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TWO TYPES FIXED POINTS OF SET–VALUED MAPPINGS AND ITERATIONS WITH ALLOWABLE RANGES

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Dedicated to Professor Hidetoshi Komiya on the occasion of his 65th birthday as a mark of longstanding friendship

Abstract. In this note, from a new perspective, we introduce some concepts related to fixed points of set-valued mappings. By considering them, we revisit existing results and present new results for set-valued mappings. Specifically, we study not only fixed points but also intrinsic fixed points of set-valued mappings. Then, under suitable conditions, we find such fixed points by using some iterations with allowable ranges.

1. Introduction and preliminaries

In 1969, Nadler [7] proved the following theorem.

Theorem 1.1. *Let (X, d) be a complete metric space and T be a mapping from X into the class* $\mathrm{CB}(X)$ *of all nonempty closed bounded subsets of X. Assume that there is $r \in [0, 1)$ satisfying the following*:

(Ns) $$H(Tx, Ty) \le rd(x, y) \quad \text{for all} \quad x, y \in X,$$

where H is the Hausdorff metric. Then, there is $z \in X$ satisfying $z \in Tz$.

In 1989, Mizoguchi–Takahashi [6] proved a generalization of Theorem 1.1 as a partial answer of Problem 9 in Reich [9]. After the remarkable works, many researches appeared in this study area. For example, some extensions of the Banach contraction principle are translated to assertions about set-valued mappings; see, for instance, Du and co–authors [3] and its references.

Inspired by the works, we present some concepts related to fixed points of set–valued mappings from a new perspective. Then, by considering them, we reconsider existing results and present new results for set-valued mappings.

In advance, we prepare some notations, some concepts and two lemmas as it is needed in our study. Then, sometimes we use them without notice.

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N and R denote the set of all positive integers and the set of all real numbers, respectively. (X, d) denotes a metric space and 2^X denotes the class of all subsets of X. For a subset C of X, $\overline{C}$ denotes the closure of C. Avoiding confusions, we denote by $\{x\}^s$ the set which consists of only one point $x \in X$.

Let T be a mapping from X into 2^X. Then, T is called a set-valued mapping from X into itself. A point $z \in X$ is called a fixed point of T if $z \in Tz$. In this note, a fixed point $z \in X$ of T satisfying $Tz = \{z\}^s$ is called an intrinsic fixed point of T. Then, $F(T)$ and $F_I(T)$ denote the set of all fixed points of T and the set of all intrinsic fixed points of T, respectively. Depending on how T is determined, every $x \in X$ may be a fixed point of T. In such cases, an intrinsic fixed point of T is often important. For reference, we present a trivial assertion which is derived from the Banach contraction principle.

Assertion 1.2. *Let (X, d) be a metric space and S be a contraction on X in the sense of Banach. Define a mapping T from X into 2^X by*

$$Tx = \overline{\{S^{n-1}x : n \in N\}} \quad \text{for each} \quad x \in X.$$

Suppose Tx is compact for all $x \in X$. Then, there is the unique intrinsic fixed point z of T. Of course, $Tz = \{z\}^s = \{Sz\}^s$.

Remark. In this assertion, Tx is compact for all $x \in X$ if X is complete.

Let $u \in X$ and $C \in 2^X$. For simplicity, we assume that C is non–empty. Set $d(u, C) = \inf_{x \in C} d(u, x)$. Then, $d(u, C)$ is called the distance from u to C. $\mathrm{CB}(X)$ denotes the class of all non–empty closed bounded subsets of X. For each $A, B \in \mathrm{CB}(X)$, define $H(A, B)$ by

$$H(A, B) = \max\big\{\sup\{d(x, B) : x \in A\}, \sup\{d(y, A) : y \in B\}\big\}.$$

Since both A and B are non–empty and bounded, $H(A, B) \in [0, \infty)$ is immediate. Furthermore, H is a metric on $\mathrm{CB}(X)$; this fact will be present later as Lemma 1.4. H is called the Hausdorff metric on $\mathrm{CB}(X)$ with respect to d.

Let (X, d) be a complete metric space and T be a mapping from X into $\mathrm{CB}(X)$. Then, we consider to find a fixed point of T. Some researchers presented iterative sequences which converge strongly to a fixed point of T under the conditions they had set. For $u \in X$, we do not know whether there is $v \in C$ satisfying $d(u, v) = d(u, C)$ even if $C \in \mathrm{CB}(X)$. This fact may cause some difficulties for our problem. Furthermore, when we consider a corresponding numerical calculation procedure, some more difficulties may appear. Then, to capture such situations in a reasonable way, we will briefly explain the concept of allowable ranges of approximation methods presented in Takeuchi [14].

Let $z_1 \in X$ and $z_2 \in Tz_1$. Observing existing results, we see the following: Under their assumptions, it is relatively easy to check that $\{z_n\}$ converges strongly to some $z_* \in Tz_*$ if we can generate a sequence $\{z_n\}$ in X such that

$$z_{n+1} \in Tz_n, \quad z_{n+1} \neq z_n, \quad d(z_{n+1}, z_{n+2}) = d(z_{n+1}, Tz_{n+1})$$

for each $n \in N$. Note that $z_n \in Tz_n$ is derived from $z_{n+1} \in Tz_n$ if $z_{n+1} = z_n$.

We now consider a corresponding numerical calculation procedure and errors caused by the procedure and a selected computer. Let $z_1 = y_1 = x_1 \in X$ and $z_2 = y_2 = x_2 \in Tz_1$. Then, we face difficulties as below:

- In general, we do not know whether z_3 as above exists.
- It may not be easy to calculate z_3 exactly even if z_3 exists.

So, by actual restrictions, we merely get x_3 which is slightly different from $z_3 (= y_3)$ even if z_3 exists. We cannot get z_4 by using z_3 because we only have x_3. Then, by using x_3, we try to get $y_4 \in X$ such that $y_4 \in Tx_3$ and $d(y_4, x_3) = d(x_3, Tx_3)$. However, again we merely get $x_4 \in X$ which is slightly different from y_4 even if y_4 exists. In addition, we know neither the size of $d(x_4, y_4)$ nor the size of $d(x_4, z_4)$ even if y_4 and z_4 exist.

Then, in this way, we can only get a sequence $\{x_n\}$ practically. Sequences $\{z_n\}$ and $\{y_n\}$ are just imaginary. Then, we face again a difficulty whether $\{x_n\}$ converges strongly. So, it is not guaranteed that $\{x_n\}$ converges strongly even if $\{z_n\}$ as above exists and converges strongly. From these reasons, we consider allowable ranges in the sense of Takeuchi [14].

In this context, an allowable range A_n for step n is a subset of X associated with the procedure. Then, in theory, the sequence $\{x_n\}$ which consists of $x_n \in A_n$ is required to converge strongly to some $x_* \in X$ satisfying $x_* \in Tx_*$. In general, we cannot get $\{A_n\}$ in advance, because usually A_{n+1} depends on x_n and A_n. Suppose we cannot get $x_{n_0+1} \in A_{n_0+1}$ by actual restrictions. Then, the procedure will be stopped. For example, the procedure has to be stopped if the size of A_{n_0+1} is smaller than the size of error caused by our equipment. Nevertheless, since $\{d(x_n, x_*)\}$ converges to 0 in theory, we can assume that we are on the right track until step n_0. So, for the procedure, we may consider that x_{n_0} is a best approximate point of $x_* \in F(T)$ even if $d(x_{n_0}, x_*)$ is unknown.

Finally, we show the following well–known lemmas without proofs.

Lemma 1.3. *Let (X, d) be a metric space and let T be a mapping from X into 2^X. Suppose $z \in X$ satisfy $Tz \neq \emptyset$. Then, the following holds*:

$$|d(u, Tz) - d(v, Tz)| \leq d(u, v) \quad \text{for any} \quad u, v \in X.$$

Lemma 1.4. *Let (X, d) be a metric space. Then, so is* $(\mathrm{CB}(X), H)$.

2. Some fixed point theorems for set–valued mappings

Let a be a function from $[0, \infty)$ into $[0, 1)$ satisfying $\limsup_{s \to t+0} a(s) < 1$ for all $t \in [0, \infty)$. The expression $\limsup_{s \to t+0} a(s) < 1$ is a little difficult to make sense of, so it might be better to use $\lim_{\varepsilon \to 0} \sup_{s \in (t, t+\varepsilon]} a(s) < 1$. Of course, $\varepsilon > 0$. Let $c \in (0, 1)$ and define a function b_c from $[0, \infty)$ into $(0, 1)$ by $b_c(t) = c \times 1 + (1 - c) a(t)$ for each $t \in [0, \infty)$. Then the following are immediate:

- $a(t) < b_c(t)$ for all $t \in [0, \infty)$.
- $\limsup_{s \to t+0} a(s) \leq \limsup_{s \to t+0} b_c(s) < 1$ for all $t \in [0, \infty)$.

For simplicity, we use $b = b_{\frac{1}{2}}$, that is, $b(t) = \frac{1}{2}(1 + a(t))$ for each $t \in [0, \infty)$.

We show a version of the Mizoguchi–Takahashi's theorem. We note that the following proof is essentially due to Suzuki [10].

Theorem 2.1. *Let (X, d) be a complete metric space and T be a mapping from X into $\mathrm{CB}(X)$. Let a be a function from $[0,\infty)$ into $[0,1)$ satisfying $\limsup_{s\to t+0} a(s) < 1$ for all $t \in [0,\infty)$. Assume*

$$\text{(MT)} \qquad H(Tx, Ty) \le a(d(x, y))d(x, y) \quad \text{for all} \quad x, y \in X.$$

Let b be the function as mentioned above. Let $x_1 \in X = A_1$ and $A_2 = Tx_1$. For each $n \in N$, generate x_{n+1} and A_{n+2} by the following procedure:

(i) $x_{n+1} \in A_{n+1}$.
(ii) *This procedure will be stopped if $x_{n+1} = x_n$.*
(iii) *A_{n+2} is the set which consists of $y \in Tx_{n+1}$ satisfying*

$$d(x_{n+1}, Tx_{n+1}) \le d(x_{n+1}, y) \le b(d(x_n, x_{n+1}))d(x_n, x_{n+1}).$$

Then the following hold:

(a) *There is $l \in N$ satisfying $x_l \in F(T)$, if the procedure stops.*
(b) *There is $u \in F(T)$ such that $\{x_n\}$ converges strongly to u, if the procedure does not stop.*

Proof. We know $A_2 = Tx_1 \ne \emptyset$. Then, we can choose an $x_2 \in A_2 \subset Tx_1$. Suppose $x_2 \ne x_1$. Then, by $x_2 \in Tx_1$, (MT) and the definition of H, we see

$$\begin{aligned} d(x_2, Tx_2) &\le \sup\{d(z, Tx_2) : z \in Tx_1\} \le H(Tx_1, Tx_2) \\ &\le a(d(x_1, x_2))d(x_1, x_2) < b(d(x_1, x_2))d(x_1, x_2) < d(x_1, x_2). \end{aligned}$$

This is summarized as below:

$$d(x_2, Tx_2) < b(d(x_1, x_2))d(x_1, x_2) < d(x_1, x_2).$$

By $d(x_2, Tx_2) = \inf_{y\in Tx_2} d(x_2, y)$, this implies $A_3 \ne \emptyset$. That is, we can choose an $x_3 \in A_3 \subset Tx_2$. Then, in this way, x_{n+1}, Tx_{n+1} and A_{n+2} can be generated until $l \in N$ satisfying $x_{l+1} = x_l$ appears.

We show (b). Suppose $x_{n+1} \ne x_n$ for all $n \in N$. Then, by the argument so far, we have $\{x_n\}$, $\{Tx_n\}$ and $\{A_n\}$. Also, we know the following:

$$\begin{aligned} &(1) \quad x_{n+1} \in A_{n+1} \subset Tx_n, \quad x_{n+1} \ne x_n, \quad d(x_{n+1}, Tx_{n+1}) \le d(x_{n+1}, x_{n+2}), \\ &\qquad d(x_{n+1}, x_{n+2}) \le b(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < d(x_n, x_{n+1}) \ \text{ for all } n \in N. \end{aligned}$$

Then, $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence in $[0,\infty)$. So, $\{d(x_n, x_{n+1})\}$ converges to some $\tau \in [0,\infty)$. Since $\limsup_{s\to\tau+0} b(s) < 1$ and $b(\tau) \in (0, 1)$, there are $r \in (0, 1)$ and $\varepsilon \in (0,\infty)$ such that $b(t) < r$ for all $t \in [\tau, \tau + \varepsilon]$. Furthermore, there is $n_0 \in N$ such that $d(x_n, x_{n+1}) \in [\tau, \tau + \varepsilon]$ for all $n \ge n_0$. Then, $b(d(x_n, x_{n+1})) < r$ for all $n \ge n_0$. So, for all $n \ge n_0$,

$$d(x_{n+1}, x_{n+2}) \le b(d(x_n, x_{n+1}))d(x_n, x_{n+1}) < rd(x_n, x_{n+1}).$$

By $r \in [0, 1)$, we know $\lim_m \frac{r^m}{1-r} = 0$. Also, for all $m, k \in N$, we see

$$d(x_{n_0+m}, x_{n_0+m+k}) \le \textstyle\sum_{j=m}^{m+k-1} d(x_{n_0+j}, x_{n_0+j+1})$$

$$< \sum_{j=m}^{m+k-1} r^j d(x_{n_0}, x_{n_0+1}) < \frac{r^m}{1-r} d(x_{n_0}, x_{n_0+1}).$$

These imply that $\{x_n\}$ is a Cauchy sequence. Then, since X is complete, $\{x_n\}$ converges to some $u \in X$. To complete the proof of (b), we show $u \in Tu$.

By Lemma 1.3, $|d(u,Tu) - d(x_n,Tu)| \leq d(x_n,u)$. Then, by $\lim_n d(x_n,u) = 0$, we see $d(u,Tu) = \lim_n d(x_n,Tu)$. By $x_{n+1} \in Tx_n$, (MT), and the definition of H,

$$\begin{aligned} d(u,Tu) &= \lim_n d(x_{n+1},Tu) \leq \lim_n H(Tx_n,Tu) \\ &\leq \lim_n b(d(x_n,u))d(x_n,u) \leq \lim_n d(x_n,u) = 0. \end{aligned}$$

Thus, since Tu is closed, we see $u \in Tu$.

We show (a). Suppose A_{l+1} was generated and $x_{l+1} = x_l$. Then, we immediately see that $x_l = x_{l+1} \in A_{l+1} \subset Tx_l$. □

Remark 2.2. Refer to Theorem 2.1. Suppose we can easily confirm whether $x_n \in Tx_n$ or not. In this case, we may stop the procedure when $l \in N$ satisfying $x_l \in Tx_l$ appears. Of course, $x_{n+1} = x_n$ implies $x_n \in Tx_n$. It may not be easy to check whether $x_n \in Tx_n$ if x_n is close to the boundary of Tx_n.

In Mizoguchi–Takahashi's original theorem, the domain of a is $(0,\infty)$, and (MT) holds for all $x,y \in X$ with $x \neq y$. However, we may consider the domain of a as $[0,\infty)$ by setting $a(0) = t_0 \in [0,1)$, and then (MT) holds for all $x,y \in X$ because $d(x,y) = 0$ implies $H(Tx,Ty) = 0$. Also, they assumed $\limsup_{s\to t+0} a(s) < 1$ for all $t \in [0,\infty)$ replacing $(0,\infty)$ in Problem 9 by $[0,\infty)$. Therefore, their theorem is a partial answer of Problem 9 in Reich [9], however, it is an almost complete answer. The original proof of Mizoguchi–Takahashi's theorem is not simple. Another proof due to Duffer–Kaneko [2] is not yet simple. Then, Suzuki replaced a by b and regarded (MT) as the following:

(MT′) $\quad H(Tx,Ty) < b(d(x,y))d(x,y) \quad$ for all $x,y \in X$ with $x \neq y$.

The simple idea of using b to create the small gap is main point of his proof.

A typical example of a in Theorem 2.1 is a monotonically non-decreasing (non-increasing) function from $[0,\infty)$ to $[0,1)$. Let $r \in [0,1)$ and a be the mapping from $[0,\infty)$ to $[0,1)$ such that $a(s) = r$ for all $s \in [0,\infty)$. Choose such an a in Theorem 2.1. Then, we have Theorem 1.1 due to Nadler.

Also, we show a version of Kannan's theorem [5] for a set–valued mapping.

Theorem 2.3. *Let (X,d) be a complete metric space and T be a mapping from X into* CB(X). *Suppose there are $r,s \in [0,1)$ satisfying $r+s \in [0,1)$ and*

(Ks) $\quad H(Tx,Ty) \leq rd(x,Tx) + sd(y,Ty) \quad$ for all $\quad x,y \in X$.

Set $\delta = \frac{1}{2}(1 + \frac{r}{1-s}) \in (\frac{r}{1-s}, 1)$. Let $x_1 \in X = A_1$ and $A_2 = Tx_1$. For each $n \in N$, generate x_{n+1} and A_{n+2} by the following procedure:

(i) $x_{n+1} \in A_{n+1}$.
(ii) *This procedure will be stopped if $x_{n+1} = x_n$.*
(iii) $A_{n+2} = \{y \in Tx_{n+1} : d(x_{n+1},Tx_{n+1}) \leq d(x_{n+1},y) \leq \delta d(x_n,x_{n+1})\}$.

Then the following hold:

(a) *There is $l \in N$ satisfying $x_l \in F(T)$, if the procedure stops.*
(b) *There is $u \in F(T)$ such that $\{x_n\}$ converges strongly to u, if the procedure does not stop.*

Proof. Note the following: By $r + s \in [0, 1)$, we know $1 = \frac{r}{r} > \frac{r}{1-s} \geq 0$, that is, $\frac{r}{1-s} \in [0, 1)$. From this, we immediately see $\delta = \frac{1}{2}(1 + \frac{r}{1-s}) \in (\frac{r}{1-s}, 1)$.

We know $A_2 = Tx_1 \neq \emptyset$. Then, we can choose an $x_2 \in A_2 \subset Tx_1$. Suppose $x_2 \neq x_1$. Then, by $x_2 \in Tx_1$, (Ks) and the definition of H, we see

$$\begin{aligned} d(x_2, Tx_2) &\leq \sup\{d(z, Tx_2) : z \in Tx_1\} \\ &\leq H(Tx_1, Tx_2) \leq rd(x_1, Tx_1) + sd(x_2, Tx_2). \end{aligned}$$

So, by $\delta \in (\frac{r}{1-s}, 1)$, $x_2 \in Tx_1$ and $x_2 \neq x_1$, it follows that

$$d(x_2, Tx_2) \leq \tfrac{r}{1-s} d(x_1, Tx_1) < \delta d(x_1, x_2) < d(x_1, x_2).$$

By $d(x_2, Tx_2) = \inf_{y \in Tx_2} d(x_2, y)$, this implies $A_3 \neq \emptyset$. That is, we can choose an $x_3 \in A_3 \subset Tx_2$. Then, in this way, x_{n+1}, Tx_{n+1} and A_{n+2} can be generated until $l \in N$ satisfying $x_{l+1} = x_l$ appears.

We show (b). Suppose $x_{n+1} \neq x_n$ for all $n \in N$. Then, by the argument so far, we have $\{x_n\}$, $\{Tx_n\}$ and $\{A_n\}$. Also, we know the following:

$$\text{(2)} \quad \begin{aligned} &x_{n+1} \in A_{n+1} \subset Tx_n, \quad x_{n+1} \neq x_n, \quad d(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, x_{n+2}), \\ &d(x_{n+1}, x_{n+2}) \leq \delta d(x_n, x_{n+1}) < d(x_n, x_{n+1}) \quad \text{for all } n \in N. \end{aligned}$$

So, we see $d(x_{m+1}, x_{m+2}) \leq \delta^m d(x_1, x_2)$. Also, by $\delta \in (\frac{r}{1-s}, 1)$, $\lim_m \frac{\delta^m}{1-\delta} = 0$. Then, by (2), we see that, for all $m, k \in N$,

$$\begin{aligned} d(x_{m+1}, x_{m+k+1}) &\leq \textstyle\sum_{j=1}^{k} d(x_{m+j}, x_{m+j+1}) \\ &< \textstyle\sum_{j=m}^{m+k-1} \delta^j d(x_1, x_2) < \frac{\delta^m}{1-\delta} d(x_1, x_2). \end{aligned}$$

These imply that $\{x_n\}$ is a Cauchy sequence. Then, since X is complete, $\{x_n\}$ converges to some $u \in X$. To complete the proof of (b), we show $u \in Tu$.

By Lemma 1.3, we know $|d(u, Tu) - d(x_n, Tu)| \leq d(x_n, u)$. Then, by $\lim_n d(x_n, u) = 0$, we see $d(u, Tu) = \lim_n d(x_n, Tu)$. So, by $x_{n+1} \in Tx_n$, (2), (Ks), and the definition of H, we see

$$\begin{aligned} d(u, Tu) &= \lim_n d(x_{n+1}, Tu) \leq \lim_n \sup\{d(z, Tu) : z \in Tx_n\} \\ &\leq \lim_n H(Tx_n, Tu) \leq \lim_n (rd(x_n, Tx_n) + sd(u, Tu)) \\ &\leq r \lim_n d(x_n, x_{n+1}) + sd(u, Tu) = sd(u, Tu). \end{aligned}$$

So, by $s \in [0, 1)$, $d(u, Tu) = 0$. Thus, since Tu is closed, we see $u \in Tu$.

We show (a). Suppose A_{l+1} was generated and $x_{l+1} = x_l$. Then, we immediately see that $x_l = x_{l+1} \in A_{l+1} \subset Tx_l$. □

We present a version of Berinde's theorem; see Berinde–Berinde [1].

Theorem 2.4. *Let (X, d) be a complete metric space and T be a mapping from X into $\mathrm{CB}(X)$. Suppose there are $r \in [0, 1)$ and $s \in [0, \infty)$ satisfying*

$$\text{(Bs)} \qquad H(Tx, Ty) \leq rd(x, y) + sd(y, Tx) \qquad \text{for all} \quad x, y \in X.$$

Set $\delta = \frac{1}{2}(1+r) \in (r,1)$. Let $x_1 \in X = A_1$ and $A_2 = Tx_1$. For each $n \in N$, generate x_{n+1} and A_{n+2} by the following procedure:

(i) $x_{n+1} \in A_{n+1}$.
(ii) *This procedure will be stopped if* $x_{n+1} = x_n$.
(iii) $A_{n+2} = \{y \in Tx_{n+1} : d(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, y) \leq \delta d(x_n, x_{n+1})\}$.

Then the following hold:

(a) *There is $l \in N$ satisfying $x_l \in F(T)$, if the procedure stops.*
(b) *There is $u \in F(T)$ such that $\{x_n\}$ converges strongly to u, if the procedure does not stop.*

Proof. We know $A_2 = Tx_1 \neq \emptyset$. Then, we can choose an $x_2 \in A_2 \subset Tx_1$. Suppose $x_2 \neq x_1$. Then, by $x_2 \in Tx_1$, (Bs) and the definition of H, we see

$$\begin{aligned} d(x_2, Tx_2) &\leq \sup\{d(z, Tx_2) : z \in Tx_1\} \leq H(Tx_1, Tx_2) \\ &\leq rd(x_1, x_2) + sd(x_2, Tx_1) = rd(x_1, x_2) < \delta d(x_1, x_2) < d(x_1, x_2). \end{aligned}$$

By $d(x_2, Tx_2) = \inf_{y \in Tx_2} d(x_2, y)$, this implies $A_3 \neq \emptyset$. That is, we can choose an $x_3 \in A_3 \subset Tx_2$. Then, in this way, x_{n+1}, Tx_{n+1} and A_{n+2} can be generated until $l \in N$ satisfying $x_{l+1} = x_l$ appears.

We show (b). Suppose $x_{n+1} \neq x_n$ for all $n \in N$. Then, by the argument so far, we have $\{x_n\}$, $\{Tx_n\}$ and $\{A_n\}$. Also, we know the following:

$$\begin{aligned} (3) \qquad & x_{n+1} \in A_{n+1} \subset Tx_n, \quad x_{n+1} \neq x_n, \quad d(x_{n+1}, Tx_{n+1}) \leq d(x_{n+1}, x_{n+2}), \\ & d(x_{n+1}, x_{n+2}) \leq \delta d(x_n, x_{n+1}) < d(x_n, x_{n+1}) \ \text{ for all } n \in N. \end{aligned}$$

By $\delta \in (r,1)$, $\lim_m \frac{\delta^m}{1-\delta} = 0$ holds. Then, the rest of the proof is similar to as in the proof of Theorem 2.3. So, we have the following:

∘ $\{x_n\}$ is a Cauchy sequence and then $\{x_n\}$ converges to some $u \in X$,
∘ $d(u, Tu) = 0$.

Thus, since Tu is closed, we see $u \in Tu$.

We show (a). Suppose A_{l+1} was generated and $x_{l+1} = x_l$. Then, we immediately see that $x_l = x_{l+1} \in A_{l+1} \subset Tx_l$. □

3. An intrinsic fixed point theorem and applications

The contents of this section is closely related to what is discussed in Takahashi and Takeuchi [13]. In advance, we prepare some concepts and basic facts as it is needed in our study. Then, we use them without notice.

For simplicity, let (X, d) be a complete metric space. Let f be a function from X into $(-\infty, \infty]$. Then, the set $D(f) = \{x \in X : f(x) < \infty\}$ is called the domain of f. For each $a \in R$, $L_{\leq a}(f)$ denotes a level set of f such that $L_{\leq a}(f) = \{x \in D(f) : f(x) \leq a\}$. f is called proper if $D(f) \neq \emptyset$. f is called lower semi–continuous if $L_{\leq a}(f)$ is closed for all $a \in R$. $\gamma^l(X)$ denotes the set of all proper lower semi–continuous functions from X into $(-\infty, \infty]$. In subsequent argument, K always denotes a non–empty closed subset of X.

Let $f \in \gamma^l(X)$ and $b \in (0,\infty)$. For each $x \in X$, define g_x by $g_x(y) = f(y)+bd(x,y)$ for each $y \in X$. Then, $g_x \in \gamma^l(X)$. $D_K(f)$ denotes $D(f) \cap K$. Define a mapping T from K into 2^X by

(ET) $$Tx = \{y \in K : f(y) + bd(x,y) \le f(x)\} \quad \text{for each} \quad x \in K.$$

Recall $g_x \in \gamma^l(X)$ and note $f(x) + bd(x,x) = f(x)$ for all $x \in X$. Then, by (ET) and properties of infimum, the following basic facts are immediate.

- ∘ Suppose $\inf_{y\in K} f(y) \in R$. Then, $\inf_{y\in K} f(y) = \inf_{y\in D_K(f)} f(y)$ and $D_K(f) \neq \emptyset$ hold. Furthermore, let K' be a non–empty subset of $D_K(f)$. Then, $\inf_{y\in K} f(y) \le \inf_{y\in K'} f(y)$ and $\inf_{y\in K'} f(y) \in R$ hold.
- ∘ $x \in Tx$ for all $x \in K$.
- ∘ $Tx \subset D_K(f) \subset K$ for all $x \in D_K(f)$ and $Tx = K$ for all $x \in K \backslash D(f)$.
- ∘ Tx is non–empty and closed for all $x \in K$.
- ∘ Suppose $z \in D_K(f)$ and $w \in Tz$. Then, $w \in Tw \subset Tz$.
  Suppose further $w \neq z$. Then, $f(w) < f(z)$.

Here we confirm only the last assertion. We already know $w \in Tw$. By $w \in Tz$, $bd(z,w) \le f(z) - f(w)$. Also, for any $y \in Tw$, $f(y) + bd(w,y) \le f(w)$. Then,

$$\begin{aligned} f(y) + bd(z,y) &\le f(y) + bd(w,y) + bd(z,w) \\ &\le f(w) + (f(z) - f(w)) = f(z) \qquad \text{for all } y \in Tw. \end{aligned}$$

Thus, we see $Tw \subset Tz$. In the case of $w \neq z$, obviously $bd(z,w) > 0$. By $f(w) + bd(z,w) \le f(z)$, we have $f(w) < f(z)$.

We present an intrinsic fixed point theorem.

Theorem 3.1. *Let (X,d) be a complete metric space and $b \in (0,\infty)$. Let K be a non–empty closed subset of X. Let $f \in \gamma^l(X)$ satisfy $\inf_{y\in K} f(y) \in R$. Let $x_1 \in D_K(f) = A_1$. Let T be the mapping from K into 2^X defined by* (ET):

$$Tx = \{y \in K : f(y) + bd(x,y) \le f(x)\} \quad \text{for each} \quad x \in K.$$

For each $n \in N$, generate A_{n+1} and x_{n+1} by the following procedure:

(i) *This procedure will be stopped, if $Tx_n = \{x_n\}^s$ $(Tx_n \backslash \{x_n\}^s = \emptyset)$.*
(ii) $A_{n+1} = \{y \in Tx_n : f(y) \le \frac{1}{2} f(x_n) + \frac{1}{2}\inf_{z\in Tx_n} f(z)\}$.
(iii) $x_{n+1} \in A_{n+1}$.

Then the following hold:

(a) *There is $l \in N$ satisfying $x_l \in F_I(T)$, if the procedure stops.*
(b) *There is $\hat{v} \in F_I(T)$ such that $\{x_n\}$ converges strongly to $\hat{v}$, if the procedure does not stop.*

Proof. By $\inf_{y\in K} f(y) \in R$, we know $D_K(f) \neq \emptyset$. Then, we can choose an $x_1 \in A_1 = D_K(f)$. We know that $x_1 \in Tx_1 \subset D_K(f)$ and Tx_1 is a non–empty closed set. Since X is complete, so is Tx_1. By $\inf_{y\in K} f(y) \in R$ and $Tx_1 \subset D_K(f)$, we see $\inf_{y\in K} f(y) \le \inf_{y\in Tx_1} f(y)$ and $\inf_{y\in Tx_1} f(y) \in R$.

Suppose $Tx_1 \neq \{x_1\}^s$. Then, there is $w \in Tx_1$ satisfying $w \neq x_1$. So, $\inf_{y\in Tx_1} f(y) \leq f(w) < f(x_1)$. By $\inf_{y\in Tx_1} f(y) \in R$, the following holds:

$$\inf_{y\in Tx_1} f(y) < \tfrac{1}{2}\inf_{y\in Tx_1} f(y) + \tfrac{1}{2} f(x_1) < f(x_1). \tag{4}$$

This implies $A_2 \neq \emptyset$. Then, we can choose an $x_2 \in A_2 \subset Tx_1$. So, we know that $x_2 \in Tx_2 \subset Tx_1 \subset D_K(f)$ and

- ∘ Tx_2 is complete, $\inf_{y\in Tx_1} f(y) \leq \inf_{y\in Tx_2} f(y)$, and $\inf_{y\in Tx_2} f(y) \in R$.

Then, in this way, A_{n+1}, x_{n+1} and Tx_{n+1} can be generated until $l \in N$ satisfying $Tx_l = \{x_l\}^s$ appears.

We show (b). Suppose $Tx_n \neq \{x_n\}^s$ for all $n \in N$. By the argument so far, we have $\{x_n\}$, $\{Tx_n\}$ and $\{A_n\}$. Also, we know the following: For all $n \in N$,

(A) $x_{n+1} \in Tx_{n+1} \subset Tx_n \subset D_K(f)$ and Tx_n is complete,
(B) $\inf_{y\in Tx_n} f(y) \leq f(x_{n+1}) \leq \frac{1}{2} f(x_n) + \frac{1}{2}\inf_{y\in Tx_n} f(y) < f(x_n)$.

Note $x_1 \in Tx_1$. Then, by (A), $\{x_n\}$ is a sequence in Tx_1. By (B), $\{f(x_n)\}$ is monotonically decreasing. Of course, $\inf_{y\in K} f(y)$ is a lower bound of $\{f(x_n)\}$. Then $\{f(x_n)\}$ converges to some $c \in R$. By (A) and (ET), for all $n, m \in N$,

$$\begin{aligned} bd(x_{n+m}, x_n) &\leq \textstyle\sum_{j=0}^{m-1} bd(x_{n+j+1}, x_{n+j}) \\ &\leq \textstyle\sum_{j=0}^{m-1} (f(x_{n+j}) - f(x_{n+j+1})) = f(x_n) - f(x_{n+m}). \end{aligned}$$

So, since $\{f(x_n)\}$ converges, by $b > 0$, $\{x_n\}$ is a Cauchy sequence in Tx_1. Then, since Tx_1 is complete, $\{x_n\}$ converges strongly to some $\hat{v} \in Tx_1 \subset D_K(f)$.

By (A), for any $j \in N$, $\{x_n\}_{n\geq j}$ is a sequence in the complete set Tx_j. Then, $\hat{v} \in \cap_{n\in N} Tx_n \subset D_K(f)$, that is, $\hat{v} \in T\hat{v} \subset \cap_{n\in N} Tx_n \subset D_K(f)$. Furthermore, by $f \in \gamma^l(X)$, we know $f(\hat{v}) \leq \liminf_n f(x_n) = \lim_n f(x_n)$.

To complete the proof of (b), we may show $T\hat{v} = \{\hat{v}\}^s$. Arguing by contradiction, assume $T\hat{v} \neq \{\hat{v}\}^s$. Then, there is $\hat{w} \in T\hat{v}$ satisfying $\hat{w} \neq \hat{v}$. So, $\hat{w} \in T\hat{v} \subset \cap_{n\in N} Tx_n$ and $f(\hat{w}) < f(\hat{v})$. By $\hat{w} \in \cap_{n\in N} Tx_n$ and (B), we see

$$2f(x_{n+1}) - f(x_n) \leq \inf\{f(y) : y \in Tx_n\} \leq f(\hat{w}) \quad \text{for all} \quad n \in N.$$

So, $\lim_n f(x_n) \leq f(\hat{w})$. We already know $f(\hat{v}) \leq \lim_n f(x_n)$ and $f(\hat{w}) < f(\hat{v})$. Thus, we meet a contradiction: $f(\hat{v}) \leq \lim_n f(x_n) \leq f(\hat{w}) < f(\hat{v})$.

Suppose x_l was generated and $Tx_l = \{x_l\}^s$. Then, obviously (a) holds. □

We apply Theorem 3.1 to prove two theorems. The following is referred to as Takahashi's minimization theorem; see Takahashi [11, 12].

Theorem 3.2. *Let (X, d) be a complete metric space and $b \in (0, \infty)$. Let K be a non–empty closed subset of X. Let $f \in \gamma^l(X)$ satisfy $\inf_{y\in K} f(y) \in R$. Suppose, for each $x \in K$, either $f(x) = \inf_{y\in K} f(y)$ or $A_x \neq \{x\}^s$ holds, where $A_x = \{y \in K : f(y) + bd(x, y) \leq f(x)\}$. Then, there is $\hat{v} \in K$ satisfying $f(\hat{v}) = \inf_{y\in K} f(y)$.*

Proof. Let T be as in Theorem 3.1. Then, there is $\hat{v} \in D_K(f) \subset K$ satisfying $A_{\hat{v}} = T\hat{v} = \{\hat{v}\}^s$. That is, $f(\hat{v}) = \inf_{y\in K} f(y)$. □

The following is the Ekeland variational principle; see Ekeland [4].

Theorem 3.3. *Let (X, d) be a complete metric space and $b \in (0, \infty)$. Let $f \in \gamma^l(X)$ satisfy* $\inf_{y \in X} f(y) \in R$. *Let $u \in X$ and $A_u = \{y \in X : f(y) + bd(u, y) \leq f(u)\}$. Then, there is $\hat{v} \in A_u$ satisfying the following*:

(E) $$f(\hat{v}) < f(y) + bd(\hat{v}, y) \quad \text{for all } \; y \in X \; \text{ with } \; y \neq \hat{v}.$$
$$f(\hat{v}) = \inf_{y \in E}\{f(y) + bd(\hat{v}, y)\}.$$
$$f(\hat{v}) \leq f(u) - bd(\hat{v}, u) \quad (\, f(\hat{v}) \leq f(u), \; f(\hat{v}) < f(u) \text{ if } u \neq \hat{v}\,).$$

Proof. By $\inf_{y \in X} f(y) \in R$, $D(f) \neq \emptyset$. Let S be the mapping from X into 2^X defined by (ET): $Sx = \{y \in E : f(y) + bd(x, y) \leq f(x)\}$ for each $x \in X$.

We consider the case of $u \in D(f)$. Obviously $Su = A_u$. Then, we know that A_u is non–empty and closed. We also know that $\inf_{y \in A_u} f(y) \in R$ and $A_u \subset D_X(f) = D(f)$. Let T be a mapping from A_u into 2^X defined by (ET):

$$Tx = \{y \in A_u : f(y) + bd(x, y) \leq f(x)\} \quad \text{for each } \; x \in A_u.$$

We know $D_{A_u}(f) = A_u$ by $A_u \subset D(f)$. By Theorem 3.1, we also know that there is $\hat{v} \in D_{A_u}(f) = A_u$ satisfying $T\hat{v} = \{\hat{v}\}^s$. Note that $\hat{v} \in A_u$ implies $f(\hat{v}) \leq f(u) - bd(\hat{v}, u)$. Of course, $f(\hat{v}) = f(\hat{v}) + bd(\hat{v}, \hat{v})$.

Suppose $y \notin A_u$. Then, $f(u) < f(y) + bd(u, y)$. So,

$$f(y) + bd(\hat{v}, y) \geq f(y) + bd(u, y) - bd(\hat{v}, u) > f(u) - bd(\hat{v}, u) \geq f(\hat{v}).$$

Suppose $y \in A_u$ and $y \neq \hat{v}$. Then, by $T\hat{v} = \{v\}^s$, we immediately see $y \notin T\hat{v}$, that is, $f(\hat{v}) < f(y) + bd(\hat{v}, y)$. Thus, we confirmed that $\hat{v} \in A_u$ satisfies (E).

We consider the case of $u \notin D(f)$. In this case, $A_u = X$ and $f(u) = \infty$. Fix any $u' \in D(f)$. We already know that there is $\hat{v} \in A_{u'} \subset A_u$ which satisfies (E) as $u = u'$. By $f(u) = \infty$, it is trivial that $\hat{v} \in A_u$ satisfies (E). ☐

Remark 3.4. We note that there are some representations of the Ekeland variational principle; for example, refer to Phelps [8]. By the argument in this section, the Ekeland variational principle can be regarded as one of useful interpretations of the intrinsic fixed point theorem (Theorem 3.1).

Let $b \in (0, \infty)$, $u \in X$ and $f \in \gamma^l(X)$ satisfy $\inf_{y \in X} f(y) \in R$. Then, by Theorem 3.3, there is $\hat{v} \in A_u$ satisfying (E). Note that we do not know whether f has a minimum point. By considering the perturbation caused by $\hat{v}$, define a mapping $g_{\hat{v}} \in \gamma^l(X)$ by $g_{\hat{v}}(y) = f(y) + bd(\hat{v}, y)$ for each $y \in X$. Then, $g_{\hat{v}}$ has the unique minimum point $\hat{v}$ even if f has no minimum point.

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