Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 11, Number 3, 2025, 145–152

A CHARACTERIZATION OF WEAK UNIFORM CONVEXITY OF COMPLETE BUSEMANN SPACES

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Abstract. This paper aims to characterize the weak uniform convexity of complete Busemann spaces. The properties of convex combinations differ between Banach spaces and more general metric spaces. Using suitable convex combinations, we prove an equivalence condition for weak uniform convexity in metric spaces that are more general than Banach spaces.

1. Introduction

The study of metric spaces without linear structures has played a vital role in various branches of pure and applied sciences. H. Busemann [7] developed a theory of non-positive curvature for path-metric spaces, based on a simple axiom of convexity of distance functions. Building on this theory, B. H. Bowditch [4] introduced Busemann spaces, which are a type of non-positively curved metric space. Busemann spaces satisfy many fundamental metric, geometric, and topological properties [21]. They have found numerous applications in optimization problems and geometric group theory [2, 5]. Against this background, it is worth exploring the properties of non-positively curved metric spaces.

Characterizations of convexity are important properties in metric spaces. In Banach spaces, a characterization of uniform convexity follows from properties of convex combinations of three points [3]. This characterization provides important properties in the geometry of Banach spaces [14, 23]. On the other hand, the convexity properties of Busemann spaces are not necessarily the same as those of Banach spaces. Since convex combinations in non-positively curved metric spaces may depend on the order of combining the points, characterizations of convexity in Busemann spaces require a new theoretical approach.

In this paper, we study a characterization of convexity of complete Busemann spaces. To ensure the existence of convex combinations, we consider Busemann spaces to be complete. Furthermore, since weak uniform convexity, which is a weaker condition than uniform convexity, exists in Busemann spaces, we primarily deal with this property. In Section 2, we present several preliminary definitions and results about convex combinations and weak uniform convexity. In Section 3, we prove a characterization of weak uniform convexity of complete Busemann spaces.

2020 *Mathematics Subject Classification.* 46B20, 46T99.
Key words and phrases. Busemann space, uniformly convex, convex combination.

2. Preliminaries

Let (X,d) be a metric space. Given distinct points $x,y\in X$, a *metric midpoint* of x and y is a point $m(x,y)\in X$ if $d\big(x,m(x,y)\big)=d\big(y,m(x,y)\big)=(1/2)d(x,y)$. A *path* in X is a continuous map $\gamma:[\alpha,\beta]\subset\mathbb{R}\to X$. Given a pair of points $x,y\in X$, we say that a path $\gamma:[\alpha,\beta]\to X$ *joins* x *and* y if $\gamma(\alpha)=x$ and $\gamma(\beta)=y$. A *geodesic path* in X is an isometry $\gamma:[\alpha,\beta]\to X$ such that $d(\gamma(s),\gamma(t))=|s-t|$ for every $s,t\in[\alpha,\beta]$. A *geodesic segment* $\gamma([\alpha,\beta])\subset X$ *from* x *to* y is the image of a geodesic path $\gamma:[\alpha,\beta]\to X$ joining x and y. Note that a geodesic segment from x to y is not necessarily unique in general. If no confusion arises, then $[x,y]$ denotes a unique geodesic segment from x to y. A *(uniquely) geodesic space* is a metric space X such that every two points in X can be joined by a (unique) geodesic path. If X is a complete metric space, then X is a geodesic space if and only if every pair of points in X has a metric midpoint [2, p. 2, Prop. 1.1.3].

Proposition 2.1 ([21, p. 62, Prop. 2.2.14])**.** *Let X be a geodesic space and $x,y,z\in X$ three pairwise distinct points. Then $d(x,y)=d(x,z)+d(z,y)$ if and only if there exists a geodesic segment from x to y containing z.*

The points on a geodesic segment are naturally parametrized by $[0,1]\subset\mathbb{R}$. For two distinct points $x,y\in X$ and a point $z\in[x,y]$, we use the notation $z=(1-t)x\oplus ty$, where $t=d(x,z)/d(x,y)$.

Lemma 2.2. *Let (X,d) be a complete uniquely geodesic space and $x,y\in X$ two distinct points. If $p=(1-t)x\oplus ty$ and $q=tx\oplus(1-t)y$ for $t\in[0,1/2]$, then $(1-s)p\oplus sq=\big(1-(t+(1-2t)s)\big)x\oplus(t+(1-2t)s)y$ for every $s\in[0,1]$.*

Proof. By assumption, $[p,q]\subset[x,y]$. By Proposition 2.1 we have

$$\begin{aligned}d(x,y)&=d(x,p)+d(p,q)+d(q,y)\\&=td(x,y)+\frac{1}{s}d(p,(1-s)p\oplus sq)+td(x,y)\end{aligned}$$

for every $s\in[0,1]$. This implies $s(1-2t)d(x,y)=d(p,(1-s)p\oplus sq)$. Note that $p\in[x,(1-s)p\oplus sq]$. Thus

$$\begin{aligned}d(x,(1-s)p\oplus sq)&=d(x,p)+d(p,(1-s)p\oplus sq)\\&=td(x,y)+s(1-2t)d(x,y)=(t+(1-2t)s)d(x,y).\end{aligned}$$

Hence $(1-s)p\oplus sq=\big(1-(t+(1-2t)s)\big)x\oplus(t+(1-2t)s)y$. □

A *Busemann space* is a geodesic space (X,d) such that for every two geodesic paths $\gamma_1:[\alpha_1,\beta_1]\subset\mathbb{R}\to X$ and $\gamma_2:[\alpha_2,\beta_2]\subset\mathbb{R}\to X$, the map $D_{\gamma_1,\gamma_2}:[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\to\mathbb{R}$ defined by

$$D_{\gamma_1,\gamma_2}(t_1,t_2)=d\big(\gamma_1(t_1),\gamma_2(t_2)\big)$$

is convex [4, pp. 576–577][21, pp. 203–204]. The definition of a Busemann space gives the following elementary facts:

(1) [21, p. 210, Prop. 8.1.4] Every Busemann space is a uniquely geodesic space.

(2) [2, p. 4, Prop. 1.1.5] Every Busemann space X has a convex metric, that is, for every $x, y, z \in X$ and $t \in [0,1]$,

$$d(x, (1-t)y \oplus tz) \leq (1-t)d(x,y) + td(x,z). \tag{2.1}$$

(3) [16, p. 40, Def. 6.5] In a Busemann space X, for every $x, y \in X$ and $t, s \in [0,1]$,

$$d\big((1-t)x \oplus ty, (1-s)x \oplus sy\big) = |t-s| d(x,y). \tag{2.2}$$

Basic examples of Busemann spaces are Euclidean spaces, strictly convex normed vector spaces, $\mathbb{R}$-trees [24], classical hyperbolic spaces [2], and Riemannian manifolds of global nonpositive sectional curvature [5].

Let (X,d) be a metric space. A *geodesic line* in X is a distance-preserving map $\gamma : \mathbb{R} \to X$. A *local geodesic* is a map $\gamma : [\alpha, \beta] \subset \mathbb{R} \to X$ with the property that for every $t \in [\alpha, \beta]$ there exists $\epsilon > 0$ such that $d\big(\gamma(s_1), \gamma(s_2)\big) = |s_1 - s_2|$ for all $s_1, s_2 \in [\alpha, \beta]$ with $|t - s_1| + |t - s_2| \leq \epsilon$. A geodesic space X is said to have *the geodesic extension property* if for every local geodesic $\gamma : [\alpha, \beta] \to X$ with $\alpha \neq \beta$, there exist $\epsilon > 0$ and a local geodesic $\gamma' : [\alpha, \beta + \epsilon] \to X$ such that $\gamma'|_{[\alpha,\beta]} = \gamma$ [5, p. 208, Def. 5.7].

Lemma 2.3 ([21, p. 212, Cor. 8.2.3])**.** *In a Busemann space, every local geodesic is a geodesic path.*

Lemma 2.4. *If X is a Busemann space, then X has the geodesic extension property if and only if every non-constant geodesic path can be extended to a geodesic line.*

Proof. This assertion is an immediate consequence of Lemma 2.3. □

The notion of uniform convexity in Banach spaces was introduced by Clarkson [8], whereas the modulus of convexity in hyperbolic spaces was coined by Goebel and Reich [17]. Similarly, the notion of uniform convexity exists in metric spaces [13], and a modulus of convexity can be defined in geodesic spaces [20, pp. 468–469]: A geodesic space (X,d) is said to be *weakly uniformly convex* if for every $a \in X$, $r > 0$, and $\epsilon \in (0,2]$ there exists a mapping $\delta(a,r,\epsilon) : X \times (0,\infty) \times (0,2] \to (0,1]$ such that for every $x, y \in X$,

$$\left.\begin{array}{l} d(a,x) \leq r \\ d(a,y) \leq r \\ d(x,y) \geq \epsilon r \end{array}\right\} \quad \Rightarrow \quad d\big(a, m(x,y)\big) \leq \big(1 - \delta(a,r,\epsilon)\big) r.$$

Such a mapping δ is called a *modulus of convexity* of X and defined by

$$\delta(a,r,\epsilon) := \inf \left\{ 1 - \frac{1}{r} d\big(a, m(x,y)\big) : x, y \in X, d(a,x) \leq r, d(a,y) \leq r, d(x,y) \geq \epsilon r \right\}.$$

See [20] for an example of a weakly uniformly convex geodesic space. Every weakly uniformly convex geodesic space X is *strictly convex*, that is, for all $a, x, y \in X$ with $x \neq y$ and all metric midpoints $m(x,y)$ we have $d\big(a, m(x,y)\big) < \max\{d(a,x), d(a,y)\}$. It follows that X is uniquely geodesic [12]. If $\delta = \delta(r, \epsilon)$ does not depend on a, we say that X is *uniformly convex.* While in Banach spaces, there exists a natural modulus of convexity for each space that depends only on ϵ, in geodesic spaces, we need to assume that the modulus depends on three variables: a, r, ϵ. Complete and weakly uniformly convex Busemann spaces satisfy the following properties.

Proposition 2.5. *Let (X, d) be a complete and weakly uniformly convex Busemann space. Fix $a \in X$ and $r > 0$. Then a modulus of convexity $\delta(a, r, \epsilon) : (0, 2] \to (0, 1]$ is a non-decreasing function of ϵ.*

Proof. Let $0 < \epsilon_1 \le \epsilon_2 \le 2$. Set $x, y \in X$ satisfying $d(a, x) \le r, d(a, y) \le r$, $d(x, y) \ge \epsilon_2 r$, and $d\big(a, (1/2)x \oplus (1/2)y\big) = \big(1 - \delta(a, r, \epsilon_2)\big)r$. Let $t = (\epsilon_2 - \epsilon_1)/2\epsilon_2$, $p = (1-t)x \oplus ty$, and $q = tx \oplus (1-t)y$. By (2.1) we have $d(a, p) \le r$ and $d(a, q) \le r$. By (2.2) we get

$$d(p, q) = |(1-t) - t|d(x, y) \ge |1 - 2t|\epsilon_2 r = \left|1 - \frac{\epsilon_2 - \epsilon_1}{\epsilon_2}\right| \epsilon_2 r = \epsilon_1 r.$$

By the definition of a modulus of convexity, we have $\delta(a, r, \epsilon_1) \le 1 - (1/r)d\big(a, (1/2)p \oplus (1/2)q\big)$. Note that $(1/2)p \oplus (1/2)q = (1/2)x \oplus (1/2)y$ by Lemma 2.2. Therefore

$$\delta(a, r, \epsilon_1) \le 1 - \frac{1}{r} d\Big(a, \frac{p}{2} \oplus \frac{q}{2}\Big) = 1 - \frac{1}{r} d\Big(a, \frac{x}{2} \oplus \frac{y}{2}\Big) = \delta(a, r, \epsilon_2).$$

□

Theorem 2.6 ([25, p. 157, Thm. 3.2])**.** *Let (X, d) be a complete Busemann space. Let $a, x, y \in X$ be three distinct points such that $d(a, x) = d(a, y) = r$. Set $\epsilon r = d(x, y)$. If X is weakly uniformly convex, then for every $t \in [0, 1]$,*

$$d\left(a, \frac{1-t}{2} a \oplus \frac{1+t}{2}\left(\frac{1}{1+t}x \oplus \frac{t}{1+t}y\right)\right) \le r\left(\frac{1+t}{2} - t\delta(a, r, \epsilon)\right).$$

3. A CHARACTERIZATION OF WEAK UNIFORM CONVEXITY

In this section, as the main result, we prove a characterization of weak uniform convexity of Busemann spaces. This extends the characterization of uniform convexity of Banach spaces.

Theorem 3.1. *Let (X, d) be a complete Busemann space having the geodesic extension property and $1 < p < \infty$. The weak uniform convexity of X is equivalent to the following property: for every $a \in X$, $r > 0$, and $\epsilon \in (0, 2]$, there exists a number $\eta_p(a, r, \epsilon) > 0$ such that, for every $t \in [0, 1]$ and three distinct points $a, x, y \in X$, the conditions $d(a, x) \le r$, $d(a, y) \le r$, and $d(x, y) \ge \epsilon r$ imply*

$$d\left(a, \frac{1-t}{2} a \oplus \frac{1+t}{2}\left(\frac{1}{1+t}x \oplus \frac{t}{1+t}\tilde{y}\right)\right)^p \le \big(1 - \eta_p(a, r, \epsilon)\big)\frac{d(a, x)^p + d(a, y)^p}{2}, \tag{3.1}$$

where $\tilde{y} \in X$ satisfies $d(a, \tilde{y}) = r$ and $y = (1-t)a \oplus t\tilde{y}$.

Proof. First, we show the sufficient condition. If $t = 1$ in (3.1), then

$$d\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right) = d\left(a, \frac{1}{2}x \oplus \frac{1}{2}\tilde{y}\right) \leq r\big(1 - \eta_p(a, r, \epsilon)\big)^{\frac{1}{p}}.$$

Therefore X is weakly uniformly convex with $\delta(a, r, \epsilon) = 1 - \sqrt[p]{1 - \eta_p(a, r, \epsilon)}$.

Next, we show the necessary condition. It suffices to prove the case $d(a, x) = r$ for x. Since X has the geodesic extension property, there exists $\tilde{y} \in X$ such that $d(a, \tilde{y}) = r$ and $y = (1-t)a \oplus t\tilde{y}$. Set $\tilde{\epsilon} := d(x, \tilde{y})/r$. By Theorem 2.6 we have

$$\frac{d\left(a, \frac{1-t}{2}a \oplus \frac{1+t}{2}\left(\frac{1}{1+t}x \oplus \frac{t}{1+t}\tilde{y}\right)\right)^p}{\frac{1}{2}\big(d(a,x)^p + d(a,y)^p\big)} \leq \frac{r^p\left(\frac{1+t}{2} - t\delta(a, r, \tilde{\epsilon})\right)^p}{\frac{1}{2}\left(r^p + \big(td(a, \tilde{y})\big)^p\right)} \leq \frac{2\left(\frac{1+t}{2} - t\delta(a, r, \tilde{\epsilon})\right)^p}{1 + t^p} =: \varphi(t).$$

Now, we have to consider two cases for ϵ:

(Case 1) If $\tilde{\epsilon} \leq \epsilon/2$, by the triangle inequality, then

$$(1-t)r = d(\tilde{y}, y) \geq d(x, y) - d(x, \tilde{y}) \geq \epsilon r - \tilde{\epsilon} r \geq \left(\epsilon - \frac{\epsilon}{2}\right) r = \frac{\epsilon r}{2}.$$

Thus $t \leq 1 - \epsilon/2$. Moreover, we have

$$\varphi(t) < \frac{2\left(\frac{1+t}{2}\right)^p}{1 + t^p} =: \varphi_1(t).$$

The real function $\varphi_1(t)$ is strictly increasing for $t \in [0, 1]$ and attains its maximum at $t = 1$ [3, p. 193, Lem. 3]. We have

$$\varphi(t) < \varphi_1(t) \leq \varphi_1\left(1 - \frac{\epsilon}{2}\right) = \frac{2(1 - \frac{\epsilon}{4})^p}{\left(1 + \left(1 - \frac{\epsilon}{2}\right)^p\right)} =: C_1(\epsilon) < 1.$$

Hence $1 - \varphi(t) > 1 - C_1(\epsilon) > 0$.

(Case 2) If $\tilde{\epsilon} > \epsilon/2$, then $\delta(a, r, \tilde{\epsilon}) \geq \delta(a, r, \epsilon/2)$ by Proposition 2.5. We have

$$\varphi(t) \leq \frac{2\left(\frac{1+t}{2} - t\delta(a, r, \epsilon/2)\right)^p}{1 + t^p} =: \varphi_2(t).$$

The maximum of $\varphi_2(t)$ is attained at $t = \big(1 - 2\delta(a, r, \epsilon/2)\big)^{\frac{1}{p-1}}$. For the sake of simplicity, we write $\delta := \delta(a, r, \epsilon/2)$. Since $\delta > 0$, we have

$$\varphi_2(t) \leq \varphi_2\big((1-2\delta)^{\frac{1}{p-1}}\big) = \frac{2\left(\frac{1 + (1-2\delta)^{\frac{1}{p-1}}}{2} - (1-2\delta)^{\frac{1}{p-1}}\delta\right)^p}{1 + \left((1-2\delta)^{\frac{1}{p-1}}\right)^p} =: C_2(\delta) < 1.$$

Hence $1 - \varphi(t) \geq 1 - \varphi_2(t) \geq 1 - C_2(\delta) > 0$.

In all cases, we can get a number $\eta_p(a,r,\epsilon)$ such that $0 < \eta_p(a,r,\epsilon) \leq \min\{1 - C_1(\epsilon), 1 - C_2(\delta)\}$. Hence

$$\begin{aligned} &1 - \frac{d\left(a, \frac{1-t}{2}a \oplus \frac{1+t}{2}\left(\frac{1}{1+t}x \oplus \frac{t}{1+t}\tilde{y}\right)\right)^p}{\frac{1}{2}\big(d(a,x)^p + d(a,y)^p\big)} \\ &\geq 1 - \varphi(t) \geq \min\{1 - C_1(\epsilon), 1 - C_2(\delta)\} \geq \eta_p(a,r,\epsilon). \end{aligned}$$

Therefore

$$d\left(a, \frac{1-t}{2}a \oplus \frac{1+t}{2}\left(\frac{1}{1+t}x \oplus \frac{t}{1+t}\tilde{y}\right)\right)^p \leq \big(1 - \eta_p(a,r,\epsilon)\big)\frac{d(a,x)^p + d(a,y)^p}{2}.$$

□

Given any two distinct points x and y in a Busemann space having the geodesic extension property, there exists a unique metric line passing through x and y. For $r \geq 0$, $(1+r)x \ominus ry$ denotes the unique point z on this metric line satisfying $d(z,x) = rd(x,y)$ and $d(z,y) = (1+r)d(x,y)$. From this, we can rewrite Theorem 3.1 in the following form.

Theorem 3.2. *Let (X,d) be a complete Busemann space having the geodesic extension property and $1 < p < \infty$. The weak uniform convexity of X is equivalent to the following property: for every $a \in X$, $r > 0$, and $\epsilon \in (0,2]$, there exists a number $\eta_p(a,r,\epsilon) > 0$ such that, for every $t \in [0,1]$ and three distinct points $a, x, y \in X$, the conditions $d(a,x) \leq r$, $d(a,y) \leq r$, and $d(x,y) \geq \epsilon r$ imply*

$$\begin{aligned} &d\left(a, \frac{1-t}{2}a \oplus \frac{1+t}{2}\left(\frac{1}{1+t}x \oplus \frac{t}{1+t}\left(\frac{1}{t}y \ominus \frac{1-t}{t}a\right)\right)\right)^p \\ &\leq \big(1 - \eta_p(a,r,\epsilon)\big)\frac{d(a,x)^p + d(a,y)^p}{2}. \end{aligned} \tag{3.2}$$

Theorem 3.1 is a generalization of the characterization of uniform convexity of Banach spaces. Since a Banach space $(E, \|\cdot\|)$ is complete and has the geodesic extension property, the conditions of Theorem 3.2 can be rewritten in the following form: $d(a,x) = \|\boldsymbol{x}\| \leq r = 1$, $d(a,y) = \|\boldsymbol{y}\| \leq 1$, and $d(x,y) = \|\boldsymbol{x} - \boldsymbol{y}\| \geq \epsilon$ for $\boldsymbol{x}, \boldsymbol{y} \in E$. We set $\tilde{\boldsymbol{y}} = \boldsymbol{y}/\|\boldsymbol{y}\|$ and $t = \|\boldsymbol{y}\|$. Hence

$$\begin{aligned} d\left(a, \frac{1-t}{2}a \oplus \frac{1+t}{2}\left(\frac{1}{1+t}x \oplus \frac{t}{1+t}\tilde{y}\right)\right) &= \left\|\frac{1+t}{2}\left(\frac{1}{1+t}\boldsymbol{x} + \frac{t}{1+t}\tilde{\boldsymbol{y}}\right)\right\| \\ = \left\|\frac{1+t}{2}\left(\frac{1}{1+t}\boldsymbol{x} + \frac{t}{1+t}\left(\frac{\boldsymbol{y}}{t}\right)\right)\right\| &= \left\|\frac{1}{2}\boldsymbol{x} + \frac{1}{2}\boldsymbol{y}\right\|. \end{aligned}$$

Therefore we obtain the following theorem.

Theorem 3.3 ([3, p. 190, Prop. 1])**.** *Let $(E, \|\cdot\|)$ be a Banach space and $1 < p < \infty$. The uniform convexity of E is equivalent to the following property: for every $\epsilon \in$*

$(0, 2]$, *there exists a number* $\eta_p(\epsilon) > 0$ *such that, for every distinct vectors* $\boldsymbol{x}, \boldsymbol{y} \in E$, *the conditions* $\|\boldsymbol{x}\| \leq 1$, $\|\boldsymbol{y}\| \leq 1$, *and* $\|\boldsymbol{x} - \boldsymbol{y}\| \geq \epsilon$ *imply*

$$\left\| \frac{\boldsymbol{x} + \boldsymbol{y}}{2} \right\|^p \leq \big(1 - \eta_p(\epsilon)\big) \frac{\|\boldsymbol{x}\|^p + \|\boldsymbol{y}\|^p}{2}.$$

Question 1. Theorem 3.3 provides important properties in the geometry of Banach spaces [14, 23]. Similarly, can Theorem 3.1 provide new geometric properties in Busemann spaces?

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Manuscript received 20 February 2025
revised 27 December 2025

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