be a geodesic of $x, y \in X$. For $t \in [0, 1]$, we denote

$$tx \oplus (1 - t)y := c((1 - t)l).$$

In other words, $z := tx \oplus (1 - t)y$ satisfies $d(x, z) = (1 - t)d(x, y)$. Let X be a geodesic metric space. A geodesic triangle is defined by the union of segment $\triangle(x, y, z) := [x, y] \cup [y, z] \cup [z, x]$. Let $\mathbb{S}^2$ be the unit sphere of the Euclidean space $\mathbb{R}^3$ and $d_{\mathbb{S}^2}$ is the spherical metric on $\mathbb{S}^2$. Then, for $x, y, z \in X$ satisfying $d(x, y) + d(y, z) + d(z, x) < 2\pi$, there exist $\overline{x}, \overline{y}, \overline{z} \in \mathbb{S}^2$ such that $d(x, y) = d_{\mathbb{S}^2}(\overline{x}, \overline{y}), d(y, z) =$

$d_{\mathbb{S}^2}(\overline{y}, \overline{z})$ and $d(z, x) = d_{\mathbb{S}^2}(\overline{z}, \overline{x})$. A point $\overline{p} \in [\overline{x}, \overline{y}]$ is called a comparison point for $p \in [x, y]$ if $d_{\mathbb{S}^2}(\overline{x}, \overline{p}) = d(x, p)$. If every p, q on the triangle $\triangle(x, y, z)$ with $d(x, y) + d(y, z) + d(z, x) < 2\pi$ and their comparison points $\overline{p}, \overline{q} \in \triangle(\overline{x}, \overline{y}, \overline{z})$ satisfy that

$$d(p, q) \leq d_{\mathbb{S}^2}(\overline{p}, \overline{q}),$$

X is called a CAT(1) space. We refer details and examples of a CAT(1) space to [2].

Theorem 2.1 (Kimura-Satô [11])**.** *Let x, y, z be points in* CAT(1) *space such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$. Let $v := tx \oplus (1 - t)y$ for some $t \in [0, 1]$. Then*

$$\cos d(v, z) \sin d(x, y) \geq \cos d(x, z) \sin(td(x, y)) + \cos d(y, z) \sin((1 - t)d(x, y)).$$

Corollary 2.1 (Kimura-Satô [12])**.** *Let x, y, z be points in* CAT(1) *space such that $d(x, y) + d(y, z) + d(z, x) < 2\pi$. Let $v := tx \oplus (1 - t)y$ for some $t \in [0, 1]$. Then*

$$\cos d(v, z) \geq t \cos d(x, z) + (1 - t) \cos d(y, z).$$

Let X be a complete CAT(1) space such that $d(v, v') < \pi/2$ for all $v, v' \in X$, and let C be a nonempty closed convex subset of X. Then for any $x \in X$, there exists a unique point $P_C x \in C$ such that

$$d(x, P_C x) = \inf_{y \in C} d(x, y).$$

That is, using similar techniques to the case of Hilbert space, we can define metric projection P_C from X onto C such that $P_C x$ is the nearest point of C to x. Let X be a metric space and $\{x_n\}$ a bounded sequence of X. The asymptotic center $AC(\{x_n\})$ of $\{x_n\}$ is defined by

$$AC(\{x_n\}) := \left\{ z \mid \limsup_{n \to \infty} d(z, x_n) = \inf_{x \in X} \limsup_{n \to \infty} d(x, x_n) \right\}.$$

We say that $\{x_n\}$ is Δ-convergent to a point z if for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, its asymptotic center consists only of z, that is, $AC(\{x_{n_i}\}) = \{z\}$. Let X be a metric space. Let T be a mapping of X into itself. Then, T is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. Hereafter we denote

$$F(T) := \{z \mid Tz = z\}$$

the set of fixed points. Then T is said to be quasinonexpansive if $d(Tx, p) \leq d(x, p)$ for all $x \in X$ and $p \in F(T)$. Using similar techniques to the case of Hilbert space, we can prove that $F(T)$ is a closed convex subset of X. T is said to be strongly quasinonexpansive if it is quasinonexpansive, and for every $p \in F(T)$ and every sequence in X satisfying that $\sup_{n \in \mathbb{N}} d(x_n, p) < \pi/2$ and $\lim_{n \to \infty}(\cos d(x_n, p)/\cos d(Tx_n, p)) = 0$, it follows that $\lim_{n \to \infty} d(x_n, Tx_n) = 0$. T is said to be Δ-demiclosed if for any Δ-convergent sequence $\{x_n\}$ in X, its Δ-limit belongs to $F(T)$ whenever $\lim_{n \to \infty} d(Tx_n, x_n) = 0$.

The notation of W-mapping is originally proposed by Takahashi. We use the same notation in the setting of geodesic space as following:

Definition 2.1 (Takahashi [17]). Let X be a geodesic metric space. Let $T_1, T_2, \dots, T_r$ be a finite number of mappings of X into itself and $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 \le \alpha_i \le 1$ for every $i = 1, 2, \dots, r$. Then, we define a mapping W of X into itself as follows:

$$\begin{aligned} U_1 &:= \alpha_1 T_1 \oplus (1-\alpha_1)I, \\ U_2 &:= \alpha_2 T_2 U_1 \oplus (1-\alpha_2)I, \\ &\cdots \\ U_r &:= \alpha_r T_r U_{r-1} \oplus (1-\alpha_r)I, \\ W &:= U_r. \end{aligned}$$

Such a mapping W is called a W-mapping generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$.

The following lemmas are important for our main result.

Lemma 2.1 (Kimura-Satô [12]). *Let T be a quasinonexpansive mapping defined on a* CAT(1) *space. For any real number $\alpha \in [0,1]$, the mapping $\alpha T \oplus (1-\alpha)I$ is quasinonexpansive.*

The proof of Lemma 2.1 is essentially obtained in [12], so we omit the proof.

Lemma 2.2 (Kimura-Satô [12]). *Let T be a nonexpansive mapping on a* CAT(1) *space. For a any real number $\alpha \in (0,1]$, the mapping $\alpha T \oplus (1-\alpha)I$ is Δ-demiclosed.*

Lemma 2.3 (Saejung-Yotkaew [15]). *Let $\{s_n\}, \{t_n\}$ be sequences of real numbers such that $s_n \ge 0$ for every $n \in \mathbb{N}$. Let $\{\gamma_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=0}^{\infty} \gamma_n = \infty$. Suppose that $s_{n+1} \le (1-\gamma_n)s_n + \gamma_n t_n$ for every $n \in \mathbb{N}$. If $\limsup_{j\to\infty} t_{n_j} \le 0$ for every subsequence $\{n_j\}$ of $\mathbb{N}$ satisfying $\liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \ge 0$, then $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.4 (Espínola-Fernández-León [4]). *Let X be a complete* CAT(1) *space, and $\{x_n\}$ be a sequence in X. If there exists $x \in X$ such that $\limsup_{n\to\infty} d(x_n, x) < \pi/2$, then $\{x_n\}$ has a Δ-convergent subsequence.*

Lemma 2.5 (He-Fang-Lopez-Li [7]). *Let X be a complete* CAT(1) *space and $p \in X$. If a sequence $\{x_n\}$ in X satisfies that $\limsup_{n\to\infty} d(x_n, p) < \pi/2$ and that $\{x_n\}$ is Δ-convergent to $x \in X$, then $d(x,p) \le \liminf_{n\to\infty} d(x_n, p)$.*

Lemma 2.6 (Kimura-Satô [12]). *Let X be a* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $\alpha \in [0,1]$ and $u, y, z \in X$. Then*

$$\begin{aligned} &1 - \cos d(\beta u \oplus (1-\beta)y, z) \\ &\le (1-\gamma)(1-\cos d(y,z)) + \gamma\left(1 - \frac{\cos d(u,z)}{\sin d(u,y)\tan(2^{-1}\beta d(u,y)) + \cos d(u,y)}\right), \end{aligned}$$

where

$$\gamma := \begin{cases} 1 - \dfrac{\sin((1-\beta)d(u,y))}{\sin(\beta d(u,y))} & (u \ne y), \\ \beta & (u = y). \end{cases}$$

3. Main result

We begin this section with the following useful lemma.

Lemma 3.1. *If $\delta \in [0, \pi/2]$ satisfies*

$$\sin \delta \geq \sin(\alpha\delta) + \sin((1-\alpha)\delta)$$

for some $\alpha \in (0,1)$, then $\delta = 0$.

Proof. It is obtained by an elementary calculation. □

Next we study the set of fixed points of a W-mapping.

Proposition 3.1. *Let X be a* CAT(1) *space. Let $T_1, T_2, \dots, T_r$ be quasinonexpansive mappings of X into itself such that $\bigcap_{i=1}^r F(T_i) \neq \emptyset$ and let $\alpha_1, \alpha_2, \dots, \alpha_r$ be real numbers such that $0 < \alpha_i < 1$ for every $i = 1, 2, \dots, r$. Let W be the W-mapping of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$. Then, $F(W) = \bigcap_{i=1}^r F(T_i)$.*

Proof. It is obvious that $\bigcap_{i=1}^r F(T_i) \subset F(W)$. So, we shall prove $F(W) \subset \bigcap_{i=1}^r F(T_i)$. Let $z \in F(W)$ and $w \in \bigcap_{i=1}^r F(T_i)$. Then it follows that

$$0 = d(z,z) = d(Wz,z) = d(\alpha_r T_r U_{r-1} z \oplus (1-\alpha_r)z, z) = \alpha_r d(z, T_r U_{r-1} z).$$

Since $0 < \alpha_r \leq 1$, we obtain $z = T_r U_{r-1} z$ and hence

$$\begin{aligned}
\cos d(z,w) &= \cos d(T_r U_{r-1} z, w) \\
&\geq \cos d(U_{r-1} z, w) \\
&= \cos d(\alpha_{r-1} T_{r-1} U_{r-2} z \oplus (1-\alpha_{r-1}) z, w) \\
&\geq \alpha_{r-1} \cos d(T_{r-1} U_{r-2} z, w) + (1-\alpha_{r-1}) \cos d(z,w) \\
&\geq \alpha_{r-1} \cos d(U_{r-2} z, w) + (1-\alpha_{r-1}) \cos d(z,w) \\
&\geq \alpha_{r-1} \cos d(\alpha_{r-2} T_{r-2} U_{r-3} z \oplus (1-\alpha_{r-2}) z, w) \\
&\quad + (1-\alpha_{r-1}) \cos d(z,w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cos d(T_{r-2} U_{r-3} z, w) + (1-\alpha_{r-1}\alpha_{r-2}) \cos d(z,w) \\
&\geq \cdots \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \cos d(T_2 U_1 z, w) \\
&\quad + (1-\alpha_{r-1}\alpha_{r-2}\cdots\alpha_2) \cos d(z,w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \cos d(U_1 z, w) \\
&\quad + (1-\alpha_{r-1}\alpha_{r-2}\cdots\alpha_2) \cos d(z,w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \cos d(\alpha_1 T_1 z \oplus (1-\alpha_1) z, w) \\
&\quad + (1-\alpha_{r-1}\alpha_{r-2}\cdots\alpha_2) \cos d(z,w) \\
&\geq \alpha_{r-1}\alpha_{r-2} \cdots \alpha_2 \alpha_1 \cos d(T_1 z, w) \\
&\quad + (1-\alpha_{r-1}\alpha_{r-2}\cdots\alpha_2\alpha_1) \cos d(z,w) \\
&\geq \cos d(z,w).
\end{aligned}$$

Then it follows that

$$d(z,w) = \cos d(T_1z,w) = d(U_1z,w) = d(\alpha_1 T_1 z \oplus (1-\alpha_1)z, w).$$

By Theorem 2.1 and Lemma 3.1 with

$$\begin{aligned}&\cos d(\alpha_1 T_1 z \oplus (1-\alpha_1)z, w)\sin d(T_1z,z)\\ &\geq \cos d(T_1z,w)\sin(\alpha_1 d(T_1z,z)) + \cos d(z,w)\sin((1-\alpha_1)d(T_1z,z)),\end{aligned}$$

we obtain $T_1z = z$. Similarly, we have

$$d(z,w) = d(T_2U_1z,w) = d(U_2z,w) = d(\alpha_2 T_2 U_1 z \oplus (1-\alpha_2)z, w).$$

By Theorem 2.1 and Lemma 3.1 with

$$\begin{aligned}&\cos d(\alpha_2 T_2 U_1 z \oplus (1-\alpha_2)z, w)\sin d(T_2U_1z,z)\\ &\geq \cos d(T_2U_1z,w)\sin(\alpha_2 d(T_2U_1z,z)) + \cos d(z,w)\sin((1-\alpha_2)d(T_2U_1z,z)),\end{aligned}$$

we obtain $T_2U_1z = z$. Since $U_1z = z$, we obtain $T_2z = z$. Using such techniques, we obtain $T_iz = z$ and $U_iz = z$ for all $i = 1, 2, \ldots, r$, and hence $z \in \bigcap_{i=1}^r F(T_i)$. This implies $F(W) \subset \bigcap_{i=1}^r F(T_i)$. Therefore we have $F(W) = \bigcap_{i=1}^r F(T_i)$. $\square$

Remark 3.1. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \ldots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$ for $n \in \mathbb{N}$. By Proposition 3.1, all the sets of fixed points $\{F(W_n)\}$ is identical.

The following Lemma 3.2 is essentially given by Kasahara [9]. For the sake of completeness, we give the proof.

Lemma 3.2 (Kasahara [9])**.** *Let $\{S_n\}$ be a sequence of quasinonexpansive mappings of a* CAT(1) *space X into itself such that $\bigcap_{n=1}^\infty F(S_n) \neq \emptyset$. Then for given real numbers $\alpha_n \in [a, 1-a] \subset (0,1)$ and $p \in \bigcap_{n=1}^\infty F(S_n)$, if $\{x_n\}$ satisfies that $\sup_{n\in\mathbb{N}} d(x_n, p) < \pi/2$ and*

$$\lim_{n\to\infty} \frac{\cos d(x_n,p)}{\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)} = 1,$$

then $\lim_{n\to\infty} d(S_nx_n, x_n) = 0$.

Proof. Let $\delta_n := d(S_nx_n, x_n)$. Assume that $\{x_n\} \subset X$ and $p \in \bigcap_{n=1}^\infty F(S_n)$ such that $\sup_{n\in\mathbb{N}} d(x_n,p) < \pi/2$ and $\lim_{n\to\infty}(\cos d(x_n,p)/\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)) = 1$, by Theorem 2.1, we have

$$\begin{aligned}&\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)\sin d(S_nx_n, x_n)\\ &\geq \cos d(S_nx_n,p)\sin(\alpha d(S_nx_n,x_n)) + \cos d(x_n,p)\sin((1-\alpha_n)d(S_nx_n,x_n))\\ &\geq \min\{\cos d(S_nx_n,p), \cos d(x_n,p)\}(\sin(\alpha_n d(S_nx_n,x_n)) + \sin((1-\alpha_n)d(S_nx_n,x_n)))\\ &= 2\cos d(x_n,p)\sin\frac{d(S_nx_n,x_n)}{2}\cos\frac{(2\alpha_n-1)d(S_nx_n,x_n)}{2}.\end{aligned}$$

Hence

$$\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)\sin\delta_n \geq 2\cos d(x_n,p)\sin\frac{\delta_n}{2}\cos\frac{(2\alpha_n-1)\delta_n}{2}.$$

We assume that $\delta_n \neq 0$. Dividing above by $2\sin(\delta_n/2)$, we have

$$\begin{aligned}\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)\cos\frac{\delta_n}{2} &\geq \cos d(x_n,p)\cos\frac{(2\alpha_n-1)\delta_n}{2}\\ &\geq \cos d(x_n,p)\cos\frac{(1-2a)\delta_n}{2}.\end{aligned}$$

Moreover, dividing above by $\cos((1-2a)\delta_n/2)$, we have

$$\cos d(x_n,p) \leq \cos d(\alpha S_n x_n \oplus (1-\alpha_n)x_n, p)\frac{\cos\dfrac{\delta_n}{2}}{\cos\dfrac{(1-2a)\delta_n}{2}}.$$

Then

$$\begin{aligned}&\cos d(x_n,p)\\ &\leq \cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)\frac{\cos\dfrac{(1-2a)\delta_n}{2}\cos(a\delta_n) - \sin\dfrac{(1-2a)\delta_n}{2}\sin(a\delta_n)}{\cos\dfrac{(1-2a)\delta_n}{2}}\\ &\leq \cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)\cos(a\delta_n).\end{aligned}$$

Thus we have that

$$\cos d(a\delta_n) \geq \frac{\cos d(x_n,p)}{\cos d(\alpha_n S_n x_n \oplus (1-\alpha_n)x_n, p)} \to 1 \ (n\to\infty),$$

which implies $\lim_{n\to\infty}\delta_n = 0$, that is, $\lim_{n\to\infty} d(S_n x_n, x_n) = 0$. $\square$

Theorem 3.1. *Let X be a complete* CAT(1) *space such that $d(v,v') < \pi/2$ for every $v, v' \in X$. Let $T_1, T_2, \dots, T_r$ be a finite number of quasinonexpansive and Δ-demiclosed mappings of X into itself such that $F := \bigcap_{i=1}^r F(T_i) \neq \emptyset$, and let $\alpha_{n,1}, a_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$, where $0 < a < 1/2$. Let W_n be the W-mappings of X into itself generated by $T_1, T_2, \dots, T_r$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^\infty \beta_n = \infty$. For a given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1-\beta_n)W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v'\in X} d(v,v') < \pi/2$;
(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_1, P_F u) < \pi/2$;
(c) $\sum_{n=1}^\infty \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_F u$.

Proof. Let $p := P_F u$ and let

$$\begin{aligned}s_n &:= 1 - \cos d(x_n, p),\\ t_n &:= 1 - \frac{\cos d(u,p)}{\sin d(u, W_n x_n)\tan(2^{-1}\beta_n d(u, W_n x_n)) + \cos d(u, W_n x_n)},\end{aligned}$$

$$\gamma_n := \begin{cases} 1 - \dfrac{\sin((1-\beta_n)d(u, W_n x_n))}{\sin(\beta_n d(u, W_n x_n))} & (u \neq W_n x_n), \\ \beta_n & (u = W_n x_n) \end{cases}$$

for $n \in \mathbb{N}$. If $\{s_n\}, \{t_n\}$ and $\{\gamma_n\}$ satisfy the conditions of Lemma 2.3, then we will have $\lim_{n\to\infty} s_n = 0$, that is, $\{x_n\}$ converges to $p = P_F u$. Thus the proof of Theorem 3.1 will be completed. First, it is obvious that $s_n \geq 0$. By Lemma 2.1, W_n is quasinonexpansive. Then, it follows from Lemma 2.6 that

$$s_{n+1} \leq (1-\gamma_n)(1 - \cos d(W_n x_n, p)) + \gamma_n t_n \leq (1-\gamma_n)s_n + \gamma_n t_n$$

for every $n \in \mathbb{N}$. Now, it is also obvious that $\{\gamma_n\}$ is a sequence in $(0,1)$. we show that $\sum_{n=1}^{\infty} \gamma_n = \infty$ holds under each condition (a),(b) and (c). We have

$$\begin{aligned} \cos d(x_{n+1}, p) &= \cos d(\beta_n u \oplus (1-\beta_n) W_n x_n, p) \\ &\geq \beta_n \cos d(u,p) + (1-\beta_n)\cos d(W_n x_n, p) \\ &\geq \beta_n \cos d(u,p) + (1-\beta_n)\cos d(x_n, p) \\ &\geq \min\{\cos d(u,p), \cos d(x_n, p)\} \end{aligned}$$

for all $n \in \mathbb{N}$. Thus we have

$$\begin{aligned} \cos d(x_n, p) &\geq \min\{\cos d(u,p), \cos d(x_1, p)\} \\ &= \cos \max\{d(u,p), d(x_1,p)\} \\ &> 0 \end{aligned}$$

for all $n \in \mathbb{N}$ and hence $\sup_{n\in\mathbb{N}} d(x_n, p) \leq \max\{d(u,p), d(x_1,p)\} < \pi/2$. For the case of (a) and (b), let $M = \sup_{n\in\mathbb{N}} d(u, W_n x_n)$. Then we show that $M < \pi/2$. For (a), it is trivial. For (b), since $\sup_{n\in\mathbb{N}} d(x_n,p) \leq \max\{d(u,p), d(x_1,p)\}$, we have

$$\begin{aligned} M &= \sup_{n\in\mathbb{N}} d(u, W_n x_n) \\ &\leq \sup_{n\in\mathbb{N}} (d(u,p) + d(p, W_n x_n)) \\ &\leq \sup_{n\in\mathbb{N}} (d(u,p) + d(p, x_n)) \\ &\leq \max\{2d(u,p), d(u,p) + d(x_1,p)\} \\ &< \frac{\pi}{2}. \end{aligned}$$

Thus, in each case of (a) and (b), we have

$$\begin{aligned} \gamma_n &\geq 1 - \frac{\sin((1-\beta_n)M)}{\sin M} \\ &= \frac{2}{\sin M} \sin\left(\frac{\beta_n}{2} M\right) \cos\left(\left(1 - \frac{\beta_n}{2}\right) M\right) \\ &\geq \beta_n \cos M. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \beta_n = \infty$, it follows that $\sum_{n=1}^{\infty} \gamma_n = \infty$. For the case of (c), we have

$$\gamma_n \geq 1 - \sin\frac{(1-\beta_n)\pi}{2} = 1 - \cos\frac{\beta_n \pi}{2} \geq \frac{\beta_n^2 \pi^2}{16}$$

for every $n \in \mathbb{N}$. Therefore, in the case of (c) we also have $\sum_{n=1}^{\infty} \gamma_n = \infty$. Finally, we show that $\limsup_{j\to\infty} t_{n_j} \leq 0$ for any subsequence $\{n_j\}$ of $\mathbb{N}$ with $\liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \geq 0$. Let $\{s_{n_j}\}$ be a subsequence of $\{s_n\}$ satisfying that $\liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \geq 0$, and put

$$\alpha := \min_{k=1,\dots,r} \left(\inf_{n\in\mathbb{N}} \alpha_{n,k} \right).$$

Then we have

$$\begin{aligned}
0 &\leq \liminf_{j\to\infty}(s_{n_j+1} - s_{n_j}) \\
&= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(x_{n_j+1}, p)) \\
&= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\beta_{n_j} u \oplus (1-\beta_{n_j}) W_{n_j} x_{n_j}, p)) \\
&\leq \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - (\beta_{n_j} \cos d(u, p) \\
&\qquad\qquad + (1-\beta_{n_j}) \cos d(W_{n_j} x_{n_j}, p)) \\
&= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(W_{n_j} x_{n_j}, p)) \\
&= \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,r} T_r U_{n_j,r-1} x_{n_j} \oplus (1-\alpha_{n_j,r}) x_{n_j}, p)) \\
&\leq \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - (\alpha_{n_j,r} \cos d(T_r U_{n_j,r-1} x_{n_j}, p) \\
&\qquad\qquad + (1-\alpha_{n_j,r}) \cos d(x_{n_j}, p))) \\
&= \liminf_{j\to\infty}(\alpha_{n_j,r} \cos d(x_{n_j}, p) - \alpha_{n_j,r} \cos d(T_r U_{n_j,r-1} x_{n_j}, p)) \\
&\leq \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(T_r U_{n_j,r-1} x_{n_j}, p)) \\
&\leq \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(U_{n_j,r-1} x_{n_j}, p)) \\
&= \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) \\
&\qquad\qquad - \cos d(\alpha_{n_j,r-1} T_{r-1} U_{n_j,r-2} x_{n_j} \oplus (1-\alpha_{n_j,r-1}) x_{n_j}, p)) \\
&\leq \alpha \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - (\alpha_{n_j,r-1} \cos d(T_{r-1} U_{n_j,r-2} x_{n_j}, p) \\
&\qquad\qquad + (1-\alpha_{n_j,r-1}) \cos d(x_{n_j}, p))) \\
&= \alpha \liminf_{j\to\infty}(\alpha_{n_j,r-1} \cos d(x_{n_j}, p) - \alpha_{n_j,r-1} \cos d(T_{r-1} U_{n_j,r-2} x_{n_j}, p)) \\
&\leq \alpha^2 \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(T_{r-1} U_{n_j,r-2} x_{n_j}, p)) \\
&\leq \cdots \\
&\leq \alpha^{r-1} \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(T_2 U_{n_j,1} x_{n_j}, p)) \\
&\leq \alpha^{r-1} \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(U_{n_j,1} x_{n_j}, p)) \\
&= \alpha^{r-1} \liminf_{j\to\infty}(\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p))
\end{aligned}$$

$$\leq \alpha^{r-1} \limsup_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p))$$
$$\leq 0.$$

Thus we have

$$\lim_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p)) = 0.$$

Using the inequality $\sup_{j\in\mathbb{N}} d(x_{n_j}, p) < \pi/2$, we also have

$$\lim_{j\to\infty} \frac{\cos d(x_{n_j}, p)}{\cos d(\alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}, p)} = 1.$$

By Lemma 3.2, it follows that

$$\lim_{j\to\infty} d(T_1 x_{n_j}, x_{n_j}) = 0.$$

Put

$$y_j^{(k)} := U_{n_j,k} x_{n_j}$$

for $k = 1, 2, \ldots, r-1$. We show that

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(k)}) = 0, \quad \lim_{j\to\infty} d(T_{k+1} y_j^{(k)}, y_j^{(k)}) = 0$$

by induction on $k = 1, 2, \ldots, r-1$. First, we consider the case $k = 1$. We have

$$\begin{aligned}
\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) &= \lim_{j\to\infty} d(x_{n_j}, U_{n_j,1} x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,1} T_1 x_{n_j} \oplus (1-\alpha_{n_j,1}) x_{n_j}) \\
&= \lim_{j\to\infty} \alpha_{n_j} d(T_1 x_{n_j}, x_{n_j}) \\
&= 0.
\end{aligned}$$

On the other hand, by the calculation above we have

$$\begin{aligned}
0 &\leq \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(U_{n_j,2} x_{n_j}, p)) \\
&= \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)) \\
&\leq \limsup_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)) \\
&\leq 0.
\end{aligned}$$

Therefore

$$\lim_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)) = 0.$$

Using the inequality $\sup_{j\in\mathbb{N}} d(x_{n_j}, p) < \pi/2$, we also have

$$\lim_{j\to\infty} \frac{\cos d(x_{n_j}, p)}{\cos d(\alpha_{n_j,2} T_2 U_{n_j,1} x_{n_j} \oplus (1-\alpha_{n_j,2}) x_{n_j}, p)} = 1.$$

By Lemma 3.2, and since $\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) = 0$,

$$\lim_{j\to\infty} d(T_2 y_j^{(1)}, y_j^{(1)}) \leq \lim_{j\to\infty} (d(T_2 y_j^{(1)}, x_{n_j}) + d(x_{n_j}, y_j^{(1)})) = 0.$$

Hence we have that case $k = 1$, that is,

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) = 0, \quad \lim_{j\to\infty} d(T_2 y_j^{(1)}, y_j^{(1)}) = 0.$$

holds. Next, assume the hypothesis with $k = l$, that is,

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(l)}) = 0, \quad \lim_{j\to\infty} d(T_{l+1} y_j^{(l)}, y_j^{(l)}) = 0$$

holds. Then by assumption, we have

$$\begin{aligned}
\lim_{j\to\infty} d(x_{n_j}, y_j^{(l+1)}) &= \lim_{j\to\infty} d(x_{n_j}, U_{n_j,l+1} x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,l+1} T_{l+1} U_{n_j,l} x_{n_j} \oplus (1-\alpha_{n_j,l+1}) x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,l+1} T_{l+1} y_j^{(l)} \oplus (1-\alpha_{n_j,l+1}) x_{n_j}) \\
&= \lim_{j\to\infty} d(x_{n_j}, \alpha_{n_j,l+1} y_j^{(l)} \oplus (1-\alpha_{n_j,l+1}) x_{n_j}) \\
&= \lim_{j\to\infty} \alpha_{n_j,l+1} d(x_{n_j}, y_j^{(l)}) \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
0 &\le \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(U_{n_j,l+2} x_{n_j}, p)) \\
&= \liminf_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1-\alpha_{n_j,l+1}) x_{n_j}, p)) \\
&= \limsup_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1-\alpha_{n_j,l+1}) x_{n_j}, p)) \\
&\le 0.
\end{aligned}$$

Therefore

$$\lim_{j\to\infty} (\cos d(x_{n_j}, p) - \cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1-\alpha_{n_j,l+1}) x_{n_j}, p)) = 0.$$

Using inequality $\sup_{j\in\mathbb{N}} d(x_{n_j} p) < \pi/2$, we have

$$\lim_{j\to\infty} \frac{\cos d(x_{n_j}, p)}{\cos d(\alpha_{n_j,l+2} T_{l+2} U_{n_j,l+1} x_{n_j} \oplus (1-\alpha_{n_j,l+1}) x_{n_j}, p)} = 1.$$

Since $\lim_{j\to\infty} d(x_{n_j}, y_j^{(l+1)}) = 0$ and by Lemma 3.2, we have

$$\begin{aligned}
\lim_{j\to\infty} d(T_{l+2} y_j^{(l+1)}, y_j^{(l+1)}) &= \lim_{j\to\infty} d(T_{l+2} y_j^{(l+1)}, x_{n_j}) \\
&= \lim_{j\to} d(T_{l+2} U_{n_j,l+1} x_{n_j}, x_{n_j}) = 0.
\end{aligned}$$

So, we have the hypothesis $k = l + 1$, that is,

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(l+1)}) = 0, \quad \lim_{j\to\infty} d(T_{l+2} y_j^{(l+1)}, y_j^{(l+1)}) = 0$$

for $k = 1, 2, \dots, r-1$. By induction, we obtain

$$\lim_{j\to\infty} d(x_{n_j}, y_j^{(k)}) = 0, \quad \lim_{j\to\infty} d(T_{k+1} y_j^{(k)}, y_j^{(k)}) = 0$$

for all $k = 1, 2, \dots, r-1$. By Lemma 2.4, let $\{x_{n_{j_k}}\}$ be a Δ-convergent subsequence of $\{x_{n_j}\}$ with the Δ-limit z such that $\lim_{k\to\infty} d(u, x_{n_{j_k}}) = \liminf_{j\to\infty} d(u, x_{n_j})$. Then, since T_1 is Δ-demiclosed and $\lim_{j\to\infty} d(x_{n_j}, T_1 x_{n_j}) = 0$, the Δ-limit z of $\{x_{n_{j_k}}\}$ belongs to $F(T_1)$. Similarly, since T_2 is Δ-demiclosed and $\lim_{j\to\infty} d(x_{n_j}, y_j^{(1)}) = \lim_{j\to\infty} d(y_j^{(1)}, T_2 y_j^{(1)}) = 0$, $\{y_{j_k}^{(1)}\}$ is Δ-convergent to z and the Δ-limit z is belongs to $F(T_2)$. Using such techniques, we obtain $z \in F(T_i)$ for all $i = 1, 2, \dots r$, and hence $z \in \bigcap_{i=1}^{r} F(T_i) = F$. Using Lemma 2.5 and the definition of the metric projection, we have

$$\begin{aligned}
\liminf_{j\to\infty} d(u, W_{n_j} x_{n_j}) &= \liminf_{j\to\infty} d(u, \alpha_{n_j,r} T_r U_{n_j,r-1} x_{n_j} \oplus (1-\alpha_{n_j}) x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, \alpha_{n_j,r} T_r y_j^{(r-1)} \oplus (1-\alpha_{n_j}) x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, \alpha_{n_j,r} y_j^{(r-1)} \oplus (1-\alpha_{n_j}) x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, \alpha_{n_j,r} x_{n_j} \oplus (1-\alpha_{n_j}) x_{n_j}) \\
&= \liminf_{j\to\infty} d(u, x_{n_j}) \\
&= \lim_{k\to\infty} d(u, x_{n_{j_k}}) \\
&\geq d(u, z) \\
&\geq d(u, P_F u).
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&\limsup_{j\to\infty} t_{n_j} \\
&= \limsup_{j\to\infty} \left(1 - \frac{\cos d(u,p)}{\sin d(u, W_{n_j} x_{n_j}) \tan(2^{-1}\beta_{n_j} d(u, W_{n_j} x_{n_j})) + \cos d(u, W_{n_j} x_{n_j})}\right) \\
&= \limsup_{j\to\infty} \left(1 - \frac{\cos d(u,p)}{0 + \cos d(u, W_{n_j} x_{n_j})}\right) \\
&= 1 - \frac{\cos d(u,p)}{\cos(\liminf_{j\to\infty} d(u, W_{n_j} x_{n_j}))} \\
&\leq 1 - \frac{\cos d(u,p)}{\cos d(u,z)} \\
&\leq 0.
\end{aligned}$$

By Lemma 2.3, we have that $\lim_{n\to\infty} s_n = 0$, that is, $\{x_n\}$ converges to $p = P_F u$, and we finish the proof. □

Remark 3.2. By Lemma 2.2, a nonexpansive mapping defined on a CAT(1) space having a fixed point is quasinonexpansive and Δ-demiclosed.

Remark 3.3. *In general, if $T_1, T_2, \dots, T_r$ are nonexpansive, then W-mapping generated by $T_1, T_2, \dots, T_r$ and $\alpha_1, \alpha_2, \dots, \alpha_r$ is not necessarily nonexpansive.*

4. Applications

Let us recall some basic notation about functions on metric space. Let X be a geodesic metric space and let f be a function from X into $(-\infty, \infty]$. We say f is lower semicontinuous if the set $\{x \in X \mid f(x) \leq a\}$ is closed for all $a \in \mathbb{R}$. The function f is said to be proper if the set $\{x \in X \mid f(x) \neq \infty\}$ is nonempty. We say f is convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in X$ and $t \in (0,1)$. Let X be a complete CAT(1) space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let f be a proper lower semicontinuous convex function from X into $(-\infty, \infty]$. A resolvent of f is defined by

$$R_f x := \operatorname*{argmin}_{y \in X} \{f(y) + \tan d(y,x) \sin d(y,x)\} \tag{4.1}$$

in [10]. Another type of the resolvent of f is defined by

$$R_f x := \operatorname*{argmin}_{y \in X} \{f(y) - \log \cos d(y,x)\} \tag{4.2}$$

in [8]. Both resolvents are quasinonexpansive, Δ-demiclosed, and satisfy $F(R_f) = \operatorname{argmin}_X f$ ([10, 8]). So, we can approximate a common minimizer of a finite number of functions by the following theorem.

Theorem 4.1. *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $f_1, f_2, \ldots, f_r$ be a finite number of convex function from X into $(-\infty, \infty]$ such that $F := \bigcap_{i=1}^r \operatorname{argmin}_X f_i \neq \emptyset$, and let $\alpha_{n,1}, a_{n,2}, \ldots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \ldots, r$, where $0 < a < 1/2$. Let R_{f_i} be a resolvent defined by either* (4.1) *or* (4.2) *for $i = 1, 2, \ldots, r$. Let W_n be the W-mappings of X into itself generated by $R_{f_1}, R_{f_2}, \ldots, R_{f_r}$ and $\alpha_{n,1}, \alpha_{n,2}, \ldots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty} \beta_n = 0$, and $\sum_{n=1}^\infty \beta_n = \infty$. For given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1-\beta_n) W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v, v') < \pi/2$;
(b) $d(u, P_F u) < \pi/4$ *and* $d(u, P_F u) + d(x_1, P_F u) < \pi/2$;
(c) $\sum_{n=1}^\infty \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_F u$.

Let us consider a more specialized situation. For a closed convex subset C of a complete CAT(1) space X, put

$$i_C(x) := \begin{cases} 0 & (x \in C) \\ \infty & (x \notin C). \end{cases}$$

This function i_C is a proper lower semicontinuous convex function. Thus the resolvent R_{i_C} of i_C is defined by either (4.1) or (4.2), and it is quasinonexpansive and Δ-demiclosed. In fact, we know $R_{i_C} = P_C$ and $F(R_{i_C}) = \operatorname{argmin} i_C = C$ for both definitions (4.1) and (4.2). Thus we can apply Theorem 3.1 and have an approximation of the nearest point in the intersection of finite family of closed convex subsets from a given point by using corresponding metric projection of each subset by the following theorem.

Theorem 4.2. *Let X be a complete* CAT(1) *space such that $d(v, v') < \pi/2$ for every $v, v' \in X$. Let $C_1, C_2, \dots, C_r$ be a finite number of closed convex subset of X such that $C := \bigcap_{i=1}^{r} C_i \neq \emptyset$, and let $\alpha_{n,1}, a_{n,2}, \dots, \alpha_{n,r}$ be real numbers for $n \in \mathbb{N}$ such that $\alpha_{n,i} \in [a, 1-a]$ for every $i = 1, 2, \dots, r$, where $0 < a < 1/2$. Let W_n be the W-mappings of X into itself generated by $P_{C_1}, P_{C_2}, \dots, P_{C_r}$ and $\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,r}$ for $n \in \mathbb{N}$. Let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \beta_n < 1$ for every $n \in \mathbb{N}, \lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. For a given points $u, x_1 \in X$, let $\{x_n\}$ be a sequence in X generated by*

$$x_{n+1} = \beta_n u \oplus (1 - \beta_n) W_n x_n$$

for $n \in \mathbb{N}$. Suppose that one of the following conditions holds:

(a) $\sup_{v,v' \in X} d(v, v') < \pi/2$;
(b) $d(u, P_C u) < \pi/4$ *and* $d(u, P_C u) + d(x_1, P_C u) < \pi/2$;
(c) $\sum_{n=1}^{\infty} \beta_n^2 = \infty$.

Then $\{x_n\}$ converges to $P_C u$.

In the introduction we mention that there exists an example which is quasinonexpansive but not strongly quasinonexpansive. The following is such an example.

Example 4.1. A closed interval $[-1, 1]$ is a complete CAT(1) space. Let $T : [-1, 1] \to [-1, 1]$ be defined by $Tx := -x$. Then $F(T) = \{0\}$. It is easy to obtain that T is quasinonexpansive and Δ-demiclosed but it is not strongly quasinonexpansive.

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