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ON A NONLINEAR INTEGRAL EQUATION

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ABSTRACT. The existence of a solution for a nonlinear integral equation of the form (P) is proved based upon the Mazur-Hukuhara fixed point theory in locally convex spaces.

1. Introduction

Given a triple of functions $h : [0,\infty) \to \mathbb{R}$, $K : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$, we consider a nonlinear integral equation of the form

$$x(t) = h(t) + \int_0^t K(t,s)f(s,x(s))ds. \qquad (P)$$

The purpose of this note is to provide a set of conditions which guarantees the existence of a unique solution of the equation (P) in a certain class of continuous functions.

The proof depends upon the Mazur-Hukuhara fixed point theorem (Mazur[4]. Hukuhara[2]) :

Let $\mathfrak{X}$ be a locally convex Hausdorff topological vector space (LCHTVS) and M a convex bounded subset of $\mathfrak{X}$. Then any compact mapping $f : M \to M$ has a fixed point.

Remark 1.1. In Goldman, Kato and Mui [1], Tosio Kato gave a constructive existence proof for a similar problem :

$$x(t) = h(t) + \int_0^t K(t-s)f(x(s))ds, \qquad (P')$$

where f is defined on $[0,\infty)$ instead of $[0,\infty)^2$. Kato also examined the uniqueness as well as some asymptotic property of the solution. However Kato's proof was "too lengthy to be included here", i.e. in [1]. As far as I know, this is the only work on mathematical economics achieved by Kato.

2. Function space $\mathfrak{C}(X, \mathbb{R})$

We begin by specifying a function space in which we look for a solution for the equation (P).

Let X be a Hausdorff topological space. $\mathfrak{C}(X, \mathbb{R})$ denotes the space of all the real-valued continuous functions, endowed with the topology of the uniform convergence

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on compacta. Then $\mathfrak{C}(X,\mathbb{R})$ is a LCHTVS, the topology of which is generated by the family $\{p_K : K \subset X \text{ is compact}\}$ of semi-norms defined by

$$p_K(u) = \sup_{x\in K} |u(x)| \quad , \quad u \in \mathfrak{C}(X,\mathbb{R}). \tag{2.1}$$

Fixing a positive number $\mu > 0$, we define $\|u\|_\mu$ for each $u \in \mathfrak{C}(X,\mathbb{R})$ by

$$\|u\|_\mu = \sup_{x\in X} |u(x)|e^{-\mu x}$$

($\|u\|_\mu$ may be infinite. I owe this idea to Krasnosel'skiĭ and Zabreĭko [3]p.229). Then

$$\mathfrak{C}_\mu(X,\mathbb{R}) = \{u \in \mathfrak{C}(X,\mathbb{R}) \mid \|u\|_\mu < \infty\}$$

is a subspace of $\mathfrak{C}(X,\mathbb{R})$.

Remark 2.1. It is not hard to show that $\|\cdot\|_\mu$ is a norm on $\mathfrak{C}_\mu(X,\mathbb{R})$ and $\mathfrak{C}_\mu(X,\mathbb{R})$ is complete with respect to this norm. This observation actually enables us to rewrite the succeeding discussion in the framework of a Banach space instead of a LCHTVS.

I would like to remind the readers' of the generalized Ascoli-Arzelà theorem :

A subset $H \subset \mathfrak{C}(X,\mathbb{R})$ is relatively compact (with respect to the topology of the uniform convergence on compacta) if

(i) *H is equi-continuous, and*
(ii) *$\{u(x)|u \in H\}$ is bounded for each $x \in X$.*

(cf. Schwartz [5]pp.78-80.)

In the remaining part of this note, we specify X as $[0,\infty)$ and assume $h \in \mathfrak{C}_\mu([0,\infty),\mathbb{R})$.

3. Lemmata

We shall find out a solution for the equation (P) in $\mathfrak{C}_\mu(X,\mathbb{R})$ by choosing $\mu > 0$ suitably.

Let S_r be the closed ball in $\mathfrak{C}_\mu([0,\infty),\mathbb{R})$ with the center 0 and the radius $r > 0$.

Lemma 3.1. *S_r is bounded in $\mathfrak{C}([0,\infty),\mathbb{R})$.*

Proof. Let p_K be a seminorm on $\mathfrak{C}([0,\infty),\mathbb{R})$ defined by (2.1). For any $u \in S_r$, $p_K(u)$ is evaluated as

$$\begin{aligned} p_K(u) &= \sup_{x\in K} |u(x)| = \sup_{x\in K} |u(x)e^{-\mu x}|\, e^{\mu x} \\ &\leq \|u\|_\mu \cdot \sup_{x\in K} e^{\mu x} \leq r \cdot \sup_{x\in K} e^{\mu x}. \end{aligned}$$

Consequently, we obtain

$$\sup_{u\in S_r} p_K(u) < \infty.$$

□

We need a couple of assumptions imposed on K and f.

Assumption 3.2. $K : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is continuous and satisfies

$$\sup_{(t,s)\in[0,\infty)^2} |K(t, s)| \equiv \kappa < \infty.$$

Assumption 3.3. $f : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|f(s, x)| \leq a + b|x| \quad , \quad s \in [0, \infty) \quad , \quad x \in \mathbb{R}$$

for some positive constants a and b.

Remark 3.4. The constants a and b can be replaced by some positive integrable functions $a(t), b(t) \in \mathfrak{L}^1([0, \infty), \mathbb{R})$. This generalization is almost trivial, and so we skip the details.

Let $T : \mathfrak{C}_\mu([0, \infty), \mathbb{R}) \to \mathfrak{C}_\mu([0, \infty), \mathbb{R})$ be a nonlinear integral operator defined by the righ-hand side of (P) :

$$T : x(\cdot) \mapsto h(t) + \int_0^t K(t, s)f(s, x(s))ds.$$

We first evaluate the magnitude $\|Tx(t)\|_\mu$ for $x(\cdot) \in \mathfrak{C}_\mu([0, \infty), \mathbb{R})$.

$$\begin{aligned}
&|e^{-\mu t}Tx(t)| \\
&= \left|e^{-\mu t}h(t) + e^{-\mu t}\int_0^t K(t, s)f(s, x(s))ds\right| \\
&\leq e^{-\mu t}|h(t)| + \left|e^{-\mu t}\int_0^t |K(t, s)(a + b\,|x(s)|)|ds\right| \\
&\leq \|h\|_\mu + e^{-\mu t}a\kappa t + e^{-\mu t}b\kappa\int_0^t |x(s)|ds.
\end{aligned}$$

It is easily shown that $0 \leq e^{-\mu t}t \leq 1/\mu e$ for all $t \in [0, \infty)$. Hence, continuing the above evaluation, we obtain

$$
\begin{aligned}
|e^{-\mu t}Tx(t)| &\leq \|h\|_\mu + \frac{a}{\mu e}\kappa + e^{-\mu t}b\kappa\int_0^t |x(s)|e^{-\mu s}\cdot e^{\mu s}ds \\
&= \gamma + e^{-\mu t}b\kappa\int_0^t |x(s)|e^{-\mu s}e^{\mu s}ds \\
&\qquad (\text{where } \gamma = \|h\|_\mu + a\kappa/\mu e) \\
&\leq \gamma + b\kappa\int_0^t \|x\|_\mu e^{-\mu(t-s)}ds \qquad (3.1)\\
&= \gamma + b\kappa\|x\|_\mu\int_0^t e^{-\mu\eta}d\eta \\
&\qquad (\text{by changing variables}) \\
&= \gamma + b\kappa\frac{1}{\mu}(1-e^{-\mu t})\|x\|_\mu \\
&= \gamma + b\kappa\frac{1}{\mu}\|x\|_\mu.
\end{aligned}
$$

Here we choose $\mu > 0$ sufficiently large so that

$$\frac{b\kappa}{\mu} \in (0,1). \tag{3.2}$$

Defining $\theta \in (0,1)$ by $1-\theta = b\kappa/\mu$, we choose $r > 0$ so large as to fulfill

$$\theta r > \gamma \quad (\gamma \text{ is defined above}). \tag{3.3}$$

If μ and r satisfy (3.2) and (3.3), we can show that $Tx \in S_r$ for any $x \in S_r$. In fact, it follows from the evaluation :

$$
\begin{aligned}
&|e^{-\mu t}Tx(t)| \leq \gamma + (1-\theta)\|x\|_\mu \quad (\text{by}(3.1)) \\
&\leq \gamma + (1-\theta)r = r - (\theta r - \gamma) < r. \quad (\text{by}(3.3))
\end{aligned}
$$

Lemma 3.5. *Under Assumptions 3.2, 3.3, we obtain $T(S_r) \subset S_r$ if μ and r satisfy (3.2) and (3.3).*

4. Fixed point Argument

We now proceed to show the relative compactness of $T(S_r)$ in $\mathfrak{C}([0,\infty),\mathbb{R})$. According to (3.1), we have

$$
|Tx(t)| \leq [\gamma + b\kappa\frac{1}{\mu}\|x\|_\mu]e^{\mu t} \leq [\gamma + b\kappa r\frac{1}{\mu}]e^{\mu t}
$$
$$\text{for any} \quad x \in S_r.$$

Hence $\{Tx(t) \mid x \in S_r\}$ is bounded for each $t \in [0,\infty)$.

Moreover $T(S_r)$ is equi-continuous. Evaluating $|Tx(t) - Tx(t^{'})\,|$ (say, $t^{'} < t$) for $x \in S_r$. we obtain

$$\begin{aligned}
&|Tx(t) - Tx(t^{'})| \\
&\le |h(t) - h(t^{'})| + \Big| \int_0^t K(t,s) f(s, x(s)) ds \\
&\qquad - \int_0^{t^{'}} K(t^{'}, s) f(s, x(s)) ds \Big| \\
&= |h(t) - h(t^{'})| + \Big| \int_{t^{'}}^t K(t,s) f(s, x(s)) ds \Big| \\
&\qquad + \int_0^{t^{'}} |K(t,s) - K(t^{'} s)| \cdot |f(s, x(s))| ds.
\end{aligned} \tag{4.1}$$

We evaluate the second and the third terms, respectively.

$$\begin{aligned}
\text{2nd term} &\le a\kappa(t - t^{'}) + b\kappa \|x\|_\mu \int_{t^{'}}^t e^{\mu s} ds \\
&\le a\kappa(t - t^{'}) + b\kappa r \frac{1}{\mu}(e^{\mu t} - e^{\mu t^{'}}).
\end{aligned} \tag{4.2}$$

We now turn to the third term. Let A be a positive number less than t. Then $[t - A, t + A] \subset [0, \infty)$, obviously. f is uniformly continuous on $[t - A, t + A] \times [0, t]$. Hence there exists some $\delta(t) \in (0, A)$, for each $\varepsilon > 0$, such that

$$|K(t,s) - K(t^{'}, s)| < \varepsilon \quad \text{if} \quad |t - t^{'}\,| < \delta(t) \quad \text{and} \quad s \in [0, t^{'}].$$

It follows that

$$\begin{aligned}
\text{3rd term} &\le \varepsilon \int_0^{t^{'}} (a + b\|x\|_\mu e^{\mu s}) ds \\
&\le a\varepsilon t^{'} + b\varepsilon r \frac{1}{\mu}(e^{\mu t^{'}} - 1) \\
&\le a\varepsilon A + b\varepsilon r \frac{1}{\mu}(e^{\mu(t+A)} - 1), \\
&\qquad \text{provided that} \quad |t^{'} - t| < \delta(t).
\end{aligned} \tag{4.3}$$

This proves the equi-continuity of $T(S_r)$ since the right-hand sides of (4.2) and (4.3) do not depend upon $x \in S_r$.

The next lemma follows from the Ascoli-Arzelà theorem stated above.

Lemma 4.1. *$T(S_r)$ is relatively compact in $\mathfrak{C}([0, \infty), \mathbb{R})$.*

We confirmed that the operator T restricted to the bounded and convex set S_r is a compact mapping into S_r. Thus we conclude that T has a fixed point in S_r by the Mazur-Hukuhara theorem.

Theorem 4.2. *The equation* (P) *has a solution in* S_r *under the Assumptions 3.2, 3.3 for a suitable combination of* r *and* μ.

5. Uniqueness

Finally, we show that there exists only one solution of the equation (P) under the additional assumption.

Assumption 5.1. There exists some constant $L > 0$ which satisfies

$$\begin{aligned} &|f(t,x) - f(t,y)| \le L|x-y| \\ &\qquad \text{for any} \quad t \in [0,\infty) \quad \text{and} \quad x, y \in \mathbb{R}. \end{aligned} \tag{5.1}$$

Let x and y be any elements of S_r. Taking account of the definition of the operator T,

$$\begin{aligned} |e^{-\mu t}(Tx(t) - Ty(t))| &= \left| e^{-\mu t} \int_0^t K(t,s)[f(t,x(s)) - f(t,y(s))]ds \right| \\ &\le e^{-\mu t}\kappa \int_0^t |f(t,x(s)) - f(t,y(s))|ds \qquad \text{(by Assumption 3.3)} \\ &\le e^{-\mu t}\kappa \int_0^t L|x(s) - y(s)|ds \qquad \text{(by Assumption 5.1)} \\ &\le e^{-\mu t}\kappa L\|x-y\|_\mu \int_0^t e^{\mu s} ds \le \kappa L e^{-\mu t} \cdot \frac{1}{\mu}(e^{\mu t} - 1)\|x-y\|_\mu \\ &= \frac{1}{\mu}\kappa L(1 - e^{-\mu t}) \cdot \|x-y\|_\mu \le \frac{1}{\mu}\kappa L\|x-y\|_\mu. \end{aligned}$$

Consequently, it follows that

$$\|Tx - Ty\|_\mu \le \frac{1}{\mu}\kappa L\|x-y\|_\mu.$$

If μ is sufficiently large, the operation T is a contraction of S_r into S_r. This implies that T has a unique fixed point of T in S_r since S_r is complete with respect to $\|\cdot\|_\mu$. Subtle locally convex space arguments are not necessary in the framework of section 5.

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