Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167

Volume 11, Number 3, 2025, 101–121

MODIFIED FIRST VARIATION FORMULAE AND GEODESICALLY MONOTONE MAPPINGS

YASUNORI KIMURA AND SHUTA SUDO

Abstract. In this paper, we propose a class of proximal-type mappings on geodesic spaces having upper-bounded curvature. We first show modified first variation formulae using tangent spaces on a geodesic space and modified logarithmic mappings.

1. Introduction

Fixed point theory is one of the most crucial topics in nonlinear analysis. In particular, the existence of fixed points of nonlinear mappings and their approximation techniques have been studied by many researchers. We have many variations of considered mappings that we investigate. Let H be a Hilbert space and T a mapping. We say that T is firmly nonexpansive [4, 10, 11, 14] if for $x, y \in H$,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle,$$

or equivalently

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 - \|Tx - x\|^2 - \|Ty - y\|^2.$$

Such a mapping is also said to be inversely strongly monotone [30, 36, 38]. It is well known that the proximal mapping J_f for a lower semicontinuous proper convex function f on H is firmly nonexpansive, which is defined by

$$J_f x = \operatorname*{Argmin}_{y \in H} \left(f(y) + \frac{1}{2}\|y - x\|^2 \right)$$

for $x \in H$. If T is firmly nonexpansive, then it is nonexpansive and nonspreading [29]. That is, for $x, y \in H$, we have

$$\|Tx - Ty\| \leq \|x - y\|$$

and

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2.$$

These mappings behave very well in fixed point theory. For instance, in some appropriate settings, they have a fixed point; see [25, 29] and the references therein. Furthermore, we have some fixed point approximation schemes for such mappings; see [12, 31, 33, 37, 39], for instance. If considered mappings have a fixed point, they have more useful properties. Assume that a firmly nonexpansive mapping T on H

2020 *Mathematics Subject Classification.* 47H10, 52A41.
Key words and phrases. Geodesic space, first variation formula, resolvent operator.
This work was partially supported by JSPS KAKENHI Grant Number JP21K03316.

has a fixed point. We denote the set of all fixed points of T by $\operatorname{Fix} T$. Then, for $x \in H$ and $y \in \operatorname{Fix} T$, we know that

$$\langle Tx - y, x - Tx \rangle \geq 0.$$

We call such a mapping a cutter mapping [2, 5, 9]. It is called a directed operator [6, 7] or a firmly quasinonexpansive mapping [40] as well.

In a general metric space (X, d), we can also define such classes of mappings. That is, we say that a mapping T on X is firmly nonexpansive if

$$2d(Tx, Ty)^2 \leq d(Tx, y)^2 + d(Ty, x)^2 - d(Tx, x)^2 - d(Ty, y)^2$$

for $x, y \in X$; it is nonexpansive if

$$d(Tx, Ty) \leq d(x, y)$$

for $x, y \in X$; it is metrically nonspreading [28] if

$$2d(Tx, Ty)^2 \leq d(Tx, y)^2 + d(Ty, x)^2$$

for $x, y \in X$. However, to define cutter mappings, we need to use inner products, namely, the underlying spaces must be linear spaces at least.

Recently, the classifications such as the above have been discussed in the setting of geodesic spaces. Particularly, Hadamard spaces are famous as complete geodesic spaces which have some reasonable structure that Hilbert spaces have. In general, a geodesic space is defined as a metric space which has geodesics connecting each two points. We know that geodesics enable us to define convex combinations, and hence we can define the convexity of subsets and functions. Moreover, if a geodesic space has upper-bounded curvature $\kappa \in \mathbb{R}$, then we call such a space a CAT(κ) space; a Hadamard space is defined as a complete CAT(0) space. In Hadamard spaces, we have fixed point existence theorems for nonexpansive mappings and nonspreading mappings; see [1, 26, 28]. In a Hadamard space H, Jost [13] and Mayer [32] proposed the proximal mapping J_f for a lower semicontinuous proper convex function f on H, which is defined by

$$J_f x = \operatorname*{Argmin}_{y \in H} \left(f(y) + \frac{1}{2} d(y, x)^2 \right)$$

for $x \in H$. In this setting, J_f is firmly nonexpansive, and hence it is nonexpansive and metrically nonspreading; see [20]. However, in general, CAT(κ) spaces do not behave like Hilbert spaces since spherical and hyperbolical surfaces are also categorised as CAT(κ) spaces. For example, the unit sphere of a Hilbert space is a CAT(1) space, and then we need to consider another technique than the Hilbert space setting. Suppose f is lower semicontinuous, proper and convex on an admissible complete CAT(1) space X. Then, we define a mapping R_f related to the proximal mapping by

$$R_f x = \operatorname*{Argmin}_{y \in X} \left(f(y) - \log \cos d(y, x) \right)$$

for $x \in X$. Then, we do not know if the mapping R_f is firmly nonexpansive, nonexpansive or metrically nonspreading. Nevertheless, we know that R_f is spherically nonspreading [15], that is,

$$2\cos d(R_f x, R_f y) \geq \cos d(R_f x, y) + \cos d(R_f y, x)$$

for $x, y \in X$. Such types of mapping are effective for fixed point theory in the spherical setting. Similarly, some hyperbolic models, such as hyperboloids and disk models, are categorised to CAT(-1) space. Suppose f is lower semicontinuous, proper and convex on a complete CAT(-1) space X. Then, we define a mapping R_f related to the proximal mapping by

$$R_f x = \operatorname*{Argmin}_{y \in X} \left(f(y) + \tanh d(y, x) \sinh d(y, x) \right)$$

for $x \in X$. Such as the case of CAT(1) spaces, the mapping R_f is hyperbolically nonspreading [16], that is,

$$2\cosh d(R_f x, R_f y) \leq \cosh d(R_f x, y) + \cosh d(R_f y, x)$$

for $x, y \in X$. Consequently, in the theory of CAT(κ) spaces, we should adopt functions matching the parameter κ.

On the other hand, focusing on the Alexandrov angle of CAT(κ) spaces, we can define tangent spaces and a metric corresponding to Riemannian manifolds. Using these notions, we introduce a function, such as inner products. Chaipunya, Kohsaka and Kumam [8] studied the proximal mapping on a Hadamard space via the notion of tangent spaces, and they show that tangent spaces are effective for the study of geodesic spaces. Motivated by this study, Kimura and Sudo [24] introduced related notions to them.

In this work, we propose some classes mappings on a CAT(κ) space for a general real number κ. These classes have a high affinity with firmly nonexpansive mappings and cutter mappings if the underlying space is a Hilbert space; they match the parameter κ. To this end, we show modified first variation formulae; see [3] or Theorem 3.1 for the original first variation formula.

2. Preliminaries

Let $\mathbb{M}^2_\kappa$ be the two-dimensional model space and D_κ the space diameter. That is,

$$\mathbb{M}^2_\kappa = \begin{cases} \dfrac{1}{\sqrt{\kappa}} \mathbb{S}^2 & (\kappa > 0); \\ \mathbb{E}^2 & (\kappa = 0); \\ \dfrac{1}{\sqrt{-\kappa}} \mathbb{H}^2 & (\kappa < 0), \end{cases}$$

and

$$D_\kappa = \operatorname{diam} \mathbb{M}^2_\kappa = \begin{cases} \infty & (\kappa \leq 0); \\ \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0). \end{cases}$$

For $\kappa \in \mathbb{R}$, a CAT(κ) space is defined as a uniquely D_κ-geodesic space satisfying the CAT(κ) inequality with the two-dimensional model space $\mathbb{M}_\kappa^2$. Namely, any two points x and y in a CAT(κ) space (M,d) with $d(x,y) < D_\kappa$ can be connected with a unique isometric mapping γ_{xy} from $[0, d(x,y)]$ to M such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(d(x,y)) = y$, and for any $x,y,z \in M$ with

$$d(y,z) + d(z,x) + d(x,y) < 2D_\kappa,$$

two points p,q on the triangle $\triangle(x,y,z)$ and their comparison points $\overline{p}$ and $\overline{q}$ on $\triangle(\overline{x},\overline{y},\overline{z})$ of $\mathbb{M}_\kappa^2$ satisfy

$$d(p,q) \le d_{\mathbb{M}_\kappa^2}(\overline{p},\overline{q}).$$

In a CAT(κ) space M, for $x,y \in M$ with $d(x,y) < D_\kappa$, there exists a unique mapping γ_{xy} mentioned above. We call it a geodesic from x to y, and then we define convex combination of x and y with a ratio $t \in [0,1]$ by

$$tx \oplus (1-t)y = \gamma_{xy}((1-t)d(x,y)).$$

In this paper, we suppose that a CAT(κ) space M is admissible [21]. Namely,

$$d(x,y) < \frac{D_\kappa}{2}$$

for any $x,y \in M$. From the definition of D_κ, for $\kappa \le 0$, every CAT(κ) space is always admissible.

Bridson and Haefliger [3] introduced the following function to characterise Riemannian metrics on the finite-dimensional model spaces:

$$f(\kappa,a) = a + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n-1}}{(2n-1)!} = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}a\right) & (\kappa > 0); \\ a & (\kappa = 0); \\ \frac{1}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa}a\right) & (\kappa < 0) \end{cases}$$

for $\kappa, a \in \mathbb{R}$. In this paper, for fixed $\kappa \in \mathbb{R}$, we denote this function by s_κ, that is,

$$s_\kappa(a) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin\left(\sqrt{\kappa}a\right) & (\kappa > 0); \\ a & (\kappa = 0); \\ \frac{1}{\sqrt{-\kappa}} \sinh\left(\sqrt{-\kappa}a\right) & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. Then, for $a \in \mathbb{R}$, we know that

$$s'_\kappa(a) = \begin{cases} \cos\left(\sqrt{\kappa}a\right) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh\left(\sqrt{-\kappa}a\right) & (\kappa < 0). \end{cases}$$

We notice that the following formulae hold: For $a \in \mathbb{R}$,

$$s'_\kappa(a)^2 + \kappa s_\kappa(a)^2 = 1.$$

For $a, b \in \mathbb{R}$,

$$\begin{aligned} s_\kappa(a+b) &= s_\kappa(a)s'_\kappa(b) + s_\kappa(b)s'_\kappa(a); \\ s_\kappa(a-b) &= s_\kappa(a)s'_\kappa(b) - s_\kappa(b)s'_\kappa(a); \\ s'_\kappa(a+b) &= s'_\kappa(a)s'_\kappa(b) - \kappa s_\kappa(a)s_\kappa(b); \\ s'_\kappa(a-b) &= s'_\kappa(a)s'_\kappa(b) + \kappa s_\kappa(a)s_\kappa(b). \end{aligned}$$

These identities imply that for $a, b \in \mathbb{R}$,

$$\begin{aligned} s_\kappa(a) + s_\kappa(b) &= 2s_\kappa\left(\frac{a+b}{2}\right)s'_\kappa\left(\frac{a-b}{2}\right); \\ s_\kappa(a) - s_\kappa(b) &= 2s'_\kappa\left(\frac{a+b}{2}\right)s_\kappa\left(\frac{a-b}{2}\right); \\ s'_\kappa(a) + s'_\kappa(b) &= 2s'_\kappa\left(\frac{a+b}{2}\right)s'_\kappa\left(\frac{a-b}{2}\right); \\ s'_\kappa(a) - s'_\kappa(b) &= -2\kappa s_\kappa\left(\frac{a+b}{2}\right)s_\kappa\left(\frac{a-b}{2}\right), \end{aligned}$$

and

$$\begin{aligned} s_\kappa(a)s'_\kappa(b) &= \frac{1}{2}(s_\kappa(a+b) + s_\kappa(a-b)); \\ s'_\kappa(a)s_\kappa(b) &= \frac{1}{2}(s_\kappa(a+b) - s_\kappa(a-b)); \\ -\kappa s_\kappa(a)s_\kappa(b) &= \frac{1}{2}\big(s'_\kappa(a+b) - s'_\kappa(a-b)\big); \\ s'_\kappa(a)s'_\kappa(b) &= \frac{1}{2}\big(s'_\kappa(a+b) + s'_\kappa(a-b)\big). \end{aligned}$$

On the other hand, Kajimura and Kimura [17] define a function c_κ by

$$c_\kappa(a) = \begin{cases} \dfrac{1-\cos\left(\sqrt{\kappa}a\right)}{\kappa} & (\kappa > 0); \\ \dfrac{1}{2}a^2 & (\kappa = 0); \\ \dfrac{\cosh\left(\sqrt{-\kappa}a\right)-1}{-\kappa} & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$; we notice that

$$c_\kappa(a) = \int_0^a s_\kappa(r)\,\mathrm{d}r.$$

Let (M, d) be an admissible CAT(κ) space. We define a function ϕ_κ on M^2 by

$$\phi_\kappa(x,y) = \int_0^{d(x,y)} s_\kappa(r)\,\mathrm{d}r = \begin{cases} \dfrac{1-\cos\left(\sqrt{\kappa}d(x,y)\right)}{\kappa} & (\kappa > 0); \\ \dfrac{1}{2}d(x,y)^2 & (\kappa = 0); \\ \dfrac{\cosh\left(\sqrt{-\kappa}d(x,y)\right)-1}{-\kappa} & (\kappa < 0) \end{cases}$$

for $x, y \in M$. Then, we know the following:

- For $x, y \in M$, $\phi_\kappa(x, y) \geq 0$;
- for $x, y \in M$, $\phi_\kappa(x, y) = 0$ if and only if $x = y$;
- for $x, y \in M$, $\phi_\kappa(x, y) = \phi_\kappa(y, x)$.

For more details about the function ϕ_κ, see [23].

3. Tangent spaces and logarithmic mappings

We next define tangent spaces on CAT(κ) spaces. We first recall the notion of the Alexandrov angle. Let M be an admissible CAT(κ) space and $p \in M$. Then, the Alexandrov angle A_p at p is defined by

$$A_p(x, y) = \limsup_{t \to 0+} \arccos\left(1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2}\right) \in [0, \pi]$$

if $p \neq x$ and $p \neq y$; $A_p(x, p) = A_p(p, x) = \pi/2$ if $p \neq x$; $A_p(p, p) = 0$. Then, we have the following formula, which is called the first variation formula:

Theorem 3.1 (Bridson–Haefliger [3, Corollary 3.5 in Chapter II.3])**.** *Let M be an admissible* CAT(κ) *space. Then,*

$$\lim_{t \to 0+} \frac{d(p, y) - d(tx \oplus (1-t)p, y)}{t} = d(p, x) \cos A_p(x, y)$$

for $p, x, y \in M$ with $p \neq y$.

Let M be an admissible CAT(κ) space and $p \in M$. For $x \in M$, let

$$[x]_p = \{y \in M \mid A_p(x, y) = 0\},$$

and let

$$D_pM = \{[x]_p \mid x \in M\}.$$

We call D_pM the direction space from p. We next define an equivalence relation on the Cartesian product

$$[0, \infty[\times D_pM.$$

Let i_p be a function from D_pM to $\{0, 1\}$ defined by

$$i_p([x]_p) = \begin{cases} 1 & ([x]_p \neq [p]_p); \\ 0 & ([x]_p = [p]_p) \end{cases}$$

for $[x]_p \in D_pM$. We define a binary relation $\simeq_p$ on $[0, \infty[\times D_pM$ by

$$(r, [x]_p) \simeq_p (s, [y]_p)$$

if one of the following holds:

- $r \cdot i_p([x]_p) = s \cdot i_p([y]_p) = 0$;
- $r \cdot i_p([x]_p) = s \cdot i_p([y]_p) > 0$ and $[x]_p = [y]_p$.

Then, this relation $\simeq_p$ is an equivalence one. We define a quotient set T_pM by

$$T_pM = ([0,\infty[\times D_pM)/{\simeq_p}.$$

For simplicity, we denote an element $[(r,[x]_p)]_{\simeq_p}$ of T_pM by $r[x]_p$. In particular, we denote $0[p]_p$ by 0_p. For $v_p = r[v]_p \in T_pM$ and $t \geq 0$, we define tv_p by

$$tv_p = (tr)[v]_p \in T_pM.$$

Particularly, for $v_p = r[v]_p \in T_pM$ and $t > 0$, we define v_p/t by

$$\frac{v_p}{t} = \left(\frac{r}{t}\right)[v]_p \in T_pM.$$

Furthermore, for $r[x]_p \in T_pM$, we denote the value $r \cdot i_p([x]_p)$ by $\|r[x]_p\|$. Then, we define a distance function d_p on T_pM by

$$d_p(r[x]_p, s[y]_p) = \sqrt{\|r[x]_p\|^2 + \|s[y]_p\|^2 - 2\|r[x]_p\|\|s[y]_p\| \cos A_p(x,y)}$$

for $r[x]_p, s[y]_p \in T_pM$. We call this metric space (T_pM, d_p) the tangent space of M at p.

Let M be an admissible CAT(κ) space and $p \in M$. We define a logarithmic mapping $\log_p$ from M to T_pM by

$$\log_p x = d(p,x)[x]_p$$

for $x \in M$. Then,

$$\left\|\log_p x\right\| = d(p,x).$$

This mapping is an analogous notion to the inverse mapping of the exponential mapping on Riemannian manifolds. We define a bifunction g_p on T_pM by

$$g_p(u_p, v_p) = \frac{\|u_p\|^2 + \|v_p\|^2 - d_p(u_p,v_p)^2}{2}$$

for $u_p, v_p \in T_pM$. Then, the following hold:

- For $v_p \in T_pM$, $g_p(v_p,v_p) = \|v_p\|^2 \geq 0$;
- for $u_p, v_p \in T_pM$, $g_p(u_p,v_p) = g_p(v_p,u_p)$;
- for $u_p, v_p \in T_pM$ and $t \geq 0$, $g_p(tu_p,v_p) = tg_p(u_p,v_p)$;
- for $x, y \in M$, $d(x,y)^2 = g_x(\log_x y, \log_x y) = g_y(\log_y x, \log_y x)$.

Further, we define another logarithmic mapping $\log_{\kappa,p}$ by

$$\log_{\kappa,p} x = s_\kappa(d(p,x))[x]_p$$

for $x \in M$. Then, we know the following:

Theorem 3.2 (Kimura–Sudo [24]). *Let M be an admissible* CAT(κ) *space. Then,*

$$g_p(\log_{\kappa,p} x, \log_{\kappa,p} y) \geq \phi_\kappa(p,x) + s'_\kappa(d(p,x))\phi_\kappa(p,y) - \phi_\kappa(x,y)$$

for $p, x, y \in M$.

Let M be an admissible CAT(κ) space and T a mapping on M. We call a point $x \in M$ a fixed point of T if

$$Tx = x,$$

and we denote the set of all fixed points of T by $\operatorname{Fix} T$. We say that T is directed [34] if it has a fixed point, and

$$g_{Tx}(\log_{Tx} y, \log_{Tx} x) \leq 0$$

for $x \in M$ and $y \in \operatorname{Fix} T$. We know that if T is directed, then it is quasinonexpansive, namely,

$$d(Tx, y) \leq d(x, y)$$

for $x \in M$ and $y \in \operatorname{Fix} T$. Therefore, if T is directed, then $\operatorname{Fix} T$ is a closed convex set. For more details, see [34].

Let H be a Hilbert space and T a directed operator in the sense of geodesic spaces. Then, for $x \in H$ and $y \in \operatorname{Fix} T$, we have

$$0 \geq g_{Tx}(\log_{Tx} y, \log_{Tx} x) = \langle y - Tx, x - Tx \rangle,$$

and hence

$$\langle Tx - y, x - Tx \rangle \geq 0.$$

It means that directed operators in the sense of geodesic spaces are a natural generalisation of cutter mappings.

4. Perturbations of resolvent operators

In general, we cannot determine whether a minimiser of the considered convex function is unique. However, for a lower semicontinuous proper convex function f on a Hilbert space H and $x \in H$, a function defined by

$$f(y) + \|y - x\|^2 \tag{4.1}$$

for $y \in H$ has a unique minimiser. We call the term '$\|y - x\|^2$' a perturbation.

In an admissible complete CAT(κ) space M, we say that a function f from M to $]-\infty, \infty]$ is convex if

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y)$$

for $x, y \in M$ and $t \in]0, 1[$. For a lower semicontinuous proper convex function f on M and $x \in M$, we consider a function f_x defined by

$$f_x(y) = f(y) + P(d(y, x))$$

for $y \in M$. Namely, we add a perturbation with a function P. On the other hand, in Hilbert spaces, we typically use quadratic functions, such as (4.1). However, unlike the setting of Hilbert spaces, different types of perturbations, rather than (4.1), are introduced in CAT(κ) spaces. For instance, we have known the following perturbations in a CAT(1) space:

(i) $1 - \cos d(y, x)$,
(ii) $-\log \cos d(y, x)$,

(iii) $\tan d(y,x) \cdot \sin d(y,x)$,
(iv) $1 - \cos d(y,x) - \log\cos d(y,x)$.

We notice that

$$\int_0^a \sin r \,\mathrm{d}r = 1 - \cos a;$$
$$\int_0^a \tan r \,\mathrm{d}r = -\log|\cos a|;$$
$$\int_0^a \left(\sin r + \frac{\tan r}{\cos r}\right) \mathrm{d}r = \tan a \cdot \sin a;$$
$$\int_0^a (\sin r + \tan r)\,\mathrm{d}r = 1 - \cos a - \log|\cos a|.$$

We also know the following perturbations in a CAT(−1) space:

(v) $\cosh d(y,x) - 1$,
(vi) $\log\cosh d(y,x)$,
(vii) $\tanh d(y,x) \cdot \sinh d(y,x)$,
(viii) $\cosh d(y,x) - 1 + \log\cosh d(y,x)$.

We notice that

$$\int_0^a \sinh r \,\mathrm{d}r = \cosh a - 1;$$
$$\int_0^a \tanh r \,\mathrm{d}r = \log\cosh a;$$
$$\int_0^a \left(\sinh r + \frac{\tanh r}{\cosh r}\right) \mathrm{d}r = \tanh a \cdot \sinh a;$$
$$\int_0^a (\sinh r + \tanh r)\,\mathrm{d}r = \cosh a - 1 + \log\cosh a.$$

The cases (i) and (v) are investigated by Sudo [34]. The cases (ii) and (vii) are introduced by Kajimura and Kimura [15, 16]. The case (iii) is done by Kimura and Kohsaka [19]. In the case (vi), Kimura and Nakadai [22] adapted the perturbation for a convex function bounded below. The cases (iv) and (viii) are investigated by Kajimura, Kimura and Kohsaka [18].

To integrate those perturbations, we use a function t_κ as follows: For $\kappa \in \mathbb{R}$ and $a \in \mathbb{R}$ with $s'_\kappa(a) \neq 0$,

$$t_\kappa(a) = \frac{s_\kappa(a)}{s'_\kappa(a)} = \begin{cases} \dfrac{\tan(\sqrt{\kappa}a)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\tanh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} & (\kappa < 0). \end{cases}$$

Then, we know that

$$\int_0^a s_\kappa(r)\,\mathrm{d}r = c_\kappa(a) = \begin{cases} \dfrac{1}{2}a^2 & (\kappa = 0); \\ \dfrac{1 - s'_\kappa(a)}{\kappa} & (\kappa \neq 0); \end{cases}$$

$$\int_0^a t_\kappa(r)\,\mathrm{d}r = \begin{cases} \dfrac{1}{2}a^2 & (\kappa = 0); \\ \dfrac{-\log|s'_\kappa(a)|}{\kappa} & (\kappa \neq 0); \end{cases}$$

$$\int_0^a \left(s_\kappa(r) + \frac{t_\kappa(r)}{s'_\kappa(r)} \right) \mathrm{d}r = t_\kappa(a) s_\kappa(a);$$

$$\int_0^a (s_\kappa(r) + t_\kappa(r))\,\mathrm{d}r = \begin{cases} a^2 & (\kappa = 0); \\ \dfrac{1 - s'_\kappa(a) - \log|s'_\kappa(a)|}{\kappa} & (\kappa \neq 0). \end{cases}$$

We focus on their integrands:

(1) $\eta_0 = s_\kappa$;
(2) $\eta_1 = t_\kappa$;
(3) $\eta_2 = s_\kappa + t_\kappa / s'_\kappa$;
(4) $\eta_3 = s_\kappa + t_\kappa$.

We notice that

$$\begin{aligned} \eta_0(a) &= s_\kappa(a) = s_\kappa(a) \times 1; \\ \eta_1(a) &= t_\kappa(a) = s_\kappa(a) \times \frac{1}{s'_\kappa(a)}; \\ \eta_2(a) &= s_\kappa(a) + \frac{t_\kappa(a)}{s'_\kappa(a)} = s_\kappa(a) \times \left(1 + \frac{1}{s'_\kappa(a)^2}\right); \\ \eta_3(a) &= s_\kappa(a) + t_\kappa(a) = s_\kappa(a) \times \left(1 + \frac{1}{s'_\kappa(a)}\right) \end{aligned}$$

for $a \in \mathbb{R}$ with $s'_\kappa(a) \neq 0$.

Let η be a function from $[0, D_\kappa/2[$ to $[0, \infty[$. We suppose the following conditions:

(P1) $\eta(a) = 0$ if and only if $a = 0$;
(P2) η is continuous.

Clearly, η_0, η_1, η_2 and η_3 satisfy both of the two conditions. On the other hand, the identity function

$$\eta_{00} \colon a \mapsto a$$

on $[0, D_\kappa/2[$ also satisfies the two conditions.

Let M be an admissible CAT(κ) space and η a function from $[0, D_\kappa/2[$ to $[0, \infty[$ satisfying the condition (P1). Fix $p \in M$. We define a mapping $\log_{\eta,p}$ from M to T_pM by

$$\log_{\eta,p} x = \eta(d(p,x))[x]_p$$

for $x \in M$. From the condition (P1), we obtain the following:

- For $p, x \in M$, $\|\log_{\eta,p} x\| = \eta(d(p,x))$;
- for $p, x \in M$, $\log_{\eta,p} x = 0_p$ if and only if $x = p$;
- for $p, x, y \in M$, $g_p(\log_{\eta,p} x, \log_p y) = \eta(d(p,x))d(p,y)\cos A_p(x,y)$.

We further obtain the following modified first variation formula:

Theorem 4.1. *Let M be an admissible* CAT(κ) *space and η a function on* $[0, D_\kappa/2[$ *satisfying the condition* (P1) *and* (P2)*. Then, for $p, x, y \in M$,*

$$\lim_{t\to 0+} \int_{d(tx\oplus(1-t)p,y)}^{d(p,y)} \frac{\eta(r)}{t}\,\mathrm{d}r = g_p(\log_p x, \log_{\eta,p} y).$$

Proof. For $p, x, y \in M$ and $t \in]0,1[$, let

$$l = d(p,y) \text{ and } l_t = d(tx \oplus (1-t)p, y).$$

Notice that $l_t \to l$ as $t \to 0+$. We first consider the case where $p \neq y$. Then, from Theorem 3.1, we have

$$\lim_{t\to 0+} \frac{l - l_t}{t} = \lim_{t\to 0+} \frac{d(p,y) - d(tx \oplus (1-t)p, y)}{t} = d(p,x)\cos A_p(x,y). \tag{4.2}$$

Fix $t \in]0,1[$ arbitrarily. We denote the antiderivative of η by E. Then,

$$\int_{l_t}^{l} \frac{\eta(r)}{t}\,\mathrm{d}r = \frac{E(l) - E(l_t)}{t} = \frac{E(l) - E(l_t)}{l - l_t} \cdot \frac{l - l_t}{t}.$$

Putting $h_t = l_t - l$, we have

$$\int_{l_t}^{l} \frac{\eta(r)}{t}\,\mathrm{d}r = \frac{E(l + h_t) - E(l)}{h_t} \cdot \frac{l - l_t}{t}. \tag{4.3}$$

Since $h_t \to 0$ as $t \to 0+$, we have

$$\lim_{t\to 0+} \frac{E(l + h_t) - E(l)}{h_t} = E'(l) = \eta(l).$$

Letting $t \to 0+$ for (4.3), we obtain from (4.2) that

$$\lim_{t\to 0+} \int_{l_t}^{l} \frac{\eta(r)}{t}\,\mathrm{d}r = \eta(l)d(p,x)\cos A_p(x,y) = g_p(\log_p x, \log_{\eta,p} y).$$

We next show the case where $p = y$. Since $\log_{\eta,p} y = 0_p$, we have

$$g_p(\log_p x, \log_{\eta,p} y) = 0.$$

Further, for fixed $t \in]0,1[$, we get

$$\begin{aligned}\int_{l_t}^{l} \frac{\eta(r)}{t}\,\mathrm{d}r &= \frac{E(l) - E(l_t)}{t} = \frac{E(0) - E(d(tx \oplus (1-t)p, y))}{t} \\ &= \frac{E(0) - E(td(x,y))}{t}.\end{aligned}$$

From l'Hôpital's rule, and the conditions (P1) and (P2), we have

$$\lim_{t\to 0+} \int_{l_t}^{l} \frac{\eta(r)}{t}\,\mathrm{d}r = \lim_{t\to 0+} \frac{E(0) - E(td(x,y))}{t} = -d(x,y) \lim_{t\to 0+} \eta(td(x,y)) = 0,$$

and hence

$$\lim_{t\to 0+}\int_{l_t}^{l}\frac{\eta(r)}{t}\,\mathrm{d}r = 0 = g_p(\log_p x, \log_{\eta,p} y).$$

This completes the proof. □

We next show the following:

Theorem 4.2. *Let M be an admissible* $\mathrm{CAT}(\kappa)$ *space and η a function on* $[0, D_\kappa/2[$ *satisfying the two conditions* (P1) *and* (P2). *For a lower semicontinuous proper convex function f on M, assume that a single-valued mapping R_f^η on M can be defined by*

$$R_f^\eta x = \operatorname*{Argmin}_{y\in M}\left(f(y) + \int_0^{d(y,x)}\eta(r)\,\mathrm{d}r\right)$$

for $x \in M$. Then, the following hold:

(i) *For $x \in M$,*

$$f(R_f^\eta x) \le \inf_{w\in M}\Big(f(w) - g_{R_f^\eta x}(\log_{R_f^\eta x} w, \log_{\eta,R_f^\eta x} x)\Big);$$

(ii) $\operatorname{Min} f = \operatorname{Fix} R_f^\eta$*;*

(iii) *if f has a minimiser, then R_f^η is directed.*

Proof. Fix $w, x \in M$ and $t \in]0,1[$ arbitrarily. Let $w_t = tw \oplus (1-t)R_f^\eta x$. Then,

$$\begin{aligned} f(R_f^\eta x) + \int_0^{d(R_f^\eta x,x)}\eta(r)\,\mathrm{d}r &\le f(w_t) + \int_0^{d(w_t,x)}\eta(r)\,\mathrm{d}r \\ &\le tf(w) + (1-t)f(R_f^\eta x) + \int_0^{d(w_t,x)}\eta(r)\,\mathrm{d}r, \end{aligned}$$

and thus

$$tf(R_f^\eta x) \le tf(w) - \int_{d(tw\oplus(1-t)R_f^\eta x,x)}^{d(R_f^\eta x,x)}\eta(r)\,\mathrm{d}r.$$

Dividing both sides by t and letting $t \to 0+$, we have

$$f(R_f^\eta x) \le f(w) - g_{R_f^\eta x}(\log_{R_f^\eta x} w, \log_{\eta,R_f^\eta x} x)$$

from Theorem 4.1. Since $w \in M$ is arbitrary,

$$f(R_f^\eta x) \le \inf_{w\in M}\Big(f(w) - g_{R_f^\eta x}(\log_{R_f^\eta x} w, \log_{\eta,R_f^\eta x} x)\Big).$$

We next show (ii). Fix $x \in \operatorname{Fix} R_f^\eta$. Then,

$$f(x) = f(R_f^\eta x) \le \inf_{w\in M}\Big(f(w) - g_{R_f^\eta x}(\log_{R_f^\eta x} w, \log_{\eta,R_f^\eta x} x)\Big) = \inf_{w\in M} f(w),$$

and hence $x \in \operatorname{Min} f$. Inversely, fix $x \in \operatorname{Min} f$. Then, from the definition of R_f^η,

$$f(R_f^\eta x) + \int_0^{d(R_f^\eta x,x)}\eta(r)\,\mathrm{d}r \le f(x) + \int_0^{d(x,x)}\eta(r)\,\mathrm{d}r = f(x),$$

and thus for the antiderivative E of η, we have

$$E(d(R_f^\eta x, x)) - E(0) = \int_0^{d(R_f^\eta x, x)} \eta(r)\,\mathrm{d}r \le f(x) - f(R_f^\eta x) \le 0,$$

which implies that

$$E(d(R_f^\eta x, x)) \le E(0).$$

By (P1), we notice that η is positive on $]0, D_\kappa/2[$, and thus E is strictly increasing. Using this fact, we have

$$d(R_f^\eta x, x) \le 0,$$

and hence $x \in \operatorname{Fix} R_f^\eta$. Therefore,

$$\operatorname{Min} f = \operatorname{Fix} R_f^\eta.$$

We show (iii). Assume that f has a minimiser, and then R_f^η has a fixed point. From (i), for fixed $x \in M$ and $y \in \operatorname{Fix} R_f^\eta = \operatorname{Min} f$, we have

$$f(R_f^\eta x) \le f(y) - g_{R_f^\eta x}(\log_{R_f^\eta x} y, \log_{\eta, R_f^\eta x} x),$$

and thus

$$g_{R_f^\eta x}(\log_{R_f^\eta x} y, \log_{\eta, R_f^\eta x} x) \le f(y) - f(R_f^\eta x) \le 0.$$

If $R_f^\eta x \ne x$, then

$$g_{R_f^\eta x}(\log_{R_f^\eta x} y, \log_{R_f^\eta x} x) = \frac{d(R_f^\eta x, x)}{\eta(d(R_f^\eta x, x))} g_{R_f^\eta x}(\log_{R_f^\eta x} y, \log_{\eta, R_f^\eta x} x) \le 0.$$

If $R_f^\eta x = x$, then we immediately obtain

$$g_{R_f^\eta x}(\log_{R_f^\eta x} y, \log_{R_f^\eta x} x) = 0.$$

Therefore, R_f^η is a directed operator. It completes the proof. □

We next consider an equilibrium problem on CAT(κ) spaces. Let M be an admissible CAT(κ) space and K a nonempty closed convex subset of M. For a given function F on K^2, we call $x \in K$ an equilibrium point of F if

$$\inf_{y \in K} F(x, y) \ge 0.$$

We denote the set of all equilibrium points of F by $\operatorname{Equil} F$. Further, we assume the following conditions:

(E1) For $x \in K$, $F(x, x) = 0$;
(E2) for $x, y \in K$, $F(x, y) + F(y, x) \le 0$;
(E3) for $x \in K$, a function $F(x, \cdot)$ on K is lower semicontinuous and convex.

In a similar way to Theorem 4.2, we prove the following result about equilibrium problems:

Theorem 4.3. *Let M be an admissible* $\mathrm{CAT}(\kappa)$ *space and η a function on* $[0, D_\kappa/2[$ *satisfying the two conditions* (P1) *and* (P2). *Let K be a nonempty closed convex subset of M. For a function F on K^2 satisfying the three conditions* (E1)–(E3), *assume that a single-valued mapping R_F^η on M can be defined by*

$$R_F^\eta x = \left\{ z \in K \;\middle|\; \inf_{y\in K} \left(F(z,y) + \int_{d(z,x)}^{d(y,x)} \eta(r)\,\mathrm{d}r \right) \geq 0 \right\}$$

for $x \in M$. Then, the following hold:

(i) *For $x \in M$,*

$$0 \leq \inf_{w\in M} \Big(F(R_F^\eta x, w) - g_{R_F^\eta x}(\log_{R_F^\eta x} w, \log_{\eta, R_F^\eta x} x) \Big);$$

(ii) $\operatorname{Equil} F = \operatorname{Fix} R_F^\eta$;

(iii) *if F has an equilibrium point, then R_F^η is directed.*

Proof. Fix $w \in K$, $x \in M$ and $t \in \,]0,1[$ arbitrarily. Let $w_t = tw \oplus (1-t)R_F^\eta x \in K$. Then, from the conditions (E1) and (E3),

$$\begin{aligned} 0 &\leq \inf_{y\in K} \left(F(R_F^\eta x, y) + \int_{d(R_F^\eta x, x)}^{d(y,x)} \eta(r)\,\mathrm{d}r \right) \\ &\leq F(R_F^\eta x, w_t) + \int_{d(R_F^\eta x, x)}^{d(w_t,x)} \eta(r)\,\mathrm{d}r \\ &\leq tF(R_F^\eta x, w) + \int_{d(R_F^\eta x, x)}^{d(w_t,x)} \eta(r)\,\mathrm{d}r, \end{aligned}$$

and thus

$$0 \leq tF(R_F^\eta x, w) - \int_{d(tw\oplus(1-t)R_F^\eta x, x)}^{d(R_F^\eta x, x)} \eta(r)\,\mathrm{d}r$$

Dividing both sides by t and letting $t \to 0+$, we have

$$0 \leq F(R_F^\eta x, w) - g_{R_F^\eta x}(\log_{R_F^\eta x} w, \log_{\eta, R_F^\eta x} x)$$

from Theorem 4.1. Since $w \in M$ is arbitrary,

$$0 \leq \inf_{w\in M} \Big(F(R_F^\eta x, w) - g_{R_F^\eta x}(\log_{R_F^\eta x} w, \log_{\eta, R_F^\eta x} x) \Big).$$

We next show (ii). Fix $x \in \operatorname{Fix} R_F^\eta$. Then,

$$0 \leq \inf_{w\in M} \Big(F(R_F^\eta x, w) - g_{R_F^\eta x}(\log_{R_F^\eta x} w, \log_{\eta, R_F^\eta x} x) \Big) = \inf_{w\in M} F(x, w),$$

and hence $x \in \operatorname{Equil} F$. Inversely, fix $x \in \operatorname{Equil} F$. Then, from the definition of R_F^η,

$$0 \leq F(R_F^\eta x, x) + \int_{d(R_F^\eta x, x)}^{d(x,x)} \eta(r)\,\mathrm{d}r = F(R_F^\eta x, x) - \int_0^{d(R_F^\eta x, x)} \eta(r)\,\mathrm{d}r,$$

and thus for the antiderivative E of η, we have

$$E(d(R_F^\eta x, x)) - E(0) = \int_0^{d(R_F^\eta x, x)} \eta(r)\,\mathrm{d}r \leq F(R_F^\eta x, x).$$

Since $F(x, R_F^\eta x) \geq 0$, from the condition (E2),

$$E(d(R_F^\eta x, x)) - E(0) \leq F(R_F^\eta x, x) \leq -F(x, R_F^\eta x) \leq 0,$$

which implies that

$$E(d(R_F^\eta x, x)) \leq E(0).$$

Recall that η is positive on $]0, D_\kappa/2[$, and thus E is strictly increasing. Therefore, we have

$$d(R_F^\eta x, x) \leq 0,$$

and hence $x \in \operatorname{Fix} R_F^\eta$. Thus,

$$\operatorname{Equil} F = \operatorname{Fix} R_F^\eta.$$

We show (iii). Assume that F has an equilibrium point, and then R_F^η has a fixed point. From (i), for fixed $x \in M$ and $y \in \operatorname{Fix} R_F^\eta = \operatorname{Equil} F$, we have

$$0 \leq F(R_F^\eta x, y) - g_{R_F^\eta x}(\log_{R_F^\eta x} y, \log_{\eta, R_F^\eta x} x),$$

and thus

$$g_{R_F^\eta x}(\log_{R_F^\eta x} y, \log_{\eta, R_F^\eta x} x) \leq F(R_F^\eta x, y) \leq -F(y, R_F^\eta x) \leq 0.$$

If $R_F^\eta x \neq x$, then

$$g_{R_F^\eta x}(\log_{R_F^\eta x} y, \log_{R_F^\eta x} x) = \frac{d(R_F^\eta x, x)}{\eta(d(R_F^\eta x, x))} g_{R_F^\eta x}(\log_{R_F^\eta x} y, \log_{\eta, R_F^\eta x} x) \leq 0.$$

If $R_F^\eta x = x$, then we immediately obtain

$$g_{R_F^\eta x}(\log_{R_F^\eta x} y, \log_{R_F^\eta x} x) = 0.$$

Therefore, R_F^η is a directed operator. It completes the proof. □

5. Geodesically monotone mapping

Let M be an admissible CAT(κ) space and ψ a real function on $[0, D_\kappa/2[$. We say a mapping T on M is vicinal with ψ if

$$\begin{aligned}(\psi(d(Ty, y)) + \psi(d(Tx, x)))\phi_\kappa(Tx, Ty) \\ \leq \psi(d(Ty, y))\phi_\kappa(Tx, y) + \psi(d(Tx, x))\phi_\kappa(Ty, x)\end{aligned}$$

for $x, y \in M$. Kohsaka [27] first introduces the vicinality of a mapping on CAT(1) spaces. After that, Kajimura and Kimura [17] have introduced them to general CAT(κ) spaces. Such mappings have some good properties for fixed point theory if a given function ψ satisfies additional conditions, such as continuity; refer to [17]. We introduce a new class of mappings. Let M be an admissible CAT(κ) space and η a function on $[0, D_\kappa/2[$ satisfying the condition (P1). We say that a mapping T on M is geodesically monotone with η if

$$g_{Tx}(\log_{Tx} Ty, \log_{\eta, Tx} x) + g_{Ty}(\log_{Ty} Tx, \log_{\eta, Ty} y) \leq 0$$

for $x, y \in M$. In the setting of Theorems 4.2 and 4.3, the mappings R_f^η and R_F^η are geodesically monotone with η, respectively. For the sake of completeness, we give proofs.

Proposition 5.1. *In the setting of Theorem 4.2, the mapping R_f^η is geodesically monotone with η.*

Proof. Let $x, y \in M$. From Theorem 4.2, we have

$$f(R_f^\eta x) \leq f(R_f^\eta y) - g_{R_f^\eta x}(\log_{R_f^\eta x} R_f^\eta y, \log_{\eta, R_f^\eta x} x)$$

and

$$f(R_f^\eta y) \leq f(R_f^\eta x) - g_{R_f^\eta y}(\log_{R_f^\eta y} R_f^\eta x, \log_{\eta, R_f^\eta y} y).$$

Adding their both sides and rearranging that equation, we obtain

$$g_{R_f^\eta x}(\log_{R_f^\eta x} R_f^\eta y, \log_{\eta, R_f^\eta x} x) + g_{R_f^\eta y}(\log_{R_f^\eta y} R_f^\eta x, \log_{\eta, R_f^\eta y} y) \leq 0.$$

It means that R_f^η is geodesically monotone with η. □

Proposition 5.2. *In the setting of Theorem 4.3, the mapping R_F^η is geodesically monotone with η.*

Proof. Let $x, y \in M$. From Theorem 4.3, we have

$$0 \leq F(R_F^\eta x, R_F^\eta y) - g_{R_F^\eta x}(\log_{R_F^\eta x} R_F^\eta y, \log_{\eta, R_F^\eta x} x)$$

and

$$0 \leq F(R_F^\eta y, R_F^\eta x) - g_{R_F^\eta y}(\log_{R_F^\eta y} R_F^\eta x, \log_{\eta, R_F^\eta y} y).$$

Adding their both sides and rearranging that equation, we obtain from (E2) that

$$g_{R_F^\eta x}(\log_{R_F^\eta x} R_F^\eta y, \log_{\eta, R_F^\eta x} x) + g_{R_F^\eta y}(\log_{R_F^\eta y} R_F^\eta x, \log_{\eta, R_F^\eta y} y) \leq 0.$$

It means that R_F^η is geodesically monotone with η. □

Let H be a Hilbert space, and suppose that T is a geodesically monotone mapping with the identity function. Then, for $x, y \in H$, we have

$$\begin{aligned} 0 &\geq g_{Tx}(\log_{Tx} Ty, \log_{\eta, Tx} x) + g_{Ty}(\log_{Ty} Tx, \log_{\eta, Ty} y) \\ &= \langle Ty - Tx, x - Tx \rangle + \langle Tx - Ty, y - Ty \rangle \\ &= -\langle Tx - Ty, x - y \rangle + \langle Tx - Ty, Tx - Ty \rangle, \end{aligned}$$

and hence

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle.$$

It means that in Hilbert spaces, a mapping is geodesically monotone if and only if it is firmly nonexpansive, or equivalently, inversely strongly monotone.

For a general geodesically monotone mapping, we obtain the following result:

Theorem 5.3. *Let M be an admissible* CAT(κ) *space and T a geodesically monotone mapping on M with a function η on $[0, D_\kappa/2[$ satisfying the condition* (P1)*. Then, the following hold:*

(i) *If T has a fixed point, then it is directed;*

(ii) *its fixed point set* $\operatorname{Fix} T$ *is closed and convex;*

(iii) *if there exists a limit*

$$\lim_{t\to 0+} \frac{\eta(t)}{s_\kappa(t)}$$

as a nonnegative real number, then T *is vicinal with* η/s_κ.

Proof. We first show (i) and (ii). We assume that T has a fixed point. Let $x \in M$ and $y \in \operatorname{Fix} T$. Since $\log_{\eta,Ty} y = 0_{Ty}$, we have

$$\begin{aligned} 0 &\geq g_{Tx}(\log_{Tx} Ty, \log_{\eta,Tx} x) + g_{Ty}(\log_{Ty} Tx, \log_{\eta,Ty} y) \\ &= g_{Tx}(\log_{Tx} y, \log_{\eta,Tx} x), \end{aligned}$$

and therefore

$$g_{Tx}(\log_{Tx} y, \log_{Tx} x) \leq 0.$$

Hence, T is directed. In this case, T is quasinonexpansive, and hence $\operatorname{Fix} T$ is closed and convex even if it is nonempty.

We next show (iii). From the assumption, we define the value of η/s_κ at 0 by

$$\frac{\eta(0)}{s_\kappa(0)} = \lim_{t\to 0+} \frac{\eta(t)}{s_\kappa(t)} \geq 0.$$

Let $x, y \in M$, and set

$$\psi_x = \frac{\eta(d(Tx,x))}{s_\kappa(d(Tx,x))} \text{ and } \psi_y = \frac{\eta(d(Ty,y))}{s_\kappa(d(Ty,y))}.$$

If $Tx = x$, then since T is quasinonexpansive,

$$\begin{aligned} (\psi_y + \psi_x)\phi_\kappa(Tx,Ty) &= \psi_y\phi_\kappa(Tx,Ty) + \psi_x\phi_\kappa(Ty,Tx) \\ &= \psi_y\phi_\kappa(x,Ty) + \psi_x\phi_\kappa(Ty,x) \\ &\leq \psi_y\phi_\kappa(x,y) + \psi_x\phi_\kappa(Ty,x) \\ &= \psi_y\phi_\kappa(Tx,y) + \psi_x\phi_\kappa(Ty,x). \end{aligned}$$

In the same way, we obtain

$$(\psi_y + \psi_x)\phi_\kappa(Tx,Ty) \leq \psi_y\phi_\kappa(Tx,y) + \psi_x\phi_\kappa(Ty,x)$$

if $Ty = y$. Suppose that $Tx \neq x$ and $Ty \neq y$. Then, from (P1), we have ψ_x and ψ_y are positive. The inequality required by the vicinality is satisfied if $Tx = Ty$. Thus, we further suppose that $Tx \neq Ty$. Since T is geodesically monotone with η, we have

$$g_{Tx}(\log_{Tx} Ty, \log_{\eta,Tx} x) + g_{Ty}(\log_{Ty} Tx, \log_{\eta,Ty} y) \leq 0. \tag{5.1}$$

Note that

$$\frac{s_\kappa(d(Tx,Ty))}{\psi_x \cdot d(Tx,Ty)} g_{Tx}(\log_{Tx} Ty, \log_{\eta,Tx} x) = g_{Tx}(\log_{\kappa,Tx} Ty, \log_{\kappa,Tx} x) \tag{5.2}$$

and

$$\frac{s_\kappa(d(Tx,Ty))}{\psi_y \cdot d(Tx,Ty)} g_{Ty}(\log_{Ty} Tx, \log_{\eta,Ty} y) = g_{Ty}(\log_{\kappa,Ty} Tx, \log_{\kappa,Ty} y). \tag{5.3}$$

Hence, from the equations (5.1), (5.2) and (5.3),

$$\psi_x g_{Tx}(\log_{\kappa,Tx} Ty, \log_{\kappa,Tx} x) + \psi_y g_{Ty}(\log_{\kappa,Ty} Tx, \log_{\kappa,Ty} y) \leq 0.$$

From Theorem 3.2, we have

$$\begin{aligned}
0 &\geq \psi_x g_{Tx}(\log_{\kappa,Tx} Ty, \log_{\kappa,Tx} x) + \psi_y g_{Ty}(\log_{\kappa,Ty} Tx, \log_{\kappa,Ty} y) \\
&\geq \psi_x \big(s_\kappa'(d(Tx,Ty))\phi_\kappa(Tx,x) + \phi_\kappa(Ty,Tx) - \phi_\kappa(Ty,x)\big) \\
&\quad + \psi_y \big(s_\kappa'(d(Ty,Tx))\phi_\kappa(Ty,y) + \phi_\kappa(Tx,Ty) - \phi_\kappa(Tx,y)\big) \\
&\geq \psi_x\phi_\kappa(Ty,Tx) - \psi_x\phi_\kappa(Ty,x) + \psi_y\phi_\kappa(Tx,Ty) - \psi_y\phi_\kappa(Tx,y),
\end{aligned}$$

and therefore

$$(\psi_y + \psi_x)\phi_\kappa(Tx,Ty) \leq \psi_y\phi_\kappa(Tx,y) + \psi_x\phi_\kappa(Ty,x).$$

Thus, the inequality required by the vicinality is satisfied. Hence, T is vicinal with η/s_κ. □

Recall that

$$\frac{\eta_0(a)}{s_\kappa(a)} = 1; \ \frac{\eta_1(a)}{s_\kappa(a)} = \frac{1}{s_\kappa'(a)}; \ \frac{\eta_2(a)}{s_\kappa(a)} = 1 + \frac{1}{s_\kappa'(a)^2}; \ \frac{\eta_3(a)}{s_\kappa(a)} = 1 + \frac{1}{s_\kappa'(a)}$$

for $a \in \mathbb{R}$. Therefore,

$$\lim_{t\to 0+} \frac{\eta_0(t)}{s_\kappa(t)} = 1; \ \lim_{t\to 0+} \frac{\eta_1(t)}{s_\kappa(t)} = 1; \ \lim_{t\to 0+} \frac{\eta_2(t)}{s_\kappa(t)} = 2; \ \lim_{t\to 0+} \frac{\eta_3(t)}{s_\kappa(t)} = 2.$$

Let M be an admissible CAT(κ) space and T a geodesically monotone mapping with s_κ. Then, T is vicinal with a constant function which is identically equal to 1. Such a mapping is said to be geodesically nonspreading [35], that is,

$$2\phi_\kappa(Tx,Ty) \leq \phi_\kappa(Tx,y) + \phi_\kappa(Ty,x)$$

for $x, y \in M$. Furthermore, suppose $\kappa = 0$. Then, since T is geodesically monotone with the identity function, for $x, y \in M$, we obtain from Theorem 3.2 that

$$\begin{aligned}
0 &\geq g_{Tx}(\log_{Tx} Ty, \log_{Tx} x) + g_{Ty}(\log_{Ty} Tx, \log_{Ty} y) \\
&\geq \frac{2d(Tx,Ty)^2 - d(Tx,y)^2 - d(Ty,x)^2 + d(Tx,x)^2 + d(Ty,y)^2}{2},
\end{aligned}$$

and hence

$$2d(Tx,Ty)^2 \leq d(Tx,y)^2 + d(Ty,x)^2 - d(Tx,x)^2 - d(Ty,y)^2.$$

Thus, T is firmly nonexpansive, and therefore it is nonexpansive and metrically nonspreading.

References

[1] M. Bačák, *Convex Analysis and Optimization in Hadamard Spaces*, De Gruyter, Berlin, 2014.

[2] H. H. Bauschke and P. L. Combettes, *A weak-to-strong convergence principle for Fejér-monotone methods in Hilbert spaces*, Math. Oper. Res. **26** (2001), 248–264.

[3] M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer-Verlag, Berlin, 1999.

[4] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201–225.

[5] A. Cegielski and Y. Censor, *Opial-type theorems and the common fixed point problem*, in: Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Springer Optim. Appl., vol. 49, Springer, New York, 2011, pp. 155–183.

[6] Y. Censor and A. Segal, *On the string averaging method for sparse common fixed-point problems*, Int. Trans. Oper. Res. **16** (2009), 481–494.

[7] Y. Censor and A. Segal, *Sparse string-averaging and split common fixed points*, Nonlinear analysis and optimization I. Nonlinear analysis, Contemp. Math., vol. 513, Amer. Math. Soc., Providence, RI, 2010, pp. 125–142.

[8] P. Chaipunya, F. Kohsaka and P. Kumam, *Monotone vector fields and generation of nonexpansive semigroups in complete* CAT(0) *spaces*, Numer. Funct. Anal. Optim. **42** (2021), 989–1018.

[9] P. L. Combettes, *Quasi-Fejérian analysis of some optimization algorithms*, in: Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications (Haifa, 2000), Stud. Comput. Math., vol. 8, North-Holland, Amsterdam, 2001, pp. 115–152.

[10] K. Goebel and W. A. Kirk, *Topics in Metric Fixed Point Theory*, Cambridge Studies in Advanced Mathematics, vol. 28, Cambridge University Press, Cambridge, 1990.

[11] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 83, Marcel Dekker, Inc., New York, 1984.

[12] B. Halpern, *Fixed points of nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[13] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), 659–673.

[14] R. E. Bruck Jr., *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341–355.

[15] T. Kajimura and Y. Kimura, *A new resolvent for convex functions in complete geodesic spaces*, RIMS Kôkyûroku **2112** (2019), 141–147.

[16] T. Kajimura and Y. Kimura, *Resolvents of convex functions in complete geodesic metric spaces with negative curvature*, J. Fixed Point Theory Appl. **21** (2019): article number 32.

[17] T. Kajimura and Y. Kimura, *A vicinal mapping on geodesic spaces*, in: Proceedings of International Conference on Nonlinear Analysis and Convex Analysis & International Conference on Optimization: Techniques and Applications –I– (Hakodate, Japan, 2019) (Y. Kimura, M. Muramatsu, W. Takahashi, and A. Yoshise, eds.), 2021, pp. 183–195.

[18] T. Kajimura, Y. Kimura, and F. Kohsaka, *Resolvents for convex functions on geodesic spaces and their nonspreadingness*, Axioms **14** (2025), 12pp.

[19] Y. Kimura and F. Kohsaka, *Spherical nonspreadingness of resolvents of convex functions in geodesic spaces*, J. Fixed Point Theory Appl. **18** (2016), 93–115.

[20] Y. Kimura and F. Kohsaka, *Two modified proximal point algorithm for convex functions in Hadamard spaces*, Linear Nonlinear Anal. **2** (2016), 69–86.

[21] Y. Kimura and F. Kohsaka, *The proximal point algorithm in geodesic spaces with curvature bounded above*, Linear Nonlinear Anal. **3** (2017), 133–148.

[22] Y. Kimura and M. Nakadai, *Resolvents of convex functions and approximation of a minimizer on a geodesic space*, RIMS Kôkyûroku **2274** (2024), 141–147.

[23] Y. Kimura and S. Sudo, *New type parallelogram laws in Banach spaces and geodesic spaces with curvature bounded above*, Arab. J. Math. **12** (2023), 389–412.

[24] Y. Kimura and S. Sudo, *Tangent spaces and a metric on geodesic spaces*, RIMS Kôkyûroku **2240** (2023), 7–19.

[25] W. A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Am. Math. Mon. **72** (1965), 1004–1006.

[26] W. A. Kirk, *Fixed point theorems in* CAT(0) *spaces and* $\mathbb{R}$*-trees*, Fixed Point Theory Appl. **4** (2004), 309–316.

[27] F. Kohsaka, *Existence and approximation of fixed points of vicinal mappings in geodesic spaces*, Pure Appl. Funct. Anal. **3** (2018), 91–106.

[28] F. Kohsaka, *Fixed points of metrically nonspreading mappings in Hadamard spaces*, Appl. Anal. Optim. **3** (2019), 213–230.

[29] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

[30] F. Liu and M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal. **6** (1998), 313–344.

[31] W. R. Mann, *Mean value method in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.

[32] U. F. Mayer, *Gradient flows on nonpositively curved metric spaces and harmonic maps*, Comm. Anal. Geom. **6** (1998), 199–253.

[33] K. Nakajo and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups*, J. Math. Anal. Appl. **279** (2003), 372–379.

[34] S. Sudo, *Monotone vector fields on a geodesic space with curvature bounded above by a general real number*, Lett. Nonlinear Anal. Appl. **3** (2025), 5–27.

[35] S. Sudo, *The proximal point algorithm for monotone vector fields on complete geodesic spaces*, Nonlinear Convex Anal. & Optim. **4** (2025), 1–15.

[36] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohama Publishers, Yokohama, 2009.

[37] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[38] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), 417–428.

[39] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486–491.

[40] I. Yamada and N. Ogura, *Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings*, Numer. Funct. Anal. Optim. **25** (2004), 619–655.

Manuscript received 5 December 2025
revised 28 December 2025

Y. Kimura
Department of Information Science, Faculty of Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: yasunori@is.sci.toho-u.ac.jp

S. Sudo
Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
E-mail address: 7523001s@st.toho-u.ac.jp