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SOME UNIFIED FIXED POINT THEOREMS OF PANT ET AL.

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Abstract. We generalize the main theorems of R. P. Pant et al. [10] in 2021 to quasi-metric spaces. Our new results are shown to have new examples and can be applied to traditional consequences of metric fixed point theory. Main theorem of [10] implies the well known fixed point theorems respectively due to Banach, Kannan, Chatterjea, Ćirić, and Suzuki. However, our new main theorem implies many other known results as particular cases.

1. Introduction

There are thousands of papers on generalizations or related works on the Banach contraction principle [2]. However, it is not well-known that the Banach principle holds for quasi-metric spaces (without assuming the symmetry).

Recently, R. P. Pant et al. [9] in 2019 generalized "the Caristi fixed point theorem by employing a weaker form of continuity and show that contractive type mappings that satisfy the conditions of our theorem provide new solutions to the Rhoades' problem on continuity at fixed point. We also obtain a Meir-Keeler type fixed point theorem which gives a new solution to the Rhoades' problem on the existence of contractive mappings that admit discontinuity at the fixed point. We prove that our theorems characterize completeness of the metric space as well as Cantor's intersection property."

Moreover, R. P. Pant et al. [10] in 2021 proved "a theorem which ensures the existence of a unique fixed point and is applicable to contractive type maps as well as maps which do not satisfy any contractive type condition. Our theorem contains the well known fixed point theorems respectively due to Banach, Kannan, Chatterjea, Ćirić and Suzuki as particular cases; and is independent of Caristi's fixed point theorem. Moreover, our theorem provides new solutions to Rhoades problem on discontinuity at the fixed point as it admits contractive mappings which are discontinuous at the fixed point. It is also shown that the weaker form of continuity employed by us is a necessary and sufficient condition for the existence of the fixed point."

In the present paper, we generalize the main theorems of R. P. Pant et al. [10] in 2021 to quasi-metric spaces. Our new results are shown to have many new examples and can be applied to traditional consequences of metric fixed point theory.

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This paper is organized as follows: Section 2 is for our routine preliminaries on quasi-metric spaces as in [1, 6]. This is same to the several ones in our recent works. In Section 3, we introduce results of Pant et al. in [9] and [10]. Section 4 deals with our main theorem in this paper which extends the corresponding one of Pant et al. [10] to quasi-metric spaces. In Section 5, we give an extension of the Banach contraction principle called the Rus-Hicks-Rhoades (RHR) theorem for a quasi-metric space (X,δ) with a selfmap $T : X \to X$ such that X is T-orbitally complete. Section 6 is to introduce new examples of the main theorem of [10, Theorem 3.6]. Recall that Pant et al. [9] gave only examples due to Banach [2], Kannan [7], Chatterjea [3], Ćirić [4], and Suzuki [18]. But we gave a huge number of examples of particular cases of Theorem 3.6.

2. Preliminaries

Recall the following

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $q : X \times X \to \mathbb{R}^+ = [0,\infty)$ verifying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $q(x, y) = q(y, x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $q(x, z) \leq q(x, y) + q(y, z)$.

A *metric* on a set X is a quasi-metric d satisfying that for all $x, y \in X$,

(c) (symmetry) $d(x, y) = d(y, x)$.

Definition 2.2 ([1, 6])**.** Let (X, q) be a quasi-metric space.
(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $n > m > N$.
(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n > N$.
(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3 ([1, 6])**.** Let (X, q) be a quasi-metric space.
(1) (X, q) is *left-complete* if every left-Cauchy sequence in X is convergent;
(2) (X, q) is *right-complete* if every right-Cauchy sequence in X is convergent;
(3) (X, q) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let $f : X \to X$ be a selfmap. The *orbit* of f at $x \in X$ is the set

$$O_f(x) = \{x, fx, \dots, f^n x, \dots\}.$$

The space (X, q) is said to be *f-orbitally complete* if every right-Cauchy sequence in $O_f(x)$ is convergent in X. A selfmap f of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} f^n x = x_0 \Longrightarrow \lim_{n\to\infty} f^{n+1} x = f x_0$$

for any $x \in X$.

Remark 2.5. Definition 2.4 also works for a topological space X and a function $q : X \times X \to [0, \infty)$ such that $q(x, y) = 0$ implies $x = y$ for $x, y \in X$.

3. General fixed point theorems of R. P. Pant et al.

We now give some relevant definitions.

Definition 3.1 ([9, 10])**.** If f is a self-mapping of a metric space (X, d) then the set $O(x, f) = \{f^n x : n = 0, 1, 2, \dots\}$ is called the orbit of f at x and f is called orbitally continuous if $u = \lim_i f^{m_i} x$ implies $fu = \lim_i f f^{m_i} x$.

Continuity implies orbital continuity but not conversely; see [9, 10].

Definition 3.2 ([9])**.** A self-mapping f of a metric space X is called k-continuous, $k = 1, 2, 3, \dots$, if $f^k x_n \to ft$ whenever $\{x_n\}$ is a sequence in X such that $f^{k-1} x_n \to t$.

It was shown in [9] that continuity of f^k and k-continuity of f are independent conditions when $k > 1$ and

continuity $\Longrightarrow$ 2-continuity $\Longrightarrow$ 3-continuity $\Longrightarrow \dots,$

but not conversely.

It is also easy to see that 1-continuity is equivalent to the usual continuity.

Definition 3.3 ([9])**.** A self-mapping f of a metric space (X, d) is called weakly orbitally continuous if the set $\{y \in X : \lim_i f^{m_i} y = u \Longrightarrow \lim_i f f^{m_i} y = fu\}$ is nonempty whenever the set $\{x \in X : \lim_i f^{m_i} x = u\}$ is nonempty.

Orbital continuity implies weak orbital continuity but not conversely. If a self-mapping of X has a fixed point then it is, obviously, weakly orbitally continuous.

The next theorem is a generalization of the Caristi fixed point theorem due to Pant et al. [9]:

Theorem 3.4 ([9])**.** *Let f be a self-map of a complete metric space (X, d). Suppose $\varphi : X \to [0, \infty)$ is a function such that for each x in X we have*

$$d(x, fx) \le \varphi(x) - \varphi(fx). \tag{vii}$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a unique fixed point.

Theorem 3.5 ([9]). *For a metric space (X, d), the following are equivalent:*

(a)*X is complete*
(b) *X satisfies Cantor's intersection property*
(c) *every k-continuous or weak orbitally continuous self-map f of X satisfying the condition:*
Given $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \max\{d(x, fx), d(y, fy)\} < \varepsilon + \delta \Longrightarrow d(fx, fy) < \varepsilon$$

has a fixed point.
(d) *every k-continuous or weak orbitally continuous self-map f of X such that there exists a function $\phi : X \to [0, \infty)$ satisfying*

$$d(x, fx) \leq \phi(x) - \phi(fx),$$

has a fixed point.

There have appeared a large number of literature on the metric completeness.

The following is the main theorem of Pant et al. [10], Theorem 2.1:

Theorem 3.6 ([10]). *Let f be a self-map of a complete metric space (X, d). Suppose $\varphi : X \to [0, \infty)$ is such that for all $x, y \in X$*

$$d(fx, fy) \leq \varphi(x) - \varphi(fx) + \varphi(y) - \varphi(fy).$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a unique fixed point.

Pant et al. [9] showed that the well known fixed point theorems due to Banach [2], Kannan [7], Chatterjea [3], Ćirić [4], and Suzuki [18] are particular cases of Theorem 3.6.

Note that all things in this section hold for quasi-metric spaces instead of metric spaces.

4. Unified fixed point theorems

In this section, we derive our main theorems in the present paper.

Theorem 4.1. *Let f be a self-map of a complete quasi-metric space (X, q). Suppose $\varphi : X \to [0, \infty)$ is such that for all $x \in X$*

$$q(fx, f^2x) \leq \varphi(x) - \varphi(fx).$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a fixed point.

Proof. Let x_0 be any point in X. Define a sequence $\{x_n\}$ in X recursively by $x_n = fx_{n-1}$, that is, $x_n = f^n x_0$. Then

$$q(x_1, x_2) = q(fx_0, fx_1) \leq \varphi(x_0) - \varphi(x_1)$$
$$q(x_2, x_3) \leq \varphi(x_1) - \varphi(x_2)$$
$$\cdots\cdots$$
$$q(x_{n-1}, x_n) \leq \varphi(x_{n-2}) - \varphi(x_{n-1})$$
$$q(x_n, x_{n+1}) \leq \varphi(x_{n-1}) - \varphi(x_n).$$

Adding these inequalities we get

$$q(x_1, x_2) + q(x_2, x_3) + \cdots + q(x_n, x_{n+1}) \leq \varphi(x_0) - \varphi(x_n) \leq \varphi(x_0).$$

Making $n \to \infty$ we obtain

$$\Sigma_{n=1}^{\infty} q(x_n, x_{n+1}) \leq \varphi(x_0).$$

This implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $t \in X$ such that $\lim_{n\to\infty} x_n = t$ and $\lim_{n\to\infty} f^k x_n = t$ for each $k \geq 1$.

Suppose f is weakly orbitally continuous. Since the sequence $\{f^n x_0\}$ is convergent for each $x_0 \in X$, weak orbital continuity of f implies that there exists $y_0 \in X$ such that $f^n y_0 \to z$ and $ff^n y_0 \to fz$ for some $z \in X$. This implies that $z = fz$ and, hence, z is a fixed point of f. If f is orbitally continuous or if f is k-continuous for some $k \geq 1$ then f is weakly orbitally continuous and the proof follows. □

Theorem 4.2. *Let f be a self-map of a complete quasi-metric space (X, q). Suppose $\varphi : X \to [0, \infty)$ is such that for all $x, y \in X$*

$$q(fx, fy) \leq \varphi(x) - \varphi(y), \tag{4.1}$$

and that $\varphi(x) = \varphi(y)$ for any fixed points $x, y \in X$ of f. If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous then f has a unique fixed point.

Proof. The existence of a fixed point follows from Theorem 4.1. For its uniqueness, if u and v are fixed points of f, then using (4.1) we get

$$q(u, v) = q(fu, fv) \leq \varphi(u) - \varphi(v) = 0.$$

Therefore $u = v$ and f has a unique fixed point. This proves the theorem. □

Remark 4.3. The weak orbital continuity is a necessary and sufficient condition for the existence of the fixed point of a mapping satisfying condition (4.1).

Our Theorem 4.2 extends the following main theorem of Pant et al. [10], Theorem 2.1:

Corollary 4.4 ([10])**.** *Let f be a self-map of a complete metric space (X, d). Suppose $\phi : X \to [0, \infty)$ is such that for all $x, y \in X$*

$$d(fx, fy) \leq \phi(x) - \phi(fx) + \phi(y) - \phi(fy).$$

If f is weakly orbitally continuous or f is orbitally continuous or f is k-continuous, then f has a unique fixed point.

5. Improvements of our previous works

In this section, we give an extension of the Banach contraction principle called the Rus-Hicks-Rhoades (RHR) theorem for a quasi-metric space (X, δ) with a selfmap $T : X \to X$ such that X is T-orbitally complete.

The following form of the RHR theorem is a consequence of correct forms of Theorems H and P in [12, 11, 13, 15, 14, 16] and useful in this article:

Theorem H$(\gamma 1)$**.** *Let (X, δ) be a quasi-metric space, and $0 < \alpha < 1$. If $f : X \to X$ is a continuous map satisfying*

$$\delta(f(x), f^2(x)) \leq \alpha\, \delta(x, f(x)) \text{ for all } x \in X \backslash \{f(x)\},$$

and X is f-orbitally complete, then f has a fixed point $v \in X$, that is, $v = f(v)$.

Recall that the Banach contraction principle was extended to multimaps by Nadler [8] in 1969. A more general form of Nadler's theorem was established by Covitz-Nadler [5] in 1970 for metric spaces.

Let (X, δ) be a quasi-metric space and $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A,\ B \in \mathrm{Cl}(X)$, set

$$(5.1)\quad H(A, B) = \begin{cases} \max\{\sup\{\delta(a, B) : a \in A\},\ \sup\{\delta(b, A) : b \in B\}\} & \text{if the maximum exists} \\ \infty & \text{if otherwise.} \end{cases}$$

where $\delta(a, B) = \inf\{\delta(a, b) : b \in B\}$.

Such a map H is called generalized Hausdorff quasimetric induced by δ. Notice that H is a quasimetric on $\mathrm{Cl}(X)$. A point $p \in X$ is called a fixed point of $T : X \to \mathrm{Cl}(X)$ if $p \in T(p)$. A function $f : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous if $\{x_n\}$ is a sequence in $O(T, x_0)$ and $x_n \to \zeta$ implies $f(\zeta) \leq \lim_n \inf f(x_n)$.

Moreover, we have the following form from the corrected Theorem H in [12, 11, 13, 15]:

Theorem H$(\delta 1)$**.** *Let (X, δ) be a complete quasi-metric space, and $0 < \alpha < 1$. Let $T : X \to \mathrm{Cl}(X)$ be a multimap such that, for any $x \in X \backslash T(x)$, there exists $y \in X \backslash \{x\}$ satisfying*

$$H(T(x), T(y)) \leq \alpha\, \delta(x, y).$$

Then T has a fixed point $v \in X$, that is, $v \in T(v)$.

The following improves the one given in [12, 11, 13, 15, 14, 16], which can be called the Rus-Hicks-Rhoades (RHR) contraction principle:

Theorem P. *Let (X, δ) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$\delta(T(x), T^2(x)) \leq \alpha\, \delta(x, T(x)) \text{ for every } x \in X,$$

where $0 < \alpha < 1$. Suppose that X is T-orbitally complete. Then,

(i) *for each* $x \in X$*, there exists a point* $x_0 \in X$ *such that*

$$\lim_{n\to\infty} T^n(x) = x_0,$$

$$q(f^n x, x_0) \le \frac{\alpha^n}{1-\alpha} q(x, fx), \quad n = 1, 2, \ldots,$$

$$q(f^n x, x_0) \le \frac{\alpha}{1-\alpha} q(f^{n-1}x, f^n x), \quad n = 1, 2, \ldots,$$

and

(ii) x_0 *is a fixed point of* T *if and only if* $T : X \to X$ *is orbitally continuous at* $x_0 \in X$*.*

The early examples of Theorem P were given by Banach [2] in 1922, Kannan [7] in 1969, and many others.

6. Examples

Pant et al. [9] showed that the well known fixed point theorems due to Banach [2], Kannan [7], Chatterjea [3], Ćirić [4], and Suzuki [18] are particular cases of Theorem 3.6.

Banach (1922): $d(fx, fy) \le a\, d(x, y),\ x, y \in X,\ 0 \le a < 1.$

Kannan (1969): $d(fx, fy) \le \frac{a}{2}[d(x, fx) + d(y, fy)],\ x, y \in X,\ 0 \le a < 1.$

Chatterjea (1972): $d(fx, fy) \le \frac{a}{2}[d(x, fy) + d(y, fx)],\ x, y \in X,\ 0 \le a < 1.$

Ćirić (1971): $d(fx, fy) \le a \max\{d(x, y), d(x, fx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, fx)]\}$, $0 \le a < 1,\ x, y \in X.$

Suzuki (2008): $\theta(r)d(x, fx) \le d(x, y) \implies d(fx, fy) \le rd(x, y),\ x, y \in X,\ 0 \le r < 1.$

Recall that $\theta : [0, 1) \to (\frac{1}{2}, 1]$ as in Suzuki [18]. Note that $d(fx, f^2x) \le r\, d(x, fx)$, see Pant et al. [10].

In our previous works [12, 11, 14], in order to show uselessness of Theorems 2 and 3 of Suzuki [18], we considered any function $\theta' : [0, \infty) \to [0, 1]$ and obtained the following:

Theorem 2′. *Replace the function* θ *in Theorems 2 and 3 of* [18] *by* θ'*. Then the conclusion of Theorem 2 holds.*

Proof. Note that, by putting $y = Tx$, T becomes an RHR map. Then by Theorems P or H(γ1), T has a fixed point and its uniqueness follows as in the proof of Theorem 2 of Suzuki [18]. □

Theorem 3′. *Replace the function* θ *in Theorem 3 of* [18] *by* θ'*. Then the conclusion of Theorem 3 of* [18] *holds.*

Proof. Note that $\theta'(r)d(x,Tx) < d(x,y)$ means T can not have a fixed point $x = y = Tx$. $\square$

Recall that there are hundreds of equivalent conditions for metric completeness.

Consequently, Theorem $2'$ gives a huge number of examples of particular cases of Theorem 3.6.

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