Linear and Nonlinear Analysis
Volume 7, Number 1, 2021, 115–140
Yokohama Publishers
ISSN 2188-8167
Copyright 2021

MEAN CONVERGENCE THEOREMS FOR GENERIC 2-GENERALIZD HYBRID MAPPINGS IN HILBERT SPACES

ATSUMASA KONDO

Dedicated to Professor Hidetoshi Komiya on the occasion of his retirement

Abstract. This paper establishes mean convergence theorems for finding attractive and fixed points of generic 2-generalized hybrid mappings. First, Baillon's type nonlinear ergodic theorems that weakly approximate attractive and fixed points are proved. The theorem is established under more general parameter conditions than the previous results. Second, we demonstrate mean convergence theorems that weakly approximate attractive and fixed points by combining Mann's and Atsushiba and Takahashi's type iterations. Finally, we present mean convergence theorems that strongly approximate attractive and fixed points by combining Halpern's and Atsushiba and Takahashi's type iterations. Our results extend many existing theorems in the literature.

1. Introduction

Throughout this paper, we use H to denote a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. The sets of natural and real numbers are represented by $\mathbb{N}$ and $\mathbb{R}$, respectively. Let T be a mapping from C into H, where C is a nonempty subset of H. The sets of fixed points and *attractive points* [37] of T are denoted by the following:

$$F(T) = \{x \in C : Tx = x\} \text{ and}$$
$$A(T) = \{x \in H : \|Ty - x\| \leq \|y - x\| \text{ for all } y \in C\},$$

respectively. A mapping $T : C \to H$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\| \text{ for all } x, y \in C.$$

Approximation methods for finding fixed points of nonexpansive mappings have been intensively studied by many researchers. In 1998, Atsushiba and Takahashi [3] introduced the following iteration:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \text{ for all } n \in \mathbb{N}, \tag{1.1}$$

where $x_1 \in C$ is given. They demonstrated weak convergence to a common fixed point of S and T, where S and T are nonexpansive commutative mappings. This iteration scheme (1.1) is derived from the ideas of Mann [28], Baillon [4], and Shimizu and Takahashi [30, 31]. Wittmann [41] proved a strong convergence theorem for

2020 *Mathematics Subject Classification.* 47H05, 47H09.

Key words and phrases. Attractive point, fixed point, generic 2-generalized hybrid mapping, mean convergence.

finding a fixed point of a nonexpansive mapping T by using Halpern's type iteration [6]

$$x_{n+1} = \lambda_n x + (1 - \lambda_n) Tx_n \quad \text{for all } n \in \mathbb{N},$$

where $x_1 = x \in C$ is given, and $\{\lambda_n\} \subset [0,1]$.

Successive studies have shown that the class of nonexpansive mappings can be generalized to include many important mappings. In 2010, Kocourek *et al.* [14] defined a wide class of mappings. A mapping $T : C \to H$ is called *generalized hybrid* [14] if $\alpha, \beta \in \mathbb{R}$ exist such that

$$\alpha \|Tx - Ty\|^2 + (1-\alpha) \|x - Ty\|^2 \le \beta \|Tx - y\|^2 + (1-\beta) \|x - y\|^2$$

for all $x, y \in C$. A class of generalized hybrid mappings contains nonexpansive mappings as a special case, where $\alpha = 1$ and $\beta = 0$. Similarly, this class of mappings includes *nonspreading mappings* [15, 16], *hybrid mappings* [36], and λ-*hybrid mappings* [1]. For generalized hybrid mappings, various types of convergence theorems for finding fixed and attractive points were established (see, e.g., Kocourek *et al.* [14], Takahashi and Takeuchi [37], Hasegawa *et al.* [7], Takahashi *et al.* [40], and Hojo and Takahashi [9].

In 2011, Maruyama *et al.* [29] further extended the class of generalized hybrid mappings. A mapping $T : C \to C$ is called a 2-*generalized hybrid mapping* if $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ exist such that

$$\begin{aligned} \alpha_1 \left\|T^2x - Ty\right\|^2 + \alpha_2 \|Tx - Ty\|^2 + (1 - \alpha_1 - \alpha_2) \|x - Ty\|^2 \\ \le \beta_1 \left\|T^2x - y\right\|^2 + \beta_2 \|Tx - y\|^2 + (1 - \beta_1 - \beta_2) \|x - y\|^2 \end{aligned}$$

for all $x, y \in C$. A 2-generalized hybrid mapping with $\alpha_1 = \beta_1 = 0$ is generalized hybrid. Kondo and Takahashi [20] introduced the following class of nonlinear mappings. A mapping $T : C \to C$ is called *normally* 2-*generalized hybrid* if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2} (\alpha_n + \beta_n) \ge 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\begin{aligned} \alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \le 0 \end{aligned}$$

for all $x, y \in C$. This class of mappings contains 2-generalized hybrid mappings, in addition to *normally generalized hybrid mappings* [39]. Hojo *et al.* [11] and Kondo [18, 19] presented examples of 2-generalized hybrid mappings and normally 2-generalized hybrid mappings that are not generalized hybrid or continuous.

In 2019, Kondo and Takahashi [22] introduced a wide class of mappings. A mapping $T : C \to C$ is called *generic* 2-*generalized hybrid* if there exist $\alpha_{ij}, \beta_i, \gamma_i \in$

$\mathbb{R}$ $(i, j = 0, 1, 2)$ such that

$$\begin{aligned} &\alpha_{00}\|x-y\|^2+\alpha_{01}\|x-Ty\|^2+\alpha_{02}\|x-T^2y\|^2\\ &\quad+\alpha_{10}\|Tx-y\|^2+\alpha_{11}\|Tx-Ty\|^2+\alpha_{12}\left\|Tx-T^2y\right\|^2\\ &\quad\quad+\alpha_{20}\left\|T^2x-y\right\|^2+\alpha_{21}\left\|T^2x-Ty\right\|^2+\alpha_{22}\left\|T^2x-T^2y\right\|^2\\ &\quad\quad\quad+\beta_0\|x-Tx\|^2+\beta_1\left\|Tx-T^2x\right\|^2+\beta_2\left\|T^2x-x\right\|^2\\ &\quad\quad\quad\quad+\gamma_0\|y-Ty\|^2+\gamma_1\left\|Ty-T^2y\right\|^2+\gamma_2\left\|T^2y-y\right\|^2\leq 0 \end{aligned} \tag{1.2}$$

for all $x, y \in C$ with some parameter conditions so that it contains normally 2-generalized hybrid mappings. We also call such a mapping an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-*generic* 2-*generalized hybrid mapping*. Kondo and Takahashi [22] proved an ergodic theorem for that class of mappings whereas they addressed Mann's type and Halpern's type convergence theorems in another paper [24]. Although their results uniformly extend the previous results in the literature, a mean convergence theorem based on the iteration (1.1) has not yet been proved for generic 2-generalized hybrid mappings.

This paper proves the mean convergence theorems for finding attractive and fixed points of generic 2-generalized hybrid mappings. First, Baillon's type nonlinear ergodic theorems that weakly approximate attractive and fixed points are demonstrated. The theorem is established under more general parameter conditions than the previous result in [22]. Second, we demonstrate the mean convergence theorems that weakly approximate attractive and fixed points by combining Mann's and Atsushiba and Takahashi's type iterations. Finally, we show that mean convergence theorems strongly approximate attractive and fixed points by combining Halpern's and Atsushiba and Takahashi's type iterations. Our results extend many existing theorems in the literature.

2. Preliminaries and lemmas

This section presents basic information and results. A systematic explanation is found in work by Takahashi [34, 35]. In a real Hilbert space H, it is known that

$$2\langle x-y,\ y\rangle \leq \|x\|^2-\|y\|^2 \leq 2\langle x-y,\ x\rangle \tag{2.1}$$

for all $x, y \in H$. For a sequence $\{x_n\}$ in H, strong and weak convergences of $\{x_n\}$ to a point $x\,(\in H)$ are denoted by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. A closed and convex subset C of H is weakly closed. It is easily verified that $x_n \rightharpoonup x$ is equivalent to the following condition. For any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, a subsequence $\left\{x_{n_j}\right\}$ of $\{x_{n_i}\}$ exists such that $x_{n_j} \rightharpoonup x$.

Let C be a nonempty, closed, and convex subset of H. The metric projection from H onto C is denoted by P_C, that is, $\|x - P_C x\| = \inf_{z\in C} \|x - z\|$ for any $x \in H$. The metric projection P_C from H onto C is nonexpansive and satisfies $\langle x - P_C x,\ P_C x - z\rangle \geq 0$ for all $x \in H$ and $z \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if $\|Tx - u\| \leq \|x - u\|$ for all $x \in C$ and $u \in F(T)$. Itoh and Takahashi [13] proved that a set of fixed points of a quasi-nonexpansive mapping is closed and convex. Kondo and Takahashi [20] demonstrated that a normally 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive.

Let T be a mapping from C into H, where C is a nonempty subset of H. Takahashi and Takeuchi [37] introduced the concept of an *attractive point*, and revealed the basic properties of the set of attractive points $A(T)$:

• $A(T)$ is closed and convex, and $A(T) \cap C \subset F(T)$ without any conditions on T;

• $F(T) \subset A(T)$ if T is quasi-nonexpansive;

• $A(T) \neq \emptyset \Longrightarrow F(T) \neq \emptyset$, if C is nonempty, closed, and convex, and T is a self-mapping defined on C.

The following lemmas are necessary to establish our main theorems.

Lemma 2.1 ([38]). *Let A be a nonempty, closed, and convex subset of H, let P_A be the metric projection from H onto A, and let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - q\| \leq \|x_n - q\|$ for all $q \in A$ and $n \in \mathbb{N}$, then $\{P_A x_n\}$ is convergent in A.*

Lemma 2.2 ([35]). *Let $x, y \in H$, and $\lambda \in \mathbb{R}$. Then, it holds that*

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda) \|y\|^2 - \lambda (1-\lambda) \|x-y\|^2 .$$

Let T be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i,j = 0,1,2)$-generic 2-generalized hybrid mapping. The next two lemmas assert that T and T^2 with fixed points are quasi-nonexpansive. We use the following notation:

$$\alpha_{i\bullet} \equiv \alpha_{i0} + \alpha_{i1} + \alpha_{i2}, \quad \alpha_{\bullet i} \equiv \alpha_{0i} + \alpha_{1i} + \alpha_{2i}, \quad \alpha_{\bullet\bullet} \equiv \sum_{i,j=0,1,2} \alpha_{ij}, \tag{2.2}$$

where $i = 0,1,2$.

Lemma 2.3 ([22]). *Let C be a nonempty subset of H, and let T be an $(\alpha_{ij}, \beta_i, \gamma_i;$ $i,j = 0,1,2)$-generic 2-generalized hybrid mapping from C into itself with $F(T) \neq \emptyset$. Suppose that T satisfies one of the following two conditions:*

(I) $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$, $\alpha_{2\bullet} \geq 0$, $\alpha_{1\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$;

(II) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} > 0$, $\gamma_0, \gamma_1, \gamma_2 \geq 0$.

Then, T is quasi-nonexpansive.

Lemma 2.4 ([24]). *Let C be a nonempty subset of H, and let T be an $(\alpha_{ij}, \beta_i, \gamma_i;$ $i,j = 0,1,2)$-generic 2-generalized hybrid mapping from C into itself with $F(T^2) \neq \emptyset$. Suppose that T satisfies one of the following two conditions:*

(i) $\alpha_{00} + \alpha_{02} + \alpha_{20} + \alpha_{22} \geq 0$, $\alpha_{10} + \alpha_{12} \geq 0$, $\alpha_{01}, \alpha_{11}, \alpha_{21} \geq 0$, $\alpha_{20} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$;

(ii) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{01} + \alpha_{21} \geq 0, \alpha_{10}, \alpha_{11}, \alpha_{12} \geq 0$, $\alpha_{02} + \alpha_{22} > 0$, $\beta_0 + \beta_1 \geq 0$, $\gamma_0, \gamma_1, \gamma_2 \geq 0$.

Then, T^2 is quasi-nonexpansive.

The following lemma is used to prove strong convergence theorems.

Lemma 2.5 ([2]; see also [42]). *Let $\{X_n\}$ be a sequence of nonnegative real numbers, let $\{Y_n\}$ be a sequence of real numbers such that $\limsup_{n\to\infty} Y_n \le 0$, and let $\{Z_n\}$ be a sequence of nonnegative real numbers such that $\sum_{n=1}^{\infty} Z_n < \infty$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. If $X_{n+1} \le (1-\lambda_n) X_n + \lambda_n Y_n + Z_n$ for all $n \in \mathbb{N}$, then $X_n \to 0$ as $n \to \infty$.*

We list two sublemmas.

Sublemma 2.1. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta \ge 0$, and let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_n - b_n \to 0$. Then, it holds that $\liminf_{n\to\infty} (\alpha a_n + \beta b_n) \ge 0$.*

The recent work by Kondo and Takahashi [24] introduced three types of attractive points of T. The sets of these attractive points are denoted as follows:

$$\begin{aligned} A_{10}(T) &\equiv A(T) \equiv \{v \in H : \|Ty - v\| \le \|y - v\| \text{ for all } y \in C\}; \\ A_{20}(T) &\equiv A\left(T^2\right) \equiv \left\{v \in H : \left\|T^2 y - v\right\| \le \|y - v\| \text{ for all } y \in C\right\}; \\ A_{21}(T) &\equiv \left\{v \in H : \left\|T^2 y - v\right\| \le \|Ty - v\| \text{ for all } y \in C\right\}. \end{aligned} \tag{2.3}$$

Notice that

$$A_{10}(T) = A_{10}(T) \cap A_{20}(T) \cap A_{21}(T). \tag{2.4}$$

This is because $A_{10}(T) \subset A_{20}(T) \cap A_{21}(T)$ (see [24]). In the rest of this paper, we use the notation $A_{10}(T)$, $A_{20}(T)$, and $A_{21}(T)$. The following sublemma is useful.

Sublemma 2.2 ([24]; see also [20]). *For $T : C \to C$ and $v \in H$, the following hold:*
(1) $v \in A_{10}(T) \iff \|Ty - y\|^2 + 2\langle Ty - y,\ y - v\rangle \le 0,\ \forall y \in C$;
(2) $v \in A_{20}(T) \iff \left\|T^2 y - y\right\|^2 + 2\left\langle T^2 y - y,\ y - v\right\rangle \le 0,\ \forall y \in C$;
(3) $v \in A_{21}(T) \iff \left\|T^2 y - Ty\right\|^2 + 2\left\langle T^2 y - Ty,\ Ty - v\right\rangle \le 0,\ \forall y \in C$.

The following two lemmas are important to prove our main theorems.

Lemma 2.6. *Let C be a nonempty subset of H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i,j = 0,1,2)$-generic 2-generalized hybrid mapping with one of the following two conditions:*
(I) $\alpha_{0\bullet} + \alpha_{1\bullet} \ge 0$, $\alpha_{2\bullet} \ge 0$, $\alpha_{1\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \ge 0$, $\gamma_0 + \gamma_1 \ge 0$, $\gamma_2 \ge 0$;
(II) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \ge 0$, $\alpha_{\bullet 2} \ge 0$, $\alpha_{\bullet 1} > 0$, $\gamma_0, \gamma_1, \gamma_2 \ge 0$, $\beta_0 + \beta_1 \ge 0$, $\beta_2 \ge 0$.
Suppose that $A_{10}(T) \ne \emptyset$. Let $\{w_n\}$ be a bounded sequence in C, and define

$$Sw_n \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^k w_n \in H.$$

Suppose that $Sw_{n_i} \rightharpoonup v\,(\in H)$, where $\{Sw_{n_i}\}$ is a subsequence of $\{Sw_n\}$. Then, $v \in A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)$. Additionally, if C is closed and convex, then $v \in F(T)$.

Proof. First, because $\{w_n\}$ is bounded and $A_{10}(T) \neq \emptyset$, sequences $\{T^{n+1}w_n\}$, $\{T^n w_n\}$, and $\{Tw_n\}$ are also bounded. Indeed, let $q \in A_{10}(T)$. Then, the following holds:

$$\left\|T^{n+1}w_n - q\right\| \leq \cdots \leq \|Tw_n - q\| \leq \|w_n - q\|.$$

As $\{w_n\}$ is bounded, the sequences $\{T^{n+1}w_n\}$, $\{T^n w_n\}$, and $\{Tw_n\}$ are also bounded, as claimed.

Case (I): Suppose that $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$, $\alpha_{2\bullet} \geq 0$, $\alpha_{1\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$, and $\gamma_2 \geq 0$. Let $y \in C$. From Sublemma 2.2-(1) and (2.4), it suffices to show that

$$\|Ty - y\|^2 + 2\langle Ty - y,\ y - v\rangle \leq 0,$$

where $v\,(\in H)$ is the weak limit of $\{Sw_{n_i}\}$. As T is $(\alpha_{ij}, \beta_i, \gamma_i;\, i,j = 0,1,2)$-generic 2-generalized hybrid, it holds that

$$\begin{aligned}
&\alpha_{00}\|y - T^k w_n\|^2 + \alpha_{01}\left\|y - T^{k+1}w_n\right\|^2 + \alpha_{02}\left\|y - T^{k+2}w_n\right\|^2 \\
&\quad + \alpha_{10}\left\|Ty - T^k w_n\right\|^2 + \alpha_{11}\left\|Ty - T^{k+1}w_n\right\|^2 + \alpha_{12}\left\|Ty - T^{k+2}w_n\right\|^2 \\
&\quad + \alpha_{20}\|T^2y - T^k w_n\|^2 + \alpha_{21}\left\|T^2y - T^{k+1}w_n\right\|^2 + \alpha_{22}\left\|T^2y - T^{k+2}w_n\right\|^2 \\
&\quad + \beta_0\|y - Ty\|^2 + \beta_1\left\|Ty - T^2y\right\|^2 + \beta_2\left\|T^2y - y\right\|^2 \\
&\quad + \gamma_0\left\|T^k w_n - T^{k+1}w_n\right\|^2 + \gamma_1\left\|T^{k+1}w_n - T^{k+2}w_n\right\|^2 \\
&\quad + \gamma_2\left\|T^{k+2}w_n - T^k w_n\right\|^2 \leq 0
\end{aligned}$$

for all $k \in \mathbb{N} \cup \{0\}$. As $\gamma_2 \geq 0$, we obtain the following:

$$\begin{aligned}
&\alpha_{00}\left\|y - T^k w_n\right\|^2 + \alpha_{01}\left\|y - T^{k+1}w_n\right\|^2 + \alpha_{02}\left\|y - T^{k+2}w_n\right\|^2 \\
&\quad + \alpha_{10}\left(\|Ty - y\|^2 + 2\left\langle Ty - y,\ y - T^k w_n\right\rangle + \left\|y - T^k w_n\right\|^2\right) \\
&\quad + \alpha_{11}\left(\|Ty - y\|^2 + 2\left\langle Ty - y,\ y - T^{k+1}w_n\right\rangle + \left\|y - T^{k+1}w_n\right\|^2\right) \\
&\quad + \alpha_{12}\left(\|Ty - y\|^2 + 2\left\langle Ty - y,\ y - T^{k+2}w_n\right\rangle + \left\|y - T^{k+2}w_n\right\|^2\right) \\
&\quad + \alpha_{20}\|T^2y - T^k w_n\|^2 + \alpha_{21}\left\|T^2y - T^{k+1}w_n\right\|^2 \\
&\quad + \alpha_{22}\left\|T^2y - T^{k+2}w_n\right\|^2 + \beta_0\|y - Ty\|^2 \\
&\quad + \beta_1\left\|Ty - T^2y\right\|^2 + \beta_2\left\|T^2y - y\right\|^2 \\
&\quad + \gamma_0\left\|T^k w_n - T^{k+1}w_n\right\|^2 + \gamma_1\left\|T^{k+1}w_n - T^{k+2}w_n\right\|^2 \leq 0.
\end{aligned}$$

Consequently,

$$
\begin{aligned}
&(\alpha_{00}+\alpha_{10})\left\|y-T^k w_n\right\|^2+(\alpha_{01}+\alpha_{11})\left\|y-T^{k+1}w_n\right\|^2\\
&\quad+(\alpha_{02}+\alpha_{12})\left\|y-T^{k+2}w_n\right\|^2+2\alpha_{10}\left\langle Ty-y,\ y-T^k w_n\right\rangle\\
&\quad+2\alpha_{11}\left\langle Ty-y,\ y-T^{k+1}w_n\right\rangle+2\alpha_{12}\left\langle Ty-y,\ y-T^{k+2}w_n\right\rangle\\
&\quad+\alpha_{20}\|T^2y-T^k w_n\|^2+\alpha_{21}\left\|T^2y-T^{k+1}w_n\right\|^2\\
&\quad+\alpha_{22}\left\|T^2y-T^{k+2}w_n\right\|^2+(\alpha_{1\bullet}+\beta_0)\|y-Ty\|^2\\
&\quad+\beta_1\left\|Ty-T^2y\right\|^2+\beta_2\left\|T^2y-y\right\|^2\\
&\quad+\gamma_0\left\|T^k w_n-T^{k+1}w_n\right\|^2+\gamma_1\left\|T^{k+1}w_n-T^{k+2}w_n\right\|^2\le 0.
\end{aligned}
$$

This yields the following:

$$
\begin{aligned}
&(\alpha_{0\bullet}+\alpha_{1\bullet})\left\|y-T^k w_n\right\|^2+(\alpha_{01}+\alpha_{11})\left(\left\|y-T^{k+1}w_n\right\|^2-\left\|y-T^k w_n\right\|^2\right)\\
&\quad+(\alpha_{02}+\alpha_{12})\left(\left\|y-T^{k+2}w_n\right\|^2-\left\|y-T^k w_n\right\|^2\right)\\
&\quad+2\left\langle Ty-y,\ \alpha_{1\bullet}y-\left(\alpha_{10}T^k w_n+\alpha_{11}T^{k+1}w_n+\alpha_{12}T^{k+2}w_n\right)\right\rangle\\
&\quad+\alpha_{2\bullet}\left\|T^2y-T^k w_n\right\|^2+\alpha_{21}\left(\left\|T^2y-T^{k+1}w_n\right\|^2-\left\|T^2y-T^k w_n\right\|^2\right)\\
&\quad+\alpha_{22}\left(\left\|T^2y-T^{k+2}w_n\right\|^2-\left\|T^2y-T^k w_n\right\|^2\right)\\
&\quad+(\alpha_{1\bullet}+\beta_0)\|Ty-y\|^2+\beta_1\left\|Ty-T^2y\right\|^2+\beta_2\left\|T^2y-y\right\|^2\\
&\quad+\gamma_0\left\|T^k w_n-T^{k+1}w_n\right\|^2+\gamma_1\left\|T^{k+1}w_n-T^{k+2}w_n\right\|^2\le 0.
\end{aligned}
$$

As $\alpha_{0\bullet}+\alpha_{1\bullet}\ge 0$ and $\alpha_{2\bullet}\ge 0$, we have the following:

$$
\begin{aligned}
&(\alpha_{01}+\alpha_{11})\left(\left\|y-T^{k+1}w_n\right\|^2-\left\|y-T^k w_n\right\|^2\right)\\
&\quad+(\alpha_{02}+\alpha_{12})\left(\left\|y-T^{k+2}w_n\right\|^2-\left\|y-T^k w_n\right\|^2\right)\\
&\quad+2\Big\langle Ty-y,\ \alpha_{1\bullet}y-\Big\{\alpha_{1\bullet}T^k w_n+\alpha_{11}\left(T^{k+1}w_n-T^k w_n\right)\\
&\quad+\alpha_{12}\left(T^{k+2}w_n-T^k w_n\right)\Big\}\Big\rangle\\
&\quad+\alpha_{21}\left(\left\|T^2y-T^{k+1}w_n\right\|^2-\left\|T^2y-T^k w_n\right\|^2\right)\\
&\quad+\alpha_{22}\left(\left\|T^2y-T^{k+2}w_n\right\|^2-\left\|T^2y-T^k w_n\right\|^2\right)\\
&\quad+(\alpha_{1\bullet}+\beta_0)\|Ty-y\|^2+\beta_1\left\|Ty-T^2y\right\|^2+\beta_2\left\|T^2y-y\right\|^2
\end{aligned}
$$

$$+ \gamma_0 \left\| T^k w_n - T^{k+1} w_n \right\|^2 + \gamma_1 \left\| T^{k+1} w_n - T^{k+2} w_n \right\|^2 \leq 0.$$

Summing these inequalities with respect to k from 0 to $n-1$, and dividing by n, we obtain

$$\begin{aligned}
&\frac{1}{n} (\alpha_{01} + \alpha_{11}) \left(\|y - T^n w_n\|^2 - \|y - w_n\|^2 \right) \\
&\quad + \frac{1}{n} (\alpha_{02} + \alpha_{12}) (\left\|y - T^{n+1} w_n\right\|^2 + \|y - T^n w_n\|^2 \\
&\quad - \|y - T w_n\|^2 - \|y - w_n\|^2) \\
&\quad + 2 \langle Ty - y,\ \alpha_{1\bullet} y - \{ \alpha_{1\bullet} S w_n + \frac{1}{n} \alpha_{11} (T^n w_n - w_n) \\
&\quad + \frac{1}{n} \alpha_{12} \left(T^{n+1} w_n + T^n w_n - T w_n - w_n \right) \} \rangle \\
&\quad + \frac{1}{n} \alpha_{21} \left(\left\|T^2 y - T^n w_n\right\|^2 - \left\|T^2 y - w_n\right\|^2 \right) \\
&\quad + \frac{1}{n} \alpha_{22} (\left\|T^2 y - T^{n+1} w_n\right\|^2 + \left\|T^2 y - T^n w_n\right\|^2 \\
&\quad - \left\|T^2 y - T w_n\right\|^2 - \left\|T^2 y - w_n\right\|^2) \\
&\quad + (\alpha_{1\bullet} + \beta_0) \|Ty - y\|^2 + \beta_1 \left\|Ty - T^2 y\right\|^2 \\
&\quad + \beta_2 \left\|T^2 y - y\right\|^2 + \gamma_0 \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^k w_n - T^{k+1} w_n \right\|^2 \\
&\quad + \gamma_1 \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^{k+1} w_n - T^{k+2} w_n \right\| \leq 0.
\end{aligned} \tag{2.5}$$

As $\{T^n w_n\}$ and $\{T^{n+1} w_n\}$ are bounded,

$$\frac{1}{n} \sum_{k=0}^{n-1} \left\| T^k w_n - T^{k+1} w_n \right\|^2 - \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^{k+1} w_n - T^{k+2} w_n \right\|^2 \to 0$$

as $n \to \infty$. As $\gamma_0 + \gamma_1 \geq 0$, from Sublemma 2.1, we have

$$\liminf_{n\to\infty} \left(\gamma_0 \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^k w_n - T^{k+1} w_n \right\|^2 + \gamma_1 \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^{k+1} w_n - T^{k+2} w_n \right\|^2 \right) \geq 0.$$

Thus, replacing n by n_i, and taking the lim inf as $i \to \infty$ in (2.5), we obtain

$$\begin{aligned}
&2\alpha_{1\bullet} \langle Ty - y,\ y - v \rangle \\
&\qquad + (\alpha_{1\bullet} + \beta_0) \|Ty - y\|^2 + \beta_1 \left\|Ty - T^2 y\right\|^2 + \beta_2 \left\|T^2 y - y\right\|^2 \leq 0.
\end{aligned}$$

Because $\beta_0, \beta_1, \beta_2 \geq 0$, we have $2\alpha_{1\bullet} \langle Ty - y,\ y - v \rangle + \alpha_{1\bullet} \|Ty - y\|^2 \leq 0$. It follows from $\alpha_{1\bullet} > 0$ that $\|Ty - y\|^2 + 2 \langle Ty - y,\ y - v \rangle \leq 0$ for all $y \in C$. From Sublemma 2.2-(1) and (2.4), we obtain $v \in A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)$.

In addition to the other assumptions, suppose that C is closed and convex. Then, $\{Sw_n\}$ is a sequence in C. As $Sw_{n_i} \rightharpoonup v$, it holds that $v \in C$. As $v \in A_{10}(T) \cap C \subset F(T)$, we have $v \in F(T)$.

Case (II). Suppose that $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} > 0$, $\gamma_0, \gamma_1, \gamma_2 \geq 0$, $\beta_0 + \beta_1 \geq 0$, and $\beta_2 \geq 0$. We can obtain the desired results by replacing y and $T^k w_n$ in (2.5). □

Under condition (I) or (II), the mapping T is quasi-nonexpansive if it has a fixed point (see Lemma 2.3). The lemma slightly generalizes Lemma 5.2 by Kondo and Takahashi [22] regarding the parameter conditions. For this point, see Theorem 3.2 in this paper, which is reproduced for convenience. The next lemma is employed to prove Theorems 3.3, 4.2, and 5.2.

Lemma 2.7. *Let C be a nonempty subset of H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i; i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping with one of the following two condition:*

(i) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{20} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$;

(ii) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{02} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$.

Suppose that $A_{20}(T) \neq \emptyset$. Let $\{w_n\}$ be a bounded sequence in C, and define

$$S' w_n \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^{2k} w_n \in H.$$

Suppose that $S' w_{n_i} \rightharpoonup v \, (\in H)$, where $\{S' w_{n_i}\}$ is a subsequence of $\{S' w_n\}$. Then, $v \in A_{20}(T)$. Additionally, if C is closed and convex, then $v \in F(T^2)$.

Proof. Because $\{w_n\}$ is bounded and $A_{20}(T) \neq \emptyset$ is assumed, a sequence $\{T^{2n} w_n\}$ is also bounded. Indeed, let $q \in A_{20}(T)$. Then, the following holds:

$$\left\|T^{2n} w_n - q\right\| \leq \left\|T^{2(n-1)} w_n - q\right\| \leq \cdots \leq \left\|T^2 w_n - q\right\| \leq \|w_n - q\|.$$

As $\{w_n\}$ is bounded, the sequence $\{T^{2n} w_n\}$ is also bounded, as claimed.

Case (i): Suppose that $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{20} + \alpha_{22} > 0$, and $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$. Let $y \in C$. From Sublemma 2.2-(2), it suffices to show that

$$\left\|T^2 y - y\right\|^2 + 2\left\langle T^2 y - y,\ y - v\right\rangle \leq 0,$$

where $v\,(\in H)$ is the weak limit of $\{S'w_{n_i}\}$. As T is $(\alpha_{ij},\beta_i,\gamma_i;\,i,j=0,1,2)$-generic 2-generalized hybrid, it follows that

$$\begin{aligned}
&\alpha_{00}\|y-T^kw_n\|^2+\alpha_{01}\left\|y-T^{k+1}w_n\right\|^2+\alpha_{02}\left\|y-T^{k+2}w_n\right\|^2\\
&\quad+\alpha_{10}\left\|Ty-T^kw_n\right\|^2+\alpha_{11}\left\|Ty-T^{k+1}w_n\right\|^2+\alpha_{12}\left\|Ty-T^{k+2}w_n\right\|^2\\
&\quad+\alpha_{20}\|T^2y-T^kw_n\|^2+\alpha_{21}\left\|T^2y-T^{k+1}w_n\right\|^2+\alpha_{22}\left\|T^2y-T^{k+2}w_n\right\|^2\\
&\quad+\beta_0\left\|y-Ty\right\|^2+\beta_1\left\|Ty-T^2y\right\|^2+\beta_2\left\|T^2y-y\right\|^2\\
&\quad+\gamma_0\left\|T^kw_n-T^{k+1}w_n\right\|^2+\gamma_1\left\|T^{k+1}w_n-T^{k+2}w_n\right\|^2\\
&\quad+\gamma_2\left\|T^{k+2}w_n-T^kw_n\right\|^2\le 0
\end{aligned}\tag{2.6}$$

for all $k\in\mathbb{N}\cup\{0\}$. As $\alpha_{10},\alpha_{11},\alpha_{12},\alpha_{01},\alpha_{21}\ge 0$ and $\gamma_0,\gamma_1,\gamma_2\ge 0$, we have

$$\begin{aligned}
&\alpha_{00}\|y-T^kw_n\|^2+\alpha_{02}\left\|y-T^{k+2}w_n\right\|^2\\
&\qquad+\alpha_{20}\|T^2y-T^kw_n\|^2+\alpha_{22}\left\|T^2y-T^{k+2}w_n\right\|^2\\
&\qquad\qquad+\beta_0\left\|y-Ty\right\|^2+\beta_1\left\|Ty-T^2y\right\|^2+\beta_2\left\|T^2y-y\right\|^2\le 0.
\end{aligned}$$

This yields

$$\begin{aligned}
&\alpha_{00}\left\|y-T^kw_n\right\|^2+\alpha_{02}\left\|y-T^{k+2}w_n\right\|^2\\
&\quad+\alpha_{20}\left(\left\|T^2y-y\right\|^2+2\left\langle T^2y-y,\ y-T^kw_n\right\rangle+\left\|y-T^kw_n\right\|^2\right)\\
&\quad+\alpha_{22}\left(\left\|T^2y-y\right\|^2+2\left\langle T^2y-y,\ y-T^{k+2}w_n\right\rangle+\left\|y-T^{k+2}w_n\right\|^2\right)\\
&\quad+\beta_0\left\|y-Ty\right\|^2+\beta_1\left\|Ty-T^2y\right\|^2+\beta_2\left\|T^2y-y\right\|^2\le 0.
\end{aligned}$$

We obtain the following:

$$\begin{aligned}
&(\alpha_{00}+\alpha_{20})\left\|y-T^kw_n\right\|^2+(\alpha_{02}+\alpha_{22})\left\|y-T^{k+2}w_n\right\|^2\\
&\quad+2\alpha_{20}\left\langle T^2y-y,\ y-T^kw_n\right\rangle+2\alpha_{22}\left\langle T^2y-y,\ y-T^{k+2}w_n\right\rangle\\
&\quad+\beta_0\left\|y-Ty\right\|^2+\beta_1\left\|Ty-T^2y\right\|^2+(\alpha_{20}+\alpha_{22}+\beta_2)\left\|T^2y-y\right\|^2\le 0;
\end{aligned}$$

thus,

$$\begin{aligned}
&(\alpha_{00}+\alpha_{20}+\alpha_{02}+\alpha_{22})\left\|y-T^kw_n\right\|^2\\
&\quad+(\alpha_{02}+\alpha_{22})\left(\left\|y-T^{k+2}w_n\right\|^2-\left\|y-T^kw_n\right\|^2\right)\\
&\quad+2\left\langle T^2y-y,\ (\alpha_{20}+\alpha_{22})\,y-\alpha_{20}T^kw_n-\alpha_{22}T^{k+2}w_n\right\rangle
\end{aligned}$$

$$+\beta_0 \|y-Ty\|^2+\beta_1 \left\|Ty-T^2y\right\|^2+(\alpha_{20}+\alpha_{22}+\beta_2)\left\|T^2y-y\right\|^2 \le 0.$$

It follows that

$$\begin{aligned}&(\alpha_{00}+\alpha_{20}+\alpha_{02}+\alpha_{22})\left\|y-T^k w_n\right\|^2\\ &\quad+(\alpha_{02}+\alpha_{22})\left(\left\|y-T^{k+2}w_n\right\|^2-\left\|y-T^k w_n\right\|^2\right)\\ &\qquad+2\left\langle T^2y-y,\ (\alpha_{20}+\alpha_{22})\,y-\{(\alpha_{20}+\alpha_{22})\,T^k w_n+\alpha_{22}(T^{k+2}w_n-T^k w_n)\}\right\rangle\\ &\qquad\quad+\beta_0\|y-Ty\|^2+\beta_1\left\|Ty-T^2y\right\|^2+(\alpha_{20}+\alpha_{22}+\beta_2)\left\|T^2y-y\right\|^2\le 0.\end{aligned}$$

As $\alpha_{00}+\alpha_{20}+\alpha_{02}+\alpha_{22}\ge 0$, we have

$$\begin{aligned}&(\alpha_{02}+\alpha_{22})\left(\left\|y-T^{k+2}w_n\right\|^2-\left\|y-T^k w_n\right\|^2\right)\\ &\quad+2\left\langle T^2y-y,\ (\alpha_{20}+\alpha_{22})\,y-\{(\alpha_{20}+\alpha_{22})\,T^k w_n+\alpha_{22}(T^{k+2}w_n-T^k w_n)\}\right\rangle\\ &\qquad+\beta_0\|y-Ty\|^2+\beta_1\left\|Ty-T^2y\right\|^2+(\alpha_{20}+\alpha_{22}+\beta_2)\left\|T^2y-y\right\|^2\le 0.\end{aligned}$$

Summing these inequalities with respect to $k=0,2,4,\cdots,2(n-1)$, and dividing by n, we obtain

$$\begin{aligned}&\frac{1}{n}(\alpha_{02}+\alpha_{22})\left(\left\|y-T^{2n}w_n\right\|^2-\|y-w_n\|^2\right)\\ &\quad+2\left\langle T^2y-y,\ (\alpha_{20}+\alpha_{22})\,y-\{(\alpha_{20}+\alpha_{22})\,S'w_n+\frac{1}{n}\alpha_{22}\left(T^{2n}w_n-w_n\right)\right\rangle\\ &\qquad+\beta_0\|y-Ty\|^2+\beta_1\left\|Ty-T^2y\right\|^2+(\alpha_{20}+\alpha_{22}+\beta_2)\left\|T^2y-y\right\|^2\le 0.\end{aligned}$$

As $\left\{T^{2n}w_n\right\}$ and $\{w_n\}$ are bounded, replacing n with n_i, and taking the limit as $i\to\infty$, we obtain

$$\begin{aligned}&2(\alpha_{20}+\alpha_{22})\left\langle T^2y-y,\ y-v\right\rangle\\ &\quad+\beta_0\|y-Ty\|^2+\beta_1\left\|Ty-T^2y\right\|^2+(\alpha_{20}+\alpha_{22}+\beta_2)\left\|T^2y-y\right\|^2\le 0.\end{aligned}$$

As $\beta_0,\beta_1,\beta_2\ge 0$, we obtain

$$2(\alpha_{20}+\alpha_{22})\left\langle T^2y-y,\ y-v\right\rangle+(\alpha_{20}+\alpha_{22})\left\|T^2y-y\right\|^2\le 0.$$

It holds from $\alpha_{20}+\alpha_{22}>0$ that $2\left\langle T^2y-y,\ y-v\right\rangle+\left\|T^2y-y\right\|^2\le 0$ for all $y\in C$. From Sublemma 2.2-(2), $v\in A_{20}(T)$.

In addition to the other assumptions, suppose that C is closed and convex. Then, $\{S'w_n\}$ is a sequence in C. As $S'w_{n_i}\rightharpoonup v$, it holds that $v\in C$. As $v\in A_{20}(T)\cap C\subset F\left(T^2\right)$, we have $v\in F\left(T^2\right)$.

Case (ii). Suppose that $\alpha_{00}+\alpha_{20}+\alpha_{02}+\alpha_{22}\ge 0,\ \alpha_{10},\alpha_{11},\alpha_{12},\alpha_{01},\alpha_{21}\ge 0$, $\alpha_{02}+\alpha_{22}>0$, and $\beta_0,\beta_1,\beta_2,\gamma_0,\gamma_1,\gamma_2\ge 0$. We can obtain the desired results by replacing y and $T^k w_n$ in (2.6). □

Under condition (i) or (ii), the mapping T^2 is quasi-nonexpansive if it has a fixed point (see Lemma 2.4).

3. Baillon's type weak convergence theorems

This section presents nonlinear ergodic theorems. The elements of the proof were developed by Takahashi [33] (see also [5, 14, 20, 27, 29, 37, 39]). First, we obtain a theorem that weakly approximates the attractive and fixed points of a generic 2-generalized hybrid mapping. The theorem generalizes that in the previous work [22].

Theorem 3.1. *Let C be a nonempty subset of H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping with $A_{10}(T) \neq \emptyset$. Let $P_{A_{10}(T)}$ be the metric projection from H onto $A_{10}(T)$. Suppose that T satisfies one of the following two conditions:*

(I) $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$, $\alpha_{2\bullet} \geq 0$, $\alpha_{1\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$, $\gamma_2 \geq 0$;
(II) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0$, $\alpha_{\bullet 2} \geq 0$, $\alpha_{\bullet 1} > 0$, $\gamma_0, \gamma_1, \gamma_2 \geq 0$, $\beta_0 + \beta_1 \geq 0$, $\beta_2 \geq 0$.

Then, for any $x \in C$, the sequence $\left\{S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ in H converges weakly to a point $\overline{x}$ of $A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)\ (\subset H)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A_{10}(T)} T^n x$. Additionally, if C is closed and convex in H, then for any $x \in C$, the sequence $\left\{S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ in C converges weakly to a point of $F(T)$.

Proof. From Takahashi and Takeuchi [37], we know that $A_{10}(T)$ is closed and convex in H. As $A_{10}(T) \neq \emptyset$ is assumed, the metric projection $P_{A_{10}(T)}$ from H onto $A_{10}(T)$ exists. Let $x \in C$, and define $S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x \in H$ for all $n \in \mathbb{N}$. As $A_{10}(T) \neq \emptyset$, $\{T^n x\}$ is bounded in C. Indeed, it holds that

$$\|T^n x - q\| \leq \left\|T^{n-1} x - q\right\| \tag{3.1}$$

for all $q \in A_{10}(T)$ and $n \in \mathbb{N}$. This demonstrates that $\{T^n x\}$ is bounded. Consequently, the sequence $\{S_n x\}$ is also bounded in H. From (3.1) and Lemma 2.1, the sequence $\left\{P_{A_{10}(T)} T^n x\right\}$ is convergent in $A_{10}(T)$. Define the following:

$$\overline{x} \equiv \lim_{n\to\infty} P_{A_{10}(T)} T^n x \in A_{10}(T).$$

Our objective is to prove that $S_n x \rightharpoonup \overline{x}$. Let $\{S_{n_i} x\}$ be a subsequence of $\{S_n x\}$. As $\{S_{n_i} x\}$ is bounded, a subsequence $\left\{S_{n_j} x\right\}$ of $\{S_{n_i} x\}$ exists such that $S_{n_j} x \rightharpoonup u$ for some $u \in H$. It suffices to demonstrate that $u = \overline{x}$. Applying Lemma 2.6 with $w_n = x$, we have $u \in A_{10}(T)$.

The sequence $\left\{\left\|T^n x - P_{A_{10}(T)} T^n x\right\|\right\}$ is monotone decreasing. Indeed, as $P_{A_{10}(T)} T^n x \in A_{10}(T)$, from (3.1), it holds that

$$\begin{aligned}\left\|T^{n+1} x - P_{A_{10}(T)} T^{n+1} x\right\| &\leq \left\|T^{n+1} x - P_{A_{10}(T)} T^n x\right\| \\ &\leq \left\|T^n x - P_{A_{10}(T)} T^n x\right\|\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus, the sequence $\left\{\left\|T^n x - P_{A_{10}(T)} T^n x\right\|\right\}$ is monotone decreasing, as claimed.

It follows from $u \in A_{10}(T)$ that

$$\left\langle T^k x - P_{A_{10}(T)} T^k x,\ P_{A_{10}(T)} T^k x - u\right\rangle \geq 0$$

for all $k \in \mathbb{N} \cup \{0\}$, and therefore,

$$\left\langle T^k x - P_{A_{10}(T)} T^k x,\ P_{A_{10}(T)} T^k x - \overline{x} + \overline{x} - u \right\rangle \geq 0.$$

As $\left\{ \left\| T^n x - P_{A_{10}(T)} T^n x \right\| \right\}$ is monotone decreasing, we have the following:

$$\begin{aligned} &\left\langle T^k x - P_{A_{10}(T)} T^k x,\ \ -(\overline{x} - u) \right\rangle \\ &\leq \left\langle T^k x - P_{A_{10}(T)} T^k x,\ P_{A_{10}(T)} T^k x - \overline{x} \right\rangle \\ &\leq \left\| T^k x - P_{A_{10}(T)} T^k x \right\| \left\| P_{A_{10}(T)} T^k x - \overline{x} \right\| \\ &\leq \left\| x - P_{A_{10}(T)} x \right\| \left\| P_{A_{10}(T)} T^k x - \overline{x} \right\|. \end{aligned}$$

Summing these inequalities with respect to k from 0 to $n-1$, we obtain

$$\begin{aligned} &\left\langle \sum_{k=0}^{n-1} T^k x - \sum_{k=0}^{n-1} P_{A_{10}(T)} T^k x,\ \ -(\overline{x} - u) \right\rangle \\ \leq \ \ &\left\| x - P_{A_{10}(T)} x \right\| \cdot \sum_{k=0}^{n-1} \left\| P_{A_{10}(T)} T^k x - \overline{x} \right\|. \end{aligned}$$

Dividing by n, we have

$$\begin{aligned} &\left\langle S_n x - \frac{1}{n} \sum_{k=0}^{n-1} P_{A_{10}(T)} T^k x,\ \ -(\overline{x} - u) \right\rangle \\ \leq \ \ &\left\| x - P_{A_{10}(T)} x \right\| \cdot \frac{1}{n} \sum_{k=0}^{n-1} \left\| P_{A_{10}(T)} T^k x - \overline{x} \right\|. \end{aligned}$$

Replacing n by n_j, and taking the limit as $j \to \infty$, we obtain

$$\langle u - \overline{x},\ \ -(\overline{x} - u) \rangle \leq 0$$

as $S_{n_j} x \rightharpoonup u$ and $P_{A_{10}(T)} T^n x \to \overline{x}$. Hence, it holds that $u = \overline{x}$. We obtain $S_n x \rightharpoonup \overline{x} \equiv \lim_{n \to \infty} P_{A_{10}(T)} T^n x \in A_{10}(T)$. It follows from $A_{10}(T) \subset A_{20}(T) \cap A_{21}(T)$ that $u \in A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)$.

In addition to the other assumptions, suppose that C is closed and convex. Because $A_{10}(T) \neq \emptyset$ is assumed, we have $F(T) \neq \emptyset$. Consequently, under a condition (I) or (II), the mapping T is quasi-nonexpansive. In this case, $\{S_n x\}$ is a sequence in C. As C is weakly closed and $S_n x \rightharpoonup \overline{x}$, $\overline{x} \in C$. Therefore, $\overline{x} \in A_{10}(T) \cap C \subset F(T)$, which completes the proof. □

Theorem 3.1 is a generalization of the previous result by Kondo and Takahashi [22]:

Theorem 3.2 ([22]). *Let C be a nonempty subset of H, and let $T : C \to C$ be a $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping with $A_{10}(T) \neq \emptyset$. Let $P_{A_{10}(T)}$ be the metric projection from H onto $A_{10}(T)$. Suppose that T satisfies one of the following conditions:*

(I)′ $\alpha_{0\bullet}+\alpha_{1\bullet} \geq 0$, $\alpha_{20}, \alpha_{21}, \alpha_{22} \geq 0$, $\alpha_{1\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0+\gamma_1 \geq 0$, $\gamma_2 \geq 0$;
(II)′ $\alpha_{\bullet 0}+\alpha_{\bullet 1} \geq 0$, $\alpha_{02}, \alpha_{12}, \alpha_{22} \geq 0$, $\alpha_{\bullet 1} > 0$, $\beta_0+\beta_1 \geq 0$, $\beta_2 \geq 0$, $\gamma_0, \gamma_1, \gamma_2 \geq 0$.

Then, for any $x \in C$, the sequence $\left\{S_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x\right\}$ converges weakly to $\overline{x} \in A_{10}(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A_{10}(T)} T^n x$. Additionally, suppose that C is closed and convex. Then, for any $x \in C$, the sequence $\{S_n x\}$ converges weakly to a fixed point $\overline{x}$ of T.

As we observe, the condition (I) (resp. (II)) is more general than (I)′ (rep. (II)′). Next, we present a weak convergence theorem for $A_{20}(T)$ and $F\left(T^2\right)$.

Theorem 3.3. *Let C be a nonempty subset of H, and let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping with $A_{20}(T) \neq \emptyset$. Let $P_{A_{20}(T)}$ be the metric projection from H onto $A_{20}(T)$. Suppose that T satisfies the following two conditions:*
(i) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{20} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$;
(ii) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{02} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$.

Then, for any $x \in C$, the sequence $\left\{S'_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^{2k} x\right\}$ in H converges weakly to a point $\overline{x}\ (\in H)$ of $A_{20}(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A_{20}(T)} T^{2n} x$. Additionally, if C is closed and convex in H, then for any $x \in C$, the sequence $\left\{S'_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^{2k} x\right\}$ in C converges weakly to a point of $F\left(T^2\right)$.

Proof. As $A_{20}(T) = A_{10}\left(T^2\right)$, it holds that $A_{20}(T)$ is closed and convex in H. As $A_{20}(T) \neq \emptyset$ is assumed, the metric projection $P_{A_{20}(T)}$ from H onto $A_{20}(T)$ exists. Let $x \in C$, and define $S'_n x \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^{2k} x \in H$ for all $n \in \mathbb{N}$. Because $A_{20}(T) \neq \emptyset$ is assumed, $\left\{T^{2n} x\right\}$ is a bounded sequence in C. Indeed, it holds that

$$\left\|T^{2(n+1)} x - q\right\| \leq \left\|T^{2n} x - q\right\| \tag{3.2}$$

for all $q \in A_{20}(T)$ and $n \in \mathbb{N} \cup \{0\}$. This indicates that $\left\{T^{2n} x\right\}$ is bounded. Thus, the sequence $\{S'_n x\}$ is also bounded in H. From (3.2) and Lemma 2.1, the sequence $\left\{P_{A_{20}(T)} T^{2n} x\right\}$ is convergent in $A_{20}(T)$. Define $\overline{x} \equiv \lim_{n\to\infty} P_{A_{20}(T)} T^{2n} x \in A_{20}(T)$. Our purpose is to prove that $S'_n x \rightharpoonup \overline{x}$. Let $\left\{S'_{n_i} x\right\}$ be a subsequence of $\{S'_n x\}$. As $\left\{S'_{n_i} x\right\}$ is bounded, a subsequence $\left\{S'_{n_j} x\right\}$ of $\left\{S'_{n_i} x\right\}$ and $u \in H$ exist such that $S'_{n_j} x \rightharpoonup u$. It suffices to demonstrate that $u = \overline{x}$. Applying Lemma 2.7 with $w_n = x$, we have $u \in A_{20}(T)$.

It is easy to verify that the sequence $\left\{\left\|T^{2n} x - P_{A_{20}(T)} T^{2n} x\right\|\right\}$ is monotone decreasing. The proof is as follows. Because $P_{A_{20}(T)} T^{2n} x \in A_{20}(T)$, from (3.2), it holds that

$$\begin{aligned}\left\|T^{2(n+1)} x - P_{A_{20}(T)} T^{2(n+1)} x\right\| &\leq \left\|T^{2(n+1)} x - P_{A_{20}(T)} T^{2n} x\right\| \\ &\leq \left\|T^{2n} x - P_{A_{20}(T)} T^{2n} x\right\|\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Thus, the sequence $\left\{\left\|T^{2n}x - P_{A_{20}(T)}T^{2n}x\right\|\right\}$ is monotone decreasing, as claimed.

As $u \in A_{20}(T)$, it follows that

$$\left\langle T^{2k}x - P_{A_{20}(T)}T^{2k}x,\ P_{A_{20}(T)}T^{2k}x - u\right\rangle \geq 0$$

for all $k \in \mathbb{N} \cup \{0\}$. This yields

$$\left\langle T^{2k}x - P_{A_{20}(T)}T^{2k}x,\ P_{A_{20}(T)}T^{2k}x - \overline{x} + \overline{x} - u\right\rangle \geq 0.$$

Using Schwarz's inequality and (3.2), we have

$$\begin{aligned}
&\left\langle T^{2k}x - P_{A_{20}(T)}T^{2k}x,\ \ -(\overline{x} - u)\right\rangle \\
&\leq \left\langle T^{2k}x - P_{A_{20}(T)}T^{2k}x,\ P_{A_{20}(T)}T^{2k}x - \overline{x}\right\rangle \\
&\leq \left\|T^{2k}x - P_{A_{20}(T)}T^{2k}x\right\| \left\|P_{A_{20}(T)}T^{2k}x - \overline{x}\right\| \\
&\leq \left\|x - P_{A_{20}(T)}x\right\| \left\|P_{A_{20}(T)}T^{2k}x - \overline{x}\right\|.
\end{aligned}$$

Summing these inequalities with respect to k from 0 to $n-1$ and dividing by n, we obtain

$$\begin{aligned}
&\left\langle S'_n x - \frac{1}{n}\sum_{k=0}^{n-1} P_{A_{20}(T)}T^{2k}x,\ \ -(\overline{x} - u)\right\rangle \\
\leq\ \ &\left\|x - P_{A_{20}(T)}x\right\| \cdot \frac{1}{n}\sum_{k=0}^{n-1}\left\|P_{A_{20}(T)}T^{2k}x - \overline{x}\right\|.
\end{aligned}$$

Replacing n with n_j, and taking the limit as $j \to \infty$, we have

$$\langle u - \overline{x},\ \ -(\overline{x} - u)\rangle \leq 0$$

because $S'_{n_j}x \rightharpoonup u$ and $P_{A_{20}(T)}T^{2n}x \to \overline{x}$, which implies that $u = \overline{x}$. We proved that $S'_n x \rightharpoonup \overline{x} \equiv \lim_{n\to\infty} P_{A_{20}(T)}T^{2n}x \in A_{20}(T)$.

Suppose, in addition to the other assumptions, that C is closed and convex. Because $A_{20}(T) = A_{10}\left(T^2\right) \neq \emptyset$ is assumed, it holds that $F\left(T^2\right) \neq \emptyset$. Under a condition (i) or (ii), the mapping T^2 is quasi-nonexpansive. Then, $\{S'_n x\}$ is a sequence in C. As C is weakly closed and $S'_n x \rightharpoonup \overline{x}$, we have that $\overline{x} \in C$. Therefore, $\overline{x} \in A_{20}(T) \cap C \subset F\left(T^2\right)$, which completes the proof. □

4. Mann's type weak convergence theorems

This section presents weak convergence theorems for finding attractive and fixed points of a generic 2-generalized hybrid mapping. Many authors have developed the proof (see [8, 9, 18, 19, 23, 25, 27]).

Theorem 4.1. *Let C be a nonempty and convex subset of H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that satisfies one of the following two conditions:*

(I) $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0$, $\alpha_{2\bullet} \geq 0$, $\alpha_{1\bullet} > 0$, $\beta_0, \beta_1, \beta_2 \geq 0$, $\gamma_0 + \gamma_1 \geq 0$, $\gamma_2 \geq 0$;

(II) $\alpha_{\bullet 0}+\alpha_{\bullet 1}\geq 0$, $\alpha_{\bullet 2}\geq 0$, $\alpha_{\bullet 1}>0$, $\gamma_0,\gamma_1,\gamma_2\geq 0$, $\beta_0+\beta_1\geq 0$, $\beta_2\geq 0$.

Suppose that $A_{10}(T)$ *is nonempty, and let* $P_{A_{10}(T)}$ *be the metric projection from* H *onto* $A_{10}(T)$. *Let* $\{\lambda_n\}$ *be a sequence of real numbers such that* $0<a\leq\lambda_n\leq b<1$ *for all* $n\in\mathbb{N}$, *where* $a,b\in\mathbb{R}$. *Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1}=\lambda_n x_n+(1-\lambda_n)\frac{1}{n}\sum_{k=0}^{n-1}T^k x_n\,(\in C)$$

for all $n\in\mathbb{N}$, *where* $x_1\in C$ *is given. Then, the sequence* $\{x_n\}$ *converges weakly to an element* $\overline{x}$ *of* $A_{10}(T)\cap A_{20}(T)\cap A_{21}(T)$, *where* $\overline{x}\equiv\lim_{n\to\infty}P_{A_{10}(T)}x_n$. *Additionally, if* C *is closed in* H, *then the sequence* $\{x_n\}$ *converges weakly to an element* $\widehat{x}$ *of* $F(T)$, *where* $\widehat{x}\equiv\lim_{n\to\infty}P_{F(T)}x_n$.

Proof. From Takahashi and Takeuchi [37], it is known that $A_{10}(T)$ is closed and convex. As it is assumed that $A_{10}(T)$ is nonempty, the metric projection $P_{A_{10}(T)}$ from H onto $A_{10}(T)$ exists. Define $Sx_n\equiv\frac{1}{n}\sum_{k=0}^{n-1}T^k x_n\,(\in C)$. Then, we have $x_{n+1}=\lambda_n x_n+(1-\lambda_n)Sx_n$. It is easy to show that

$$\|Sx_n-q\|\leq\|x_n-q\| \tag{4.1}$$

for all $q\in A_{10}(T)$ and $n\in\mathbb{N}$. Indeed, using $q\in A_{10}(T)$, we have

$$\begin{aligned}\|Sx_n-q\|&=\left\|\frac{1}{n}\sum_{k=0}^{n-1}T^k x_n-q\right\|=\frac{1}{n}\left\|\sum_{k=0}^{n-1}T^k x_n-nq\right\|\\&=\frac{1}{n}\left\|\sum_{k=0}^{n-1}\left(T^k x_n-q\right)\right\|\leq\frac{1}{n}\sum_{k=0}^{n-1}\left\|T^k x_n-q\right\|\\&\leq\frac{1}{n}\sum_{k=0}^{n-1}\|x_n-q\|=\|x_n-q\|.\end{aligned}$$

This result reveals that (4.1) holds. Using this, we can demonstrate that

$$\|x_{n+1}-q\|\leq\|x_n-q\| \tag{4.2}$$

for all $q\in A_{10}(T)$ and $n\in\mathbb{N}$. Indeed, it follows from (4.1) that

$$\begin{aligned}\|x_{n+1}-q\|&=\|\lambda_n x_n+(1-\lambda_n)Sx_n-q\|\\&=\|\lambda_n(x_n-q)+(1-\lambda_n)(Sx_n-q)\|\\&\leq\lambda_n\|x_n-q\|+(1-\lambda_n)\|Sx_n-q\|\\&\leq\lambda_n\|x_n-q\|+(1-\lambda_n)\|x_n-q\|\\&=\|x_n-q\|.\end{aligned}$$

The relationship (4.2) means that $\{\|x_n-q\|\}$ is monotone decreasing. Thus, $\{\|x_n-q\|\}$ is convergent in $\mathbb{R}$, and $\{x_n\}$ is bounded. From (4.2) and Lemma 2.1, $\left\{P_{A_{10}(T)}x_n\right\}$ is convergent in $A_{10}(T)$. We denote the limit point by $\overline{x}$, that is, $\overline{x}\equiv\lim_{n\to\infty}P_{A_{10}(T)}x_n$.

Next, we demonstrate that

$$\lambda_n (1-\lambda_n) \|x_n - Sx_n\|^2 \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \tag{4.3}$$

for all $q \in A_{10}(T)$ and $n \in \mathbb{N}$. Indeed, using Lemma 2.2 and (4.1), we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|\lambda_n (x_n - q) + (1-\lambda_n)(Sx_n - q)\|^2 \\
&= \lambda_n \|x_n - q\|^2 + (1-\lambda_n)\|Sx_n - q\|^2 - \lambda_n (1-\lambda_n)\|x_n - Sx_n\|^2 \\
&\le \lambda_n \|x_n - q\|^2 + (1-\lambda_n)\|x_n - q\|^2 - \lambda_n (1-\lambda_n)\|x_n - Sx_n\|^2 \\
&= \|x_n - q\|^2 - \lambda_n (1-\lambda_n)\|x_n - Sx_n\|^2,
\end{aligned}$$

which implies that (4.3) holds. As the sequence $\{\|x_n - q\|\}$ is convergent, from (4.3), we have that $x_n - Sx_n \to 0$. Our goal is to demonstrate that $x_n \rightharpoonup \overline{x} \left(\equiv \lim_{n\to\infty} P_{A_{10}(T)} x_n\right)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. As $\{x_{n_i}\}$ is bounded, a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and an element $u \in H$ exist such that $x_{n_j} \rightharpoonup u$. As $x_n - Sx_n \to 0$, it follows that $Sx_{n_j} \rightharpoonup u$. From Lemma 2.6, $u \in A_{10}(T)$. We prove that $u = \overline{x}$. It follows from $u \in A_{10}(T)$ that

$$\left\langle x_n - P_{A_{10}(T)} x_n,\ P_{A_{10}(T)} x_n - u \right\rangle \ge 0.$$

As $x_{n_j} \rightharpoonup u$ and $P_{A_{10}(T)} x_n \to \overline{x}$, we have $\langle u - \overline{x},\ \overline{x} - u\rangle \ge 0$. Therefore, we obtain $u = \overline{x}$. We have demonstrated that

$$x_n \rightharpoonup \overline{x} \left(\equiv \lim_{n\to\infty} P_{A_{10}(T)} x_n\right) \in A_{10}(T).$$

Because $A_{10}(T) \subset A_{20}(T) \cap A_{21}(T)$, we obtain $x_n \rightharpoonup \overline{x} \in A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)$.

Additionally, suppose that C is closed in H. In this case, $\{Sx_n\}$ is a sequence in C. As C is weakly closed and $Sx_n \rightharpoonup \overline{x}$, we have $\overline{x} \in C$. Therefore, $\overline{x} \in A_{10}(T) \cap C \subset F(T)$. Thus, $F(T)$ is nonempty. Under condition (I) or (II), the mapping T is quasi-nonexpansive. Thus, $F(T)$ is closed and convex. Hence, the metric projection $P_{F(T)}$ from H onto $F(T)$ exists. In the same way as the proof of (4.2), we can obtain $\|x_{n+1} - q\| \le \|x_n - q\|$ for all $q \in F(T)$ and $n \in \mathbb{N}$. It follows from Lemma 2.1 that $\{P_{F(T)} x_n\}$ converges strongly to an element $\widehat{x}$ of $F(T)$, that is, $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T)} x_n$. We show that

$$\overline{x} \left(= \lim_{n\to\infty} P_{A(T)} x_n\right) = \widehat{x} \left(= \lim_{n\to\infty} P_{F(T)} x_n\right).$$

As $\overline{x} \in F(T)$, it follows from a property of the metric projection that

$$\langle x_n - P_{F(T)} x_n,\ P_{F(T)} x_n - \overline{x}\rangle \ge 0$$

for all $n \in \mathbb{N}$. As $x_n \rightharpoonup \overline{x}$ and $P_{F(T)} x_n \to \widehat{x}$, we have $\langle \overline{x} - \widehat{x}, \widehat{x} - \overline{x}\rangle \ge 0$, which means that $\widehat{x} = \overline{x}$. This implies that $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(T)} x_n \in F(T)$. This completes the proof. □

The next theorem shows how to construct sequences that converge weakly to points of $A_{20}(T)$ and $F\left(T^2\right)$.

Theorem 4.2. *Let C be a nonempty and convex subset of H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that satisfies one of the following two conditions:*

(i) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{20} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$;

(ii) $\alpha_{00} + \alpha_{20} + \alpha_{02} + \alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{02} + \alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$.

Suppose that $A_{20}(T)$ is nonempty, and let $P_{A_{20}(T)}$ be the metric projection from H onto $A_{20}(T)$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < 1$ for all $n \in \mathbb{N}$, where $a, b \in \mathbb{R}$. Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n x_n + (1 - \lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^{2k} x_n \, (\in C)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges weakly to an element $\overline{x}$ of $A_{20}(T)$, where $\overline{x} \equiv \lim_{n\to\infty} P_{A_{20}(T)} x_n$. Additionally, if C is closed in H, then the sequence $\{x_n\}$ converges weakly to an element $\widehat{x}$ of $F(T^2)$, where $\widehat{x} \equiv \lim_{n\to\infty} P_{F(T^2)} x_n$.

Proof. As $A_{20}(T) = A_{10}(T^2)$, $A_{20}(T)$ is closed and convex. As $A_{20}(T) \neq \emptyset$ is assumed, the metric projection $P_{A_{20}(T)}$ from H onto $A_{20}(T)$ exists. Define $S' x_n \equiv \frac{1}{n} \sum_{k=0}^{n-1} T^{2k} x_n \, (\in C)$. Then, $x_{n+1} = \lambda_n x_n + (1 - \lambda_n) S' x_n$. It is easily ascertained that

$$\left\| S' x_n - q \right\| \leq \| x_n - q \| \tag{4.4}$$

for all $q \in A_{20}(T)$ and $n \in \mathbb{N}$. Indeed, as $q \in A_{20}(T) = A_{10}(T^2)$, it follows that

$$\begin{aligned}
\left\| S' x_n - q \right\| &= \left\| \frac{1}{n} \sum_{k=0}^{n-1} T^{2k} x_n - q \right\| = \frac{1}{n} \left\| \sum_{k=0}^{n-1} T^{2k} x_n - nq \right\| \\
&= \frac{1}{n} \left\| \sum_{k=0}^{n-1} (T^{2k} x_n - q) \right\| \leq \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^{2k} x_n - q \right\| \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \left\| T^{2(k-1)} x_n - q \right\| \leq \cdots \\
&\leq \frac{1}{n} \sum_{k=0}^{n-1} \| x_n - q \| = \| x_n - q \| .
\end{aligned}$$

Therefore, (4.4) holds, as claimed. Using (4.4), we have

$$\| x_{n+1} - q \| \leq \| x_n - q \| \tag{4.5}$$

for all $q \in A_{20}(T)$ and $n \in \mathbb{N}$. Indeed, it holds that

$$\begin{aligned}
\| x_{n+1} - q \| &= \left\| \lambda_n x_n + (1 - \lambda_n) S' x_n - q \right\| \\
&= \left\| \lambda_n (x_n - q) + (1 - \lambda_n) \left(S' x_n - q \right) \right\| \\
&\leq \lambda_n \| x_n - q \| + (1 - \lambda_n) \left\| S' x_n - q \right\|
\end{aligned}$$

$$\leq \lambda_n \|x_n - q\| + (1 - \lambda_n) \|x_n - q\|$$
$$= \|x_n - q\| .$$

Therefore, $\{\|x_n - q\|\}$ is convergent in $\mathbb{R}$, and $\{x_n\}$ is bounded. Furthermore, from (4.5) and Lemma 2.1, $\{P_{A_{20}(T)} x_n\}$ is convergent in $A_{20}(T)$. Denote the limit by $\overline{x}$, in other words, $\overline{x} \equiv \lim_{n\to\infty} P_{A_{20}(T)} x_n \in A_{20}(T)$.

Next, we verify that

$$\lambda_n (1 - \lambda_n) \left\|x_n - S' x_n\right\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \tag{4.6}$$

for all $q \in A_{20}(T)$ and $n \in \mathbb{N}$. Indeed, it follows from Lemma 2.2 and (4.4) that

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \left\|\lambda_n (x_n - q) + (1 - \lambda_n) \left(S' x_n - q\right)\right\|^2 \\
&= \lambda_n \|x_n - q\|^2 + (1 - \lambda_n) \left\|S' x_n - q\right\|^2 - \lambda_n (1 - \lambda_n) \left\|x_n - S' x_n\right\|^2 \\
&\leq \lambda_n \|x_n - q\|^2 + (1 - \lambda_n) \|x_n - q\|^2 - \lambda_n (1 - \lambda_n) \left\|x_n - S' x_n\right\|^2 \\
&= \|x_n - q\|^2 - \lambda_n (1 - \lambda_n) \left\|x_n - S' x_n\right\|^2 .
\end{aligned}$$

Thus, (4.6) holds, as claimed. As the sequence $\{\|x_n - q\|\}$ is convergent, from (4.6), it holds that $x_n - S' x_n \to 0$. Our aim is to show that $x_n \rightharpoonup \overline{x} \left(\equiv \lim_{n\to\infty} P_{A_{20}(T)} x_n\right)$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. As $\{x_{n_i}\}$ is bounded, a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and an element $u \in H$ exist such that $x_{n_j} \rightharpoonup u$. As $x_n - S' x_n \to 0$, we have $S' x_{n_j} \rightharpoonup u$. From Lemma 2.7, $u \in A_{20}(T)$. We prove that $u = \overline{x}$. As $u \in A_{20}(T)$, it follows that

$$\left\langle x_n - P_{A_{20}(T)} x_n,\ P_{A_{20}(T)} x_n - u \right\rangle \geq 0.$$

As $x_{n_j} \rightharpoonup u$ and $P_{A_{20}(T)} x_n \to \overline{x}$, we have $\langle u - \overline{x},\ \overline{x} - u\rangle \geq 0$. We obtain $u = \overline{x}$; hence, $x_n \rightharpoonup \overline{x} \left(\equiv \lim_{n\to\infty} P_{A_{20}(T)} x_n\right) \in A_{20}(T)$.

In addition to the other assumptions, suppose that C is closed in H. As $x_n \rightharpoonup \overline{x} \,(\in A_{20}(T))$, and C is weakly closed, it holds that

$$\overline{x} \in C \cap A_{20}(T) = C \cap A_{20}(T)\, A_{10}\left(T^2\right) \subset F\left(T^2\right).$$

Hence, $F\left(T^2\right)$ is nonempty. Under condition (i) or (ii), the mapping T^2 is quasi-nonexpansive. Consequently, $F\left(T^2\right)$ is closed and convex, and the metric projection $P_{F(T^2)}$ from H onto $F\left(T^2\right)$ exists. As in the proof of (4.4), it holds that

$$\left\|S' x_n - q\right\| \leq \|x_n - q\|$$

for all $q \in F\left(T^2\right)$ and $n \in \mathbb{N}$. Using this, we can obtain

$$\|x_{n+1} - q\| \leq \|x_n - q\|$$

for all $q \in F\left(T^2\right)$ and $n \in \mathbb{N}$ in the same way as (4.5). Consequently, from Lemma 2.1, $\{P_{F(T^2)} x_n\}$ converges strongly to an element $\widehat{x}$ of $F\left(T^2\right)$, that is, $\widehat{x} = \lim_{n\to\infty} P_{F(T^2)} x_n$. We prove that

$$\overline{x} \left(= \lim_{n\to\infty} P_{A_{20}(T)} x_n\right) = \widehat{x} \left(= \lim_{n\to\infty} P_{F(T^2)} x_n\right).$$

As $\overline{x} \in F(T^2)$, we have

$$\langle x_n - P_{F(T^2)}x_n,\ P_{F(T^2)}x_n - \overline{x}\rangle \geq 0$$

for all $n \in \mathbb{N}$. As $x_n \rightharpoonup \overline{x}$ and $P_{F(T^2)}x_n \to \widehat{x}$, we have $\langle \overline{x} - \widehat{x},\ \widehat{x} - \overline{x}\rangle \geq 0$; thus, $\widehat{x} = \overline{x}$. Therefore, $\{x_n\}$ converges weakly to $\widehat{x} = \lim_{n\to\infty} P_{F(T^2)}x_n$, which ends the proof. □

5. Halpern's type strong convergence theorems

This section presents strong convergence theorems for finding attractive and fixed points of a generic 2-generalized hybrid mapping. The proof has been developed in many studies (see [8, 9, 10, 17, 18, 19, 21, 26, 27, 32]).

Theorem 5.1. *Let C be a nonempty and convex subset of H. Let $T : C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i, j = 0, 1, 2)$-generic 2-generalized hybrid mapping that satisfies one of the following two conditions:*

(I) $\alpha_{0\bullet} + \alpha_{1\bullet} \geq 0,\ \alpha_{2\bullet} \geq 0,\ \alpha_{1\bullet} > 0,\ \beta_0, \beta_1, \beta_2 \geq 0,\ \gamma_0 + \gamma_1 \geq 0,\ \gamma_2 \geq 0;$
(II) $\alpha_{\bullet 0} + \alpha_{\bullet 1} \geq 0,\ \alpha_{\bullet 2} \geq 0,\ \alpha_{\bullet 1} > 0,\ \gamma_0, \gamma_1, \gamma_2 \geq 0,\ \beta_0 + \beta_1 \geq 0,\ \beta_2 \geq 0.$

Suppose that $A_{10}(T)$ is nonempty, and let $P_{A_{10}(T)}$ be the metric projection from H onto $A_{10}(T)$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0, 1)$ that satisfies $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{y_n\}$ be a sequence in C such that $y_n \to y\ (\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n y_n + (1 - \lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n\ (\in C)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges strongly to an element $\overline{y}$ of $A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)$, where $\overline{y} \equiv P_{A_{10}(T)}y$. Additionally, if C is closed in H, then $\{x_n\}$ converges strongly to a fixed point $\widehat{y} = P_{F(T)}y \in F(T)$, where $P_{F(T)}$ is the metric projection from H onto $F(T)$.

Proof. As $A_{10}(T)$ is a nonempty, closed, and convex subset of H, the metric projection $P_{A_{10}(T)}$ from H onto $A_{10}(T)$ exists. Define $Sx_n \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^k x_n\ (\in C)$. Then, we have $x_{n+1} = \lambda_n y_n + (1 - \lambda_n) Sx_n$. It can be proved that

$$\|Sx_n - q\| \leq \|x_n - q\| \tag{5.1}$$

for all $q \in A_{10}(T)$ and $n \in \mathbb{N}$ in the same way as the proof of (4.1).

Next, we prove that $\{x_n\}$ is bounded using mathematical induction. Let $q \in A_{10}(T)$, and define

$$M \equiv \max\left\{\sup_{n\in\mathbb{N}} \|y_n - q\|,\ \ \|x_1 - q\|\right\}.$$

As $\{y_n\}$ is bounded, M is a real number. We show that $\|x_n - q\| \leq M$ for all $n \in \mathbb{N}$. (i) It holds for $n = 1$. (ii) Assume that

$$\|x_k - q\| \leq M \tag{5.2}$$

for some $k \in \mathbb{N}$, where k is arbitrarily chosen. From (5.1) and (5.2), it follows that

$$\begin{aligned} \|x_{k+1} - q\| &= \|\lambda_k y_k + (1-\lambda_k) Sx_k - q\| \\ &\leq \lambda_k \|y_k - q\| + (1-\lambda_k)\|Sx_k - q\| \\ &\leq \lambda_k M + (1-\lambda_k)\|x_k - q\| \\ &\leq \lambda_k M + (1-\lambda_k) M = M. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded, as claimed. From (5.1), $\{Sx_n\}$ is also bounded. Consequently, it holds that

$$\begin{aligned} \|x_{n+1} - Sx_n\| &= \|\lambda_n y_n + (1-\lambda_n) Sx_n - Sx_n\| \\ &= \lambda_n \|y_n - Sx_n\| \to 0 \end{aligned} \tag{5.3}$$

as $n \to 0$ because $\lambda_n \to 0$ is assumed.

Define $X_n \equiv \|x_n - \overline{y}\|^2$, where $\overline{y} \equiv P_{A_{10}(T)}y$. Our aim is to prove that $X_n \to 0$. From (2.1) and (5.1), it holds that

$$\begin{aligned} X_{n+1} &\equiv \|x_{n+1} - \overline{y}\|^2 \\ &\equiv \|\lambda_n y_n + (1-\lambda_n) Sx_n - \overline{y}\|^2 \\ &= \|\lambda_n (y_n - \overline{y}) + (1-\lambda_n)(Sx_n - \overline{y})\|^2 \\ &= (1-\lambda_n)^2 \|Sx_n - \overline{y}\|^2 + 2\lambda_n \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle \\ &\leq (1-\lambda_n)\|x_n - \overline{y}\|^2 + 2\lambda_n \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle \\ &\equiv (1-\lambda_n) X_n + 2\lambda_n \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle. \end{aligned}$$

From Lemma 2.5, it suffices to demonstrate that

$$\lim\sup_{n\to\infty} \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle \leq 0.$$

From (5.3), it suffices to prove that $\limsup_{n\to\infty} \langle y_n - \overline{y},\ Sx_n - \overline{y}\rangle \leq 0$. As $\{y_n\}$ and $\{Sx_n\}$ are bounded, we assume, without loss of generality, that there are subsequences $\{y_{n_i}\}$ of $\{y_n\}$ and $\{Sx_{n_i}\}$ of $\{Sx_n\}$ such that

$$\limsup_{n\to\infty} \langle y_n - \overline{y},\ Sx_n - \overline{y}\rangle = \lim_{i\to\infty} \langle y_{n_i} - \overline{y},\ Sx_{n_i} - \overline{y}\rangle$$

and $Sx_{n_i} \rightharpoonup u$ for some $u \in H$. From Lemma 2.6, $u \in A_{10}(T)$. As $y_n \to y$ and $\overline{y} \equiv P_{A_{10}(T)}y$, we obtain

$$\begin{aligned} \limsup_{n\to\infty} \langle y_n - \overline{y},\ Sx_n - \overline{y}\rangle &= \lim_{i\to\infty} \langle y_{n_i} - \overline{y},\ Sx_{n_i} - \overline{y}\rangle \\ &= \langle y - \overline{y},\ u - \overline{y}\rangle \leq 0. \end{aligned}$$

Thus, we obtain $X_n \to 0$, and equivalently, $x_n \to \overline{y}$. As $\overline{y} \equiv P_{A_{10}(T)}y \in A_{10}(T)$ and $A_{10}(T) \subset A_{20}(T) \cap A_{21}(T)$, we obtain $\overline{y} \in A_{10}(T) \cap A_{20}(T) \cap A_{21}(T)$.

Additionally, suppose that C is closed in H. We demonstrate that $x_n \to \widehat{y}\left(\equiv P_{F(T)}y\right)$. Because $x_n \to \overline{y} \equiv P_{A_{10}(T)}y$ and C is closed, it holds that $\overline{y} \in C \cap P_{A_{10}(T)}$. Thus, $\overline{y} \in F(T)$; hence, $F(T) \neq \emptyset$. Because T is quasi-nonexpansive, $F(T)$ is closed and convex. Consequently, the metric projection $P_{F(T)}$ from H onto $F(T)$ exists. We prove that $(\widehat{y} \equiv) P_{F(T)}y = \overline{y}\left(\equiv P_{A_{10}(T)}y\right)$. Because $\overline{y} \in F(T)$, it suffices to demonstrate that $\|y-\overline{y}\| \leq \|y-v\|$ for all $v \in F(T)$. Let $v \in F(T)$. As T is quasi-nonexpansive, it holds that $F(T) \subset A_{10}(T)$. Thus, we have

$$\begin{aligned} \|y-\overline{y}\| &= \inf\{\|y-q\| : q \in A_{10}(T)\} \\ &\leq \inf\{\|y-q\| : q \in F(T)\} \\ &\leq \|y-v\|. \end{aligned}$$

This result means that $\overline{y} = P_{F(T)}y \, (\equiv \widehat{y})$, which completes the proof. □

The following is a strong convergence theorem that approximate points of $A_{20}(T)$ and $F(T^2)$.

Theorem 5.2. *Let C be a nonempty and convex subset of H. Let $T: C \to C$ be an $(\alpha_{ij}, \beta_i, \gamma_i;\ i,j=0,1,2)$-generic 2-generalized hybrid mapping that satisfies one of the following two conditions:*

(i) $\alpha_{00}+\alpha_{20}+\alpha_{02}+\alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{20}+\alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$;

(ii) $\alpha_{00}+\alpha_{20}+\alpha_{02}+\alpha_{22} \geq 0$, $\alpha_{10}, \alpha_{11}, \alpha_{12}, \alpha_{01}, \alpha_{21} \geq 0$, $\alpha_{02}+\alpha_{22} > 0$, $\beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2 \geq 0$.

Suppose that $A_{20}(T)$ is nonempty, and let $P_{A_{20}(T)}$ be the metric projection from H onto $A_{20}(T)$. Let $\{\lambda_n\}$ be a sequence of real numbers in the interval $[0,1)$ that satisfies $\lambda_n \to 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Let $\{y_n\}$ be a sequence in C such that $y_n \to y \, (\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$x_{n+1} = \lambda_n y_n + (1-\lambda_n) \frac{1}{n} \sum_{k=0}^{n-1} T^{2k} x_n \, (\in C)$$

for all $n \in \mathbb{N}$, where $x_1 \in C$ is given. Then, the sequence $\{x_n\}$ converges strongly to an element $\overline{y}$ of $A_{20}(T)$, where $\overline{y} \equiv P_{A_{20}(T)}y$. Additionally, if C is closed in H, then $\{x_n\}$ converges strongly to a fixed point $\widehat{y} = P_{F(T^2)}y \in F(T^2)$, where $P_{F(T^2)}$ is the metric projection from H onto $F(T^2)$.

Proof. As $A_{20}(T) = A_{10}(T^2)$, $A_{20}(T)$ is closed and convex. From the assumption $A_{20}(T) \neq \emptyset$, the metric projection $P_{A_{20}(T)}$ from H onto $A_{20}(T)$ exists. Define $S'x_n \equiv \frac{1}{n}\sum_{k=0}^{n-1} T^{2k}x_n \, (\in C)$. Then, $x_{n+1} = \lambda_n y_n + (1-\lambda_n) S'x_n$. We can demonstrate that

$$\|S'x_n - q\| \leq \|x_n - q\| \tag{5.4}$$

for all $q \in A_{20}(T)$ and $n \in \mathbb{N}$ in the same way as the proof of (4.4).

We prove that $\{x_n\}$ is bounded. For this aim, we use mathematical induction. Taking $q \in A_{20}(T)$ arbitrarily, we define

$$M \equiv \max\left\{\sup_{n\in\mathbb{N}} \|y_n - q\|,\ \|x_1 - q\|\right\}.$$

As $\{y_n\}$ is bounded, M is a real number. We show that $\|x_n - q\| \leq M$ for all $n \in \mathbb{N}$. (i) It is true for $n=1$. (ii) Assume that

$$\|x_k - q\| \leq M \tag{5.5}$$

for $k \in \mathbb{N}$. From (5.4) and (5.5), we have

$$\begin{aligned}\|x_{k+1}-q\| &= \left\|\lambda_k y_k + (1-\lambda_k) S' x_k - q\right\| \\ &\le \lambda_k \|y_k - q\| + (1-\lambda_k)\left\|S' x_k - q\right\| \\ &\le \lambda_k M + (1-\lambda_k)\|x_k - q\| \\ &\le \lambda_k M + (1-\lambda_k) M = M.\end{aligned}$$

This shows that $\{x_n\}$ is bounded, as claimed. From (5.4), $\{S'x_n\}$ is also bounded. Using this, we have

$$\begin{aligned}\left\|x_{n+1}-S'x_n\right\| &= \left\|\lambda_n y_n + (1-\lambda_n) S' x_n - S' x_n\right\| \\ &= \lambda_n \left\|y_n - S' x_n\right\| \to 0\end{aligned} \tag{5.6}$$

as $n \to 0$.

Denote $X_n \equiv \|x_n - \overline{y}\|^2$, where $\overline{y} \equiv P_{A_{20}(T)} y$. We prove that $X_n \to 0$. From (2.1) and (5.4), it holds that

$$\begin{aligned}X_{n+1} &\equiv \|x_{n+1}-\overline{y}\|^2 \\ &\equiv \left\|\lambda_n y_n + (1-\lambda_n) S' x_n - \overline{y}\right\|^2 \\ &= \left\|\lambda_n (y_n - \overline{y}) + (1-\lambda_n)\left(S' x_n - \overline{y}\right)\right\|^2 \\ &= (1-\lambda_n)^2 \left\|S' x_n - \overline{y}\right\|^2 + 2\lambda_n \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle \\ &\le (1-\lambda_n)\|x_n - \overline{y}\|^2 + 2\lambda_n \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle \\ &\equiv (1-\lambda_n) X_n + 2\lambda_n \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle .\end{aligned}$$

From Lemma 2.5, it suffices to demonstrate that

$$\limsup_{n\to\infty} \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle \le 0.$$

From (5.3), it suffices to show that

$$\limsup_{n\to\infty} \langle y_n - \overline{y},\ x_{n+1} - \overline{y}\rangle = \limsup_{n\to\infty} \left\langle y_n - \overline{y},\ S' x_n - \overline{y}\right\rangle \le 0.$$

As $\{y_n\}$ and $\{S'x_n\}$ are bounded, we assume, without loss of generality, that subsequences $\{y_{n_i}\}$ of $\{y_n\}$ and $\{S'x_{n_i}\}$ of $\{S'x_n\}$ exist such that

$$\limsup_{n\to\infty} \left\langle y_n - \overline{y},\ S' x_n - \overline{y}\right\rangle = \lim_{i\to\infty} \left\langle y_{n_i} - \overline{y},\ S' x_{n_i} - \overline{y}\right\rangle$$

and $S'x_{n_i} \rightharpoonup u$ for some $u \in H$. From Lemma 2.7, $u \in A_{20}(T)$. Because $y_n \to y$, $S'x_{n_i} \rightharpoonup u$, and $\overline{y} \equiv P_{A_{20}(T)} y$, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \left\langle y_n - \overline{y},\ S' x_n - \overline{y}\right\rangle &= \lim_{i\to\infty} \left\langle y_{n_i} - \overline{y},\ S' x_{n_i} - \overline{y}\right\rangle \\ &= \langle y - \overline{y},\ u - \overline{y}\rangle \le 0.\end{aligned}$$

Therefore, it holds that $X_n \to 0$, which means that $x_n \to \overline{y}$.

Additionally, suppose that C is closed in H. Our next purpose is to prove that $x_n \to \widehat{y}\left(\equiv P_{F(T^2)} y\right)$. Because $x_n \to \overline{y} \equiv P_{A_{20}(T)} y$ and C is closed, it holds that $\overline{y} \in C \cap P_{A_{20}(T)} \subset F\left(T^2\right)$. Consequently, $F\left(T^2\right) \neq \emptyset$. Under condition (i) or (ii), T^2 is quasi-nonexpansive. Therefore, $F\left(T^2\right)$ is closed and convex, and the metric projection $P_{F(T^2)}$ from H onto $F\left(T^2\right)$ exists. We prove that $(\widehat{y} \equiv) P_{F(T^2)}y = \overline{y}\left(\equiv P_{A_{20}(T)}y\right)$. As $\overline{y} \in F\left(T^2\right)$, it suffices to demonstrate that $\|y-\overline{y}\| \leq \|y-v\|$ for all $v \in F\left(T^2\right)$. Choose $v \in F\left(T^2\right)$ arbitrarily. As T^2 is quasi-nonexpansive, it follows that $F\left(T^2\right) \subset A_{20}(T)$. Thus, we have

$$\begin{aligned} \|y-\overline{y}\| &= \inf\{\|y-q\| : q \in A_{20}(T)\} \\ &\leq \inf\left\{\|y-q\| : q \in F\left(T^2\right)\right\} \\ &\leq \|y-v\|. \end{aligned}$$

This implies that $\overline{y} = P_{F(T^2)}y\,(\equiv \widehat{y})$. Thus, the proof is complete. □

Acknowledgements

This study is supported by the Ryousui Gakujutsu Foundation of Shiga University.

References

[1] K. Aoyama, S. Iemoto, F. Kohsaka and W. Takahashi, *Fixed point and ergodic theorems for λ-hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 335–343.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350–2360.

[3] S. Atsushiba and W. Takahashi, *Approximating common fixed points of two nonexpansive mappings in Banach spaces*, Bull. Austral. Math. Soc. **57** (1998), 117–127.

[4] J. B. Baillon, *Un Théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Ser. A–B **280** (1975), 1511–1514.

[5] S.-M. Guu, and W. Takahashi, *Existence and approximation of attractive points of the widely more generalized hybrid mappings in Hilbert spaces*, Abstract and Applied Analysis **2013** (2013): Article ID 904164, 10 pages.

[6] B. Halpern, *Fixed points of nonexpanding maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[7] K. Hasegawa, Ken, T. Komiya and W. Takahashi, *Fixed point theorems for general contractive mappings in metric spaces and estimating expressions*, Sci. Math. Jponicae **74** (2011), 15–27.

[8] M. Hojo, A. Kondo and W. Takahashi, *Weak and strong convergence theorems for commutative normally 2-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **4** (2018), 117–134.

[9] M. Hojo and W. Takahashi, *Weak and strong convergence theorems for generalized hybrid mappings in Hilbert spaces*, Scientiae Math. Japonicae **73** (2011), 31–40.

[10] M. Hojo and W. Takahashi, *Halpern-type strong convergence theorem for widely more generalized hybrid mappings in Hilbert spaces and applications*, J. Nonlinear Convex Anal. **16** (2015), 1761–1771.

[11] M. Hojo, W. Takahashi and I. Termwuttipong, *Strong convergence theorems for 2-generalized hybrid mappings in Hilbert spaces*, Nonlinear Anal. **75** (2012), 2166–2176.

[12] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[13] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), 493–508.

[14] P. Kocourek, W. Takahashi and J. C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert Spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.
[15] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.
[16] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. **91** (2008), 166–177.
[17] H. Komiya and W. Takahashi, *Strong Convergence Theorems for an Infinite Family of Demimetric Mappings in a Hilbert Space*, J. Convex Anal. **24** (2017), 1357–1373.
[18] A. Kondo, *Strong approximation using hybrid methods to find common fixed points of noncommutative nonlinear mappings in Hilbert spaces*, to appear in J. Nonlinear Convex Anal.
[19] A. Kondo, *Mean convergence theorems using hybrid methods to find common fixed points of noncommutative nonlinear mappings in Hilbert spaces*, J. Appl. Math. Comput., published online, doi: https://doi.org/10.1007/s12190-021-01527-8
[20] A. Kondo and W. Takahashi, *Attractive point and weak convergence theorems for normally N-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **3** (2017), 297–310.
[21] A. Kondo and W. Takahashi, *Strong convergence theorems of Halpern's type for normally 2-generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **19** (2018), 617–631.
[22] A. Kondo and W. Takahashi, *Attractive point and nonlinear ergodic theorems for generic 2-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **5** (2019), 87-103.
[23] A. Kondo and W. Takahashi, *Approximation of a common attractive point of noncommutative normally 2-generalized hybrid mappings in Hilbert spaces*, Linear Nonlinear Anal. **5** (2019), 279–297.
[24] A. Kondo and W. Takahashi, *Attractive Point, Weak and Strong Convergence Theorems for Generic 2-Generalized Hybrid Mappings in Hilbert Spaces*, Linear Nonlinear Anal. **6** (2020), 103–133.
[25] A. Kondo and W. Takahashi, *Weak convergence theorems to common attractive points of normally 2-generalized hybrid mappings with errors*, J. Nonlinear Convex Anal. **21** (2020), 2549–2570.
[26] A. Kondo and W. Takahashi, *Strong convergence theorems for finding common attractive points of normally 2-generalized hybrid mappings and applications*, Linear Nonlinear Anal. **6** (2020), 421–438.
[27] Y. Kurokawa and W. Takahashi, *Weak and strong convergence theorems for nonspreading mappings in Hilbert spaces*, Nonlinear Anal. **73** (2010), 1562–1568.
[28] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[29] T. Maruyama, W. Takahashi and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 185–197.
[30] T. Shimizu and W. Takahashi, *Strong convergence theorem for asymptotically nonexpansive mappings*, Nonlinear Anal. **26** (1996), 265–272.
[31] T. Shimizu and W. Takahashi, *Strong convergence to common fixed points of families of nonexpansive mappings*, J. Math. Anal. Appl. **211** (1997), 71–83.
[32] T. Suzuki, *A sufficient and necessary condition for Halpern-type strong convergence to fixed points of nonexpansive mappings*, Proc. Amer. Math. Soc., **135** (2007), 99–106.
[33] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc. **81** (1981), 253–256.
[34] W. Takahashi, *Nonlinear Functional Analysis. Fixed Point Theory and its Applications*, Yokohama Publishes, Yokohama, 2000.
[35] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishes, Yokohama, 2009.
[36] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[37] W. Takahashi and Y. Takeuchi, *Nonlinear ergodic theorem without convexity for generalized hybrid mappings in a Hilbert space*, J. Nonlinear Convex Anal. **12** (2011), 399–406.

[38] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.
[39] W. Takahashi, N. C. Wong, J. C. Yao, *Attractive point and weak convergence theorems for new generalized hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **13** (2012), 745–757.
[40] W. Takahashi, N. C. Wong, J. C. Yao, *Attractive points and Halpern-type strong convergence theorems in Hilbert spaces*, J. Fixed Point Theory Appl. **17** (2015), 301–311.
[41] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.
[42] H. K. Xu, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109–113.

Manuscript received 18 March 2021
revised 27 November 2021

Atsumasa Kondo
Department of Economics, Shiga University, Banba 1-1-1, Hikone, Shiga 522-0069, Japan
E-mail address: a-kondo@biwako.shiga-u.ac.jp