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TOWARD THE STUDY OF THE LANGEVIN EQUATION INVOLVING THE (k,φ)-HILFER DERIVATIVE VIA BROWDER-GUPTA APPROACH

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Abstract. This typescript is devoted to the study of some topological structure of the solution sets for a class of nonlinear (k,χ)-Hilfer fractional Langevin equations. In this context, the nonlinearity, which acts on an infinite-dimensional Banach space, is assumed to satisfy Nagumo-type growth conditions. The obtained Aronszajn-type result for the problem at hand is derived via the nonlinear alternative for condensing maps combined with the Browder-Gupta strategy. Finally, an example is provided to illustrate our findings.

1. Introduction

It is well known that the theory of fractional differential equations has widespread applications in many scientific fields. Consequently, many scholars are dedicated to investigating this subject [14]. A multitude definitions for fractional differential derivatives and integral operators have been introduced [23]. These operators exhibit several distinctions and properties. Recently, a novel branch of the theory known as "fractional calculus with respect to functions" has emerged [3]. This operator appears in various concrete models. For example, it is present in several anomalous diffusions, including ultra-slow processes [16], financial crisis [20], random walks [12], Verhulst model [7] and Heston model [6]. Therefore, considerable attention has been devoted to the qualitative and quantitative properties of solutions to various types of differential problems driven by fractional derivative with respect to functions [25, 26].

The classical mathematical Langevin model is highly significant for illustrating how particles interact within their surrounding medium and the stochastic forces or fluctuations that cause their erratic movements. Nevertheless, the reliance on the specific relationship between a particle's position and velocity has prompted the development of the fractional Langevin model, aimed at describing anomalous diffusion phenomena [15]. While the above-mentioned motivational models have a great advantage, the difficulty of the corresponding mathematical model may significantly increase, complicating the study of the existence of solutions. Therefore, investigating the qualitative aspect for the nonlinear (k,χ)-Hilfer Langevin equation became important.

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Motivated by the preceding discussions, this manuscript presents a new qualitative study for the following nonlinear fractional Langevin equation involving (k,χ)-Hilfer fractional derivative (FD)

$$\begin{cases} \left({}^{H}_{k}\mathfrak{D}^{\vartheta,\kappa;\chi}_{a^+} + \aleph\, {}^{H}_{k}\mathcal{D}^{\vartheta-k,\kappa;\chi}_{a^+}\right)\mathfrak{z}(\xi) = \mathfrak{f}(\xi,\mathfrak{z}(\xi)), \quad \xi \in \mathcal{J} := [a,b], \\ \mathfrak{z}(a) = \mathfrak{z}'(a) = 0, \end{cases} \tag{1.1}$$

where $k>0$, $k<\vartheta<2k$, $\kappa\in[0,1]$, ${}^{H}_{k}\mathfrak{D}^{\vartheta,\kappa;\chi}_{a^+}$ and ${}^{H}_{k}\mathfrak{D}^{\vartheta-k,\kappa;\chi}_{a^+}$ denotes the (k,χ)-Hilfer fractional derivative of order ϑ and $\vartheta-k$, respectively and type κ (where χ is a positive function and $\chi'(\xi)>0$), $\mathfrak{f}:\mathcal{J}\times\mathbb{G}\to\mathbb{G}$ is a given appropriate function specified later, $(\mathbb{G},\|\cdot\|)$ be a real Banach space and $\aleph>0$.

We should emphasize that if the imposed hypotheses for existence of solutions of the system at hand do not guarantee uniqueness, it becomes essential to address the topological characteristics of the solution set. Moreover, providing the topological structure of solution sets, which encompasses properties such as acyclicity, compactness, and the R_δ property (see Definition 2.12), represents a highly intriguing fact of the qualitative aspect of integral/differential equations and inclusions, owing to its potential applications [4]. Specifically, the R_δ property can serve to establish the invariance of a reachability set under nonlinear perturbations in the associated control system [11].

As a distinctive aspect of our investigation, we can highlight the following:

(1) Providing some conditions under which problem (1.1) is uniquely solvable.
(2) A new Aronszajn-type result is established, addressing an unresolved question concerning the Langevin fractional equation in a general setting, specifically, when the nonlinear forcing term acts on an infinite-dimensional Banach space. This is achieved through a combination of the nonlinear alternative for condensing maps and the Browder-Gupta technique.
(3) Our findings generalize, improve and extend the results demonstrated in [2, 26].

The rest of the manuscript is structured as follows: In Section 2, we gather some necessary background required for the subsequent sections. In Section 3, we will address the aforementioned points (1)-(2). Finally, to validate our abstract result, some illustrative examples are provided.

2. Preliminary results

In what follows, we equip the space $C(\mathcal{J},\mathbb{G})$ of continuous functions $f:\mathcal{J}\to\mathbb{G}$ with the supnorm

$$\|f\|_\infty = \sup_{\xi\in\mathcal{J}}\|f(\xi)\|, \quad \text{for all } f\in C(\mathcal{J},\mathbb{G}).$$

Consider the space $L^{\alpha}_{\chi}(\mathcal{J},\mathbb{G})$ $(1\leq\alpha<\infty)$ of Bochner-integrable functions f on $\mathcal{J}$ with the norm

$$\|f\|_{L^{\alpha}_{\chi}}=\left(\int_a^b \chi'(s)\|f\|^{\alpha}ds\right)^{\frac{1}{\alpha}}. \tag{2.1}$$

If $\chi(\xi)=\xi$ the space $L^{\alpha}_{\chi}(\mathcal{J},\mathbb{G})$ coincides with the usual $L^{\alpha}(\mathcal{J},\mathbb{G})$.

Set

$$\mathbb{S}^1_{+}(\mathcal{J},\mathbb{R})=\{\chi\ :\ \chi\in C^1(\mathcal{J},\mathbb{R}) \text{ and } 0<\chi'(\xi) \text{ for all } \xi\in\mathcal{J}\}.$$

Let $\chi\in\mathbb{S}^1_{+}(\mathcal{J},\mathbb{R})$ for $\xi,s\in\mathcal{J}$, $(s<\xi)$, we define

$$\phi(\xi,s)=\chi(\xi)-\chi(s) \quad \text{and} \quad \phi(\xi,s)^{\vartheta}=(\chi(\xi)-\chi(s))^{\vartheta}.$$

Definition 2.1 ([10]). For $\alpha,\theta,k>0$, the k-gamma function is given by

$$\Gamma_k(\alpha)=\int_0^{\infty}\xi^{\alpha-1}e^{-\frac{\xi^k}{k}}d\xi.$$

We have also some useful following relations.

$$\Gamma_k(\alpha)=k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right),\ \Gamma_k(\alpha+k)=\alpha\Gamma_k(\alpha),\ \Gamma_k(k)=\Gamma(1)=1.$$

Definition 2.2 ([17]). Let $\vartheta,k>0$ and $\chi\in\mathbb{S}^1_{+}(\mathcal{J},\mathbb{R})$. The (k,χ)-Riemann-Liouville fractional integral (RLFI) of a function f of order ϑ is given by

$${}_k\mathfrak{I}^{\vartheta,\chi}_{a^+}f(\xi)=\frac{1}{k\Gamma_k(\vartheta)}\int_a^{\xi}\chi'(s)\phi(\xi,s)^{\frac{\vartheta}{k}-1}f(s)ds,$$

Lemma 2.3 ([17]). *Let $\vartheta_1,k>0$ and let $\nu\in\mathbb{R}$ such that $\nu>-k$. Then*

$${}_k\mathfrak{I}^{\vartheta,\chi}_{a^+}\ \phi(\xi,a)^{\frac{\nu}{k}}=\frac{\Gamma_k(\nu)}{\Gamma_k(\vartheta+\nu+k)}\phi(t,a)^{\frac{\vartheta+\nu}{k}}.$$

Definition 2.4 ([17]). Let $k>0$ and $\chi\in\mathbb{S}^1_{+}(\mathcal{J},\mathbb{R})$. The (k,χ)-Hilfer FD of $f\in C^n(\mathcal{J},\mathbb{R})$ of order $(n-1)k<\vartheta<nk$ and type $0\leq\kappa\leq1$ is given as

$${}^{H}_{k}\mathfrak{D}^{\vartheta,\kappa;\chi}_{a^+}f(\xi)\ ={}_k\mathfrak{I}^{\kappa(kn-\vartheta);\chi}_{a^+}\left(\frac{1}{\chi'(\xi)}\frac{d}{d\xi}\right)^n{}_k\mathfrak{I}^{(1-\kappa)(kn-\vartheta);\chi}_{a^+}f(\xi).$$

where $C^n(\mathcal{J},\mathbb{R})$ is the spaces of n-times continuously differentiable functions on $\mathcal{J}$.

Theorem 2.5 ([17]). *Let $f\in C^n_{\gamma,\chi}(\mathcal{J},\mathbb{R})$ and $\gamma=\frac{1}{k}(\vartheta+\kappa(nk-\vartheta))$ where $(n-1)k<\vartheta<nk$, $0\leq\kappa\leq1$, $n\in\mathbb{N}$ and $k>0$, then*

$$\begin{aligned}&\left({}_k\mathfrak{I}^{\vartheta;\chi}_{a^+}{}^{H}_{k}\mathfrak{D}^{\vartheta,\kappa;\chi}_{a^+}f\right)(\xi)\\ &\quad=f(\xi)-\sum_{j=1}^{n}\frac{\phi(\xi,a)^{\gamma-j}}{k^{j-n}\Gamma_k(k(\gamma-j+1))}\left\{\left(\frac{1}{\chi'(\xi)}\frac{d}{d\xi}\right)^{n-j}\left({}_k\mathfrak{I}^{k(n-\gamma);\chi}_{a^+}f(\xi)\right)\right\}_{\xi=a}.\end{aligned}$$

where $C^n_{\gamma,\chi}(\mathcal{J},\mathbb{R})$ the weighted space

$C^n_{\gamma,\chi}(\mathcal{J},\mathbb{R})$

$$= \left\{ f : \left(\frac{1}{\chi'(\xi)} \frac{d}{d\xi} \right)^{n-1} f(\xi) \in C(\mathcal{J}, \mathbb{R}) \text{ and } \left(\frac{1}{\chi'(\xi)} \frac{d}{d\xi} \right)^{n} f(\xi) \in C_{\gamma,\chi}(\mathcal{J}, \mathbb{R}) \right\}$$

and

$$C_{\gamma,\chi}(\mathcal{J}, \mathbb{R}) = \left\{ f : f \in C((a,b], \mathbb{R}) \text{ and } \phi(\cdot, a)^{1-\gamma} f(\cdot) \in C(\mathcal{J}, \mathbb{R}) \right\}.$$

Definition 2.6 ([5])**.** The Kuratowski measure of noncompactness (MNC) $\mathcal{O}$ of a bounded set $\mathcal{A}$ in a Banach space $\mathbb{G}$ is defined as:

$$\mathcal{O}(\mathcal{A}) := \inf \left\{ \varepsilon > 0 : \mathcal{A} = \bigcup_{j=1}^{n} \mathcal{A}_j \text{ and } \operatorname{diam}(\mathcal{A}_j) \leq \varepsilon \text{ for } 1 \leq j \leq n \right\}.$$

Lemma 2.7 ([5, 18])**.** *Let $\mathbb{U}, \mathbb{V} \subset \mathbb{G}$ be two bounded subsets. Then $\mathcal{O}(\cdot)$ satisfies*

(1) $\mathcal{O}(\mathbb{U}) = 0$ *if and only if* $\mathbb{U}$ *is relatively compact;*
(2) $\mathbb{U} \subset \mathbb{V}$ *implies that* $\mathcal{O}(\mathbb{U}) \leq \mathcal{O}(\mathbb{V})$*;*
(3) $\mathcal{O}(\mathbb{U} \cup \mathbb{V}) = \max\{\mathcal{O}(\mathbb{U}), \mathcal{O}(\mathbb{V})\}$*;*
(4) $\mathcal{O}(\mathbb{U}) = \mathcal{O}(\overline{\mathbb{U}}) = \mathcal{O}(\operatorname{conv}(\mathbb{U}))$*, where* $\overline{\mathbb{U}}$ *and* $\operatorname{conv} \mathbb{U}$ *represent the closure and the convex hull of* $\mathbb{U}$*, respectively;*
(5) $\mathcal{O}(\mathbb{U} + \mathbb{V}) \leq \mathcal{O}(\mathbb{U}) + \mathcal{O}(\mathbb{V})$*, where* $\mathbb{U} + \mathbb{V} = \{u + v : u \in \mathbb{U}, v \in \mathbb{V}\}$*;*
(6) $\mathcal{O}(d\mathbb{U}) = |d|\mathcal{O}(\mathbb{U})$*, for any* $d \in \mathbb{R}$*;*
(7) $\mathcal{O}(\mathbb{U} + u) \leq \mathcal{O}(\mathbb{U})$*, for any* $u \in \mathbb{G}$*.*
(8) *For any bounded* $\mathbb{U}$*, there exists a countable set* $\widetilde{\mathbb{U}} \subset A$*, such that*

$$\mathcal{O}(\mathbb{U}) \leq 2\mathcal{O}(\widetilde{\mathbb{U}}).$$

Lemma 2.8 ([13])**.** *Assume that* $\{w_n\}_{n=1}^{+\infty} \subset L^1(\mathcal{J}, \mathbb{G})$ *such that* $\|w_n(t)\| \leq \varsigma(t)$ *for almost all* $t \in J$ *and* $n \geq 1$*, where* $\varsigma \in L^1(\mathcal{J}, \mathbb{R}_+)$*. Then,* $t \mapsto \mathcal{O}(\{w_n(t)\}_{n=1}^{+\infty})$ *is integrable and*

$$\mathcal{O}\left(\left\{ \int_0^t w_n(s) ds \right\}_{n=1}^{+\infty} \right) \leq 2 \int_0^t \mathcal{O}(\{w_n(s)\}_{n=1}^{+\infty}) ds. \tag{2.2}$$

Lemma 2.9 ([9])**.** *Let* $\mathbb{U}$ *be a bounded closed subset of a Banach space* $\mathbb{K}$ *and* $\mathfrak{F} : \mathbb{U} \to \mathbb{K}$ *be a condensing map. Then* $I - \mathfrak{F}$ *is proper and* $I - \mathfrak{F}$ *maps closed subsets of* $\mathbb{U}$ *onto closed sets.*

Recall that the continuous map $I - \mathfrak{F}$ is proper if for every compact $\mathbb{J} \subset \mathbb{K}$, the set $(I - \mathfrak{F})^{-1}(\mathcal{J})$ is compact.

Next, we review certain concepts from geometric topology, see [4, 11].

Definition 2.10. A subset $\mathfrak{A} \subset \mathbb{K}$ is a retract of a Banach space $\mathbb{K}$ if there exists a continuous map $r : \mathfrak{A} \to \mathbb{K}$ such that $r(w) = w$, for all $w \in \mathfrak{A}$.

Definition 2.11. A set $\mathfrak{A}$ is called contractible provided there exists a continuous homotopy $\mathcal{H} : \mathfrak{A} \times [0,1] \to \mathfrak{A}$ and $w_0 \in \mathfrak{A}$ such that

(1) $\mathcal{H}(w, 0) = w$, for all $w \in \mathfrak{A}$.

(2) $\mathcal{H}(w,1) = w_0$, for all $w \in \mathfrak{A}$.

Definition 2.12. A compact nonvoid space $\mathfrak{X}$ is called an R_δ-set provided there exists a decreasing sequence of compact nonvoid contractible spaces $(\mathfrak{X}_m)_{m\in\mathbb{N}}$ such that

$$\mathfrak{X} = \bigcap_{m=1}^{\infty} \mathfrak{X}_m.$$

Lemma 2.13 ([11]). *Let $\mathbb{Y}$, $\mathbb{X}$ be two normed spaces and $\mathfrak{F} : \mathbb{X} \to \mathbb{Y}$ be a continuous map. Then, for all $\varepsilon > 0$, there is a locally Lipschitz map $\mathfrak{F}_\varepsilon : \mathbb{X} \to \mathbb{Y}$ such that*

$$\|\mathfrak{F}(w) - \mathfrak{F}_\varepsilon(w)\| < \varepsilon, \text{ for all } w \in \mathbb{X}.$$

Theorem 2.14 ([8]). *Let $\mathbb{X}$ be a normed space, $\mathbb{K}$ a Banach space and $\mathfrak{F} : \mathbb{X} \to \mathbb{K}$ a proper map. Suppose that for all $\epsilon > 0$, a proper map $\mathfrak{F}_\epsilon : \mathbb{X} \to \mathbb{K}$ is given, and the following two assertions are met:*

(a) $\|\mathfrak{F}_\epsilon(w) - \mathfrak{F}(w)\| < \epsilon$, *for every* $w \in \mathbb{X}$.
(b) *For every $\epsilon > 0$ and $y \in \mathbb{K}$ in a neighborhood of the origin such that $\|y\| \le \epsilon$, the equation $\mathfrak{F}_\epsilon(w) = y$ has exactly one solution w_ϵ.*

Then, $\mathfrak{F}^{-1}(0)$ is an R_δ-set.

Theorem 2.15 ([21]). *Let $\mathbb{K}$ be a Banach space, and $\mathbb{U} \subset \mathbb{K}$ be closed convex with $0 \in \mathbb{U}$. Let $\mathcal{W} : \mathbb{U} \to \mathbb{U}$ be a condensing map. Then either*

(1) *$\mathcal{W}$ admits a fixed point, or*
(2) *the set $\mathbb{F} = \{\mathfrak{z} \in \mathbb{U} : \ u = \rho\mathcal{W}\mathfrak{z}, \ 0 < \rho < 1\}$ is unbounded.*

The previous theorem provides the compactness of the solution set, the proof is similar to the one given by [5, Theorem 1.6.12].

A slight modification of [19, Theorem 1] yields the following result:

Lemma 2.16. *Let $k > 0, \frac{\vartheta}{k}, p > 1$, $q = \frac{p}{p-1}$, $\chi \in \mathbb{S}^1_+(\mathcal{J}, \mathbb{R}_+)$, and A, B, F, w are non-negative, continuous functions on the interval $\mathcal{J}$. Let $\eta : [0,\infty) \to [0,\infty)$ be a continuous, positive, non-decreasing function. If w satisfies the inequality*

$$w(t) \le A(\xi) + B(\xi)\int_a^\xi \chi'(s)\phi(\xi,s)^{\frac{\vartheta}{k}-1}F(s)\eta(w(s))ds, \quad \xi \in \mathcal{J},$$

then

$$w(\xi) \le \left[\Lambda^{-1}\left(\Lambda(A_1(\xi)) + B_1(\xi)\int_a^\xi F(s)^q\chi'(s)ds\right)\right]^{1/q}, \quad \xi \in \mathcal{J},$$

where $A_1(\xi) = 2^{q-1}\sup_{a\le s\le\xi} A(s)^q$, $B_1(\xi) = 2^{q-1}\mathfrak{Z}_p\sup_{a\le s\le\xi} B(s)^q$, $\mathfrak{Z}_p = \left(\frac{\phi(b,a)^{p\left(\frac{\vartheta}{k}-1\right)+1}}{p\left(\frac{\vartheta}{k}-1\right)+1}\right)^{1/p}$, $\Lambda(x) = \int_{x_0}^x \frac{ds}{[\eta(x^{1/q})]^q}$, for $x_0, x > 0$ and Λ^{-1} is the inverse of Λ.

3. Main results

Before stating our result, we need to present the auxiliary lemma.

Lemma 3.1. *Let* $\mathfrak{z} \in C^2_{\gamma,\chi}(\mathcal{J},\mathbb{R})$ *and* $g \in C(\mathcal{J},\mathbb{R})$. *The fractional linear differential equation*

$$\begin{cases} \left({}^H_k\mathfrak{D}^{\vartheta,\kappa;\chi}_{a^+} + \aleph\, {}^H_k\mathcal{D}^{\vartheta-k,\kappa;\chi}_{a^+}\right)\mathfrak{z}(\xi) = g(\xi), \quad \xi \in [a,b], \\ \mathfrak{z}(a) = \mathfrak{z}'(a) = 0, \end{cases} \tag{3.1}$$

has a solution given by

$$\mathfrak{z}(\xi) = \frac{\vartheta - k}{k^2\Gamma_k(\vartheta-k)} \int_a^\xi \chi'(s) e^{-\frac{\aleph}{k}\phi(\xi,s)} \left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2} g(\tau)d\tau\right) ds. \tag{3.2}$$

Proof. Taking the (k,χ)-RLFI ${}_k\mathfrak{I}^{\vartheta;\chi}_{a^+}$ ato both sides of (3.1), we obtain,

$${}_k\mathfrak{I}^{\vartheta;\chi}_{a^+}\left({}^H_k\mathfrak{D}^{\vartheta,\kappa;\chi}_{a^+}(\mathfrak{z}(\xi))\right) + \aleph\, {}_k\mathfrak{I}^{k;\chi}_{a^+}\left({}_k\mathfrak{I}^{\vartheta-k;\chi}_{a^+}\, {}^H_k\mathcal{D}^{\vartheta-k,\kappa;\chi}_{a^+}(\mathfrak{z}(\xi))\right) = {}_k\mathfrak{I}^{\vartheta;\chi}_{a^+} g(\xi).$$

Using Theorem 2.5, this implies that

$$\mathfrak{z}(\xi) - d_1\phi(\xi,a)^{\gamma-1} - d_2\phi(\xi,a)^{\gamma-2} + \aleph\, {}_k\mathfrak{I}^{k;\chi}_{a^+}\left(\mathfrak{z}(\xi) - d_3\phi(\xi,a)^{\gamma_0-1}\right) = {}_k\mathfrak{I}^{\vartheta;\chi}_{a^+} g(\xi).$$

where $0 < \gamma_0 = \frac{\vartheta-k+\kappa(2k-(\vartheta-k))}{k} < 1$. Thus

$$\mathfrak{z}(\xi) - d_1\phi(\xi,a)^{\gamma-1} - d_2\phi(\xi,a)^{\gamma-2} + \tfrac{\aleph}{k}\int_a^\xi \chi'(s)\left(\mathfrak{z}(s) - d_3\phi(s,a)^{\gamma_0-1}\right) ds = {}_k\mathfrak{I}^{\vartheta;\chi}_{a^+} g(\xi).$$

The condition $\mathfrak{z}(a) = 0$, leads to $d_2 = 0$. We can take the first ordinary derivative of

$$\mathfrak{z}(\xi) - d_1\phi(\xi,a)^{\gamma-1} + \tfrac{\aleph}{k}\int_a^\xi \chi'(s)\left(\mathfrak{z}(s) - d_3\phi(s,a)^{\gamma_0-1}\right) ds = {}_k\mathfrak{I}^{\vartheta;\chi}_{a^+} g(\xi),$$

we have

$$\begin{aligned}\mathfrak{z}'(\xi) + \frac{\aleph}{k}\chi'(\xi)\mathfrak{z}(\xi) &= d_1(\gamma-1)\chi'(\xi)\phi(\xi,a)^{\gamma-2} \\ &\quad + \frac{\aleph d_3}{k}\chi'(\xi)\phi(\xi,a)^{\gamma_0-1} + \left(\frac{\vartheta}{k}-1\right)\chi'(\xi)\, {}_k\mathfrak{I}^{\vartheta-k;\chi}_{a^+} g(\xi).\end{aligned}$$

The condition $\mathfrak{z}(a) = \mathfrak{z}'(a) = 0$, implies that $d_1 = d_3 = 0$.

New, let $\mathfrak{z}(\xi) = e^{-\frac{\aleph}{k}\chi(\xi)}\mathfrak{u}(\xi)$, then

$$\mathfrak{z}'(\xi) = -\frac{\aleph}{k}\chi'(\xi)e^{-\frac{\aleph}{k}\chi(\xi)}\mathfrak{u}(\xi) + \mathfrak{u}'(\xi)e^{-\frac{\aleph}{k}\chi(\xi)}.$$

So

$$e^{-\frac{\aleph}{k}\chi(\xi)}\mathfrak{u}'(\xi) = \left(\frac{\vartheta}{k}-1\right)\chi'(\xi)\, {}_k\mathfrak{I}^{\vartheta-k;\chi}_{a^+} g(\xi),$$

accordingly,

$$\mathfrak{u}'(\xi) = \left(\frac{\vartheta}{k}-1\right)\chi'(\xi)e^{\frac{\aleph}{k}\chi(\xi)}\, {}_k\mathfrak{I}^{\vartheta-k;\chi}_{a^+} g(\xi).$$

Integrating the last Equation, it follows that

$$\mathfrak{u}(\xi) = \mathfrak{u}(a) + \left(\frac{\vartheta}{k}-1\right)\int_a^\xi \chi'(s)e^{\frac{\aleph}{k}\chi(s)}\, {}_k\mathfrak{I}^{\vartheta-k;\chi}_{a^+} g(s)ds,$$

condition $\mathfrak{z}(a)$ implies $\mathfrak{u}(a)$; hence

$$\mathfrak{z}(\xi) = \left(\frac{\vartheta}{k} - 1\right) e^{-\frac{\aleph}{k}\chi(\xi)} \int_a^{\xi} \chi'(s) e^{\frac{\aleph}{k}\chi(s)} {}_k\mathfrak{I}_{a^+}^{\vartheta-k;\chi} g(s) ds,$$

which completes the proof. □

To give an Aronszajn-type result for problem (1.1), Theorem 2.15 and 2.14 are applied. Henceforth, the set of all solutions of problem (1.1) will be denoted by $\mathfrak{S}(\mathcal{J}, \mathfrak{f})$. Assume that:

(H1) The function $\mathfrak{f} : \mathcal{J} \times \mathbb{G} \to \mathbb{G}$ is Carath 迦 odory.

(H2) There exists a real-valued function $K \in L_{\chi}^{1/\alpha}(\mathcal{J}, \mathbb{R}_+)$ and a constant $0 < \alpha + 1 < \frac{\vartheta}{k}$ such that

$$\|\mathfrak{f}(\xi, v) - \mathfrak{f}(\xi, u)\| \leq K(t)\|v - u\|,$$

for all $v, u \in \mathbb{G}$ and for each $\xi \in \mathfrak{J}$.

(H3) There exists a functions $G, Q \in C(\mathcal{J}, \mathbb{R}_+)$, and nondecreasing continuous functions $\eta : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\|\mathfrak{f}(\xi, v)\| \leq G(\xi)\eta(\|v\|) + Q(\xi),$$

for all $t \in \mathcal{J}$ and each $v \in \mathbb{G}$.

(H4) There exists a function $\Theta \in C(\mathcal{J}, \mathbb{R}_+)$, such that for each $t \in \mathcal{J}$,

$$\mathcal{O}(\mathfrak{f}(\xi, \mathfrak{z}(\xi))) \leq \Theta(t)\mathcal{O}(\mathbb{V}(\xi)),$$

where $\mathbb{V}$ is bounded in $C(\mathcal{J}, \mathbb{G})$.

For easy computations, we let $\Theta^* = \sup_{\xi \in \mathcal{J}} \Theta(\xi)$, $G^* = \sup_{\xi \in \mathcal{J}} G(\xi)$ and $Q^* = \sup_{\xi \in \mathcal{J}} Q(\xi)$.

3.1. **Uniqueness Result.** This subsection presents the contractibility of $\mathfrak{S}(\mathcal{J}, \mathfrak{f})$, where the classical contraction principle is applied.

Theorem 3.2. *Suppose that (H1)-(H2) hold. Additionally, it is assumed that*

$$\mho_\alpha := \frac{e^{\frac{\aleph}{k}\phi(b,a)}(\vartheta - k)\phi(b,a)^{\frac{\vartheta}{k}-\alpha}\|K\|_{L_{\chi}^{\frac{1}{\alpha}}}}{k\Gamma_k(\vartheta - k)(\vartheta - k\alpha)} \left(\frac{k(1-\alpha)}{\vartheta - k(1+\alpha)}\right)^{1-\alpha} < 1. \tag{3.3}$$

Then, the set $\mathfrak{S}(\mathcal{J}, \mathfrak{f})$ is a singleton in $C(\mathcal{J}, \mathbb{G})$. Moreover, $\mathfrak{S}(\mathcal{J}, \mathfrak{f})$ is contractible, hence an acyclic space.

Proof. Thanks to Lemma 3.1, let us define the operator $\mathcal{W} : C(\mathcal{J}, \mathbb{G}) \to C(\mathcal{J}, \mathbb{G})$ given by

(3.4)

$$(\mathcal{W}\mathfrak{z})(\xi) = \frac{\vartheta - k}{k^2\Gamma_k(\vartheta - k)} \int_a^{\xi} \chi'(s) e^{-\frac{\aleph}{k}\phi(\xi,s)} \left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau, \mathfrak{z}(\tau))d\tau\right) ds,$$

$$\xi \in \mathcal{J}.$$

Evidently, the solution set of (1.1) coincides with the fixed point of $\mathcal{W}$. We need to show that the operator $\mathcal{W}$ is a contraction.

Since $\frac{1}{\alpha}+\frac{\alpha-1}{\alpha}=1$, then $\chi'(\tau)=(\chi'(\tau))^{\frac{1}{\alpha}}(\chi'(\tau))^{\frac{\alpha-1}{\alpha}}$, using the H目lder inequality, we obtain

$$\begin{aligned}
&\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}K(\tau)d\tau\\
&\le \left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta-2k}{k(1-\alpha)}}d\tau\right)^{1-\alpha}\left(\int_a^s \chi'(\tau)(K(\tau))^{\frac{1}{\alpha}}d\tau\right)^{\alpha} \qquad (3.5)\\
&\le \left(\frac{k(1-\alpha)}{\vartheta-k(1+\alpha)}\right)^{1-\alpha}\phi(s,a)^{\frac{\vartheta}{k}-(1+\alpha)}\|K\|_{L_\chi^{\frac{1}{\alpha}}},
\end{aligned}$$

and by $e^{-\frac{\aleph}{k}\phi(\xi,a)}\le 1$ for all $\xi\in\mathcal{J}$, we get

$$e^{-\frac{\aleph}{k}\phi(\xi,s)}=e^{-\frac{\aleph}{k}\phi(\xi,a)}e^{\frac{\aleph}{k}\phi(s,a)}\le e^{\frac{\aleph}{k}\phi(s,a)}\le e^{\frac{\aleph}{k}\phi(b,a)},\quad \text{for all } a<s\le\xi. \qquad (3.6)$$

For each $\mathfrak{z},\mathfrak{r}\in C(\mathfrak{J},\mathbb{G})$ and all $\xi\in\mathfrak{J}$, using (H2) combined (3.5) and (3.6), we can get

$$\begin{aligned}
&\|(\mathcal{W}\mathfrak{z})(\xi)-(\mathcal{W}\mathfrak{r})(\xi)\|\\
&\le \frac{\vartheta-k}{k^2\Gamma_k(\vartheta-k)}e^{\frac{\aleph}{k}\phi(b,a)}\int_a^{\xi}\chi'(s)\left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}K(\tau)\|\mathfrak{z}(\tau)-\mathfrak{r}(\tau)\|d\tau\right)ds\\
&\le \frac{\vartheta-k}{k^2\Gamma_k(\vartheta-k)}e^{\frac{\aleph}{k}\phi(b,a)}\|\mathfrak{z}-\mathfrak{r}\|_\infty\left(\frac{k(1-\alpha)}{\vartheta-k(1+\alpha)}\right)^{1-\alpha}\|K\|_{L_\chi^{\frac{1}{\alpha}}}\int_a^{\xi}\chi'(s)\phi(s,a)^{\frac{\vartheta}{k}-1-\alpha}ds\\
&\le \frac{e^{\frac{\aleph}{k}\phi(b,a)}(\vartheta-k)\phi(\xi,a)^{\frac{\vartheta}{k}-\alpha}\|K\|_{L_\chi^{\frac{1}{\alpha}}}}{k\Gamma_k(\vartheta-k)(\vartheta-k\alpha)}\left(\frac{k(1-\alpha)}{\vartheta-k(1+\alpha)}\right)^{1-\alpha}\|\mathfrak{z}-\mathfrak{r}\|_\infty.
\end{aligned}$$

So, one has

$$\|\mathcal{W}\mathfrak{z}-\mathcal{W}\mathfrak{r}\|_\infty\le\mho_\alpha\|\mathfrak{z}-\mathfrak{r}\|_\infty.$$

Thus, $\mathcal{W}$ is a contraction due to the condition (3.3). By Banach contraction principle, we can deduce that $\mathcal{W}$ admits an unique fixed point, implying that $\mathfrak{S}(\mathcal{J},\mathfrak{f})=\{\widehat{\mathfrak{u}}\}$.

Now, we introduce the homotopy $\mathcal{H}:\mathfrak{S}(\mathcal{J},\mathfrak{f})\times[0,1]\to\mathfrak{S}(\mathcal{J},\mathfrak{f})$ by

$$\mathcal{H}(\mathfrak{z},\mu)(\xi)=\begin{cases}\mathfrak{z}(t), & \text{for } a<\xi\le a+(b-a)\mu,\\ \widehat{\mathfrak{u}}(t), & \text{for } a+(b-a)\mu<\xi\le b.\end{cases}$$

In particular,

$$\mathcal{H}(\mathfrak{z},\mu)=\begin{cases}\mathfrak{z}, & \text{for } \mu=1,\\ \widehat{\mathfrak{u}}, & \text{for } \mu=0.\end{cases}$$

We have to prove that $\mathcal{H}(\cdot,\cdot)$ is contractive. To this end, we will show that it is continuous. Let $(\mathfrak{z}_n,\mu_n)\in\mathfrak{S}(\mathcal{J},\mathfrak{f})\times[0,1]$ be such that $(\mathfrak{z}_n,\mu_n)\to(\mathfrak{z},\mu)$. We have

$$\mathcal{H}(\mathfrak{z}_n,\mu_n)(\xi)=\begin{cases}\mathfrak{z}_n(\xi), & \text{for } a<\xi\le a+(b-a)\mu_n,\\ \widehat{\mathfrak{u}}(\xi), & \text{for } a+(b-a)\mu_n<\xi\le b.\end{cases}$$

Next we divide the proof into the following cases:

(i) If $\lim_{n\to\infty} \mu_n = 0$, then

$$\mathcal{H}(\mathfrak{z},0)(\xi) = \widehat{\mathfrak{u}}(\xi), \quad \text{for} \quad \xi \in \mathcal{J}.$$

Thus,

$$\begin{aligned}\|\mathcal{H}(\mathfrak{z}_n,\mu_n)(\xi) - \mathcal{H}(\mathfrak{z},\mu)(\xi)\| &\leq \|\mathcal{H}(\mathfrak{z}_n,\mu_n)(\xi) - \mathcal{H}(\mathfrak{z},\mu)(\xi)\|_{[a,a+(b-a)\mu_n]} \\ &\quad + \|\mathcal{H}(\mathfrak{z}_n,\mu_n)(\xi) - \mathcal{H}(\mathfrak{z},\mu)(\xi)\|_{[a+(b-a)\mu_n,b]} \\ &\leq \|\mathfrak{z}_n(\xi) - \mathfrak{z}(\xi)\|_{[a,a+(b-a)\mu_n]} + \|\widehat{\mathfrak{u}}(\xi) - \widehat{\mathfrak{u}}(\xi)\|_{[a+(b-a)\mu_n,b]} \\ &\leq \|\mathfrak{z}_n(\xi) - \mathfrak{z}(\xi)\|_{[a,a+(b-a)\mu_n]}.\end{aligned}$$

Hence

$$\|\mathcal{H}(\mathfrak{z}_n,\mu_n) - \mathcal{H}(\mathfrak{z},\mu)\|_\infty \leq \|\mathfrak{z}_n - \mathfrak{z}\|_\infty,$$

which tends to 0 as $n \to \infty$.

(ii) The case $\lim_{n\to\infty} \mu_n = 1$ is addressed similarly.

(iii) If $\mu_n \neq 0$ and $0 < \lim_{n\to\infty} \mu_n = \mu < 1$, then, we can distinguish between two sub-cases:

(1). $\mathfrak{z}_n \in \mathfrak{S}(\mathcal{J},\mathfrak{f})$. This implies that for $\xi \in [a, a+(b-a)\mu_n]$

$$\mathfrak{z}_n(\xi) = \frac{\vartheta - k}{k^2\Gamma_k(\vartheta - k)} \int_a^\xi \chi'(s) e^{-\frac{\aleph}{k}\phi(\xi,s)} \left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2} \mathfrak{f}(\tau,\mathfrak{z}_n(\tau)) d\tau \right) ds.$$

Using (H1) we find that for $\xi \in \mathcal{J}$,

$$\mathfrak{z}(t) = \frac{\vartheta - k}{k^2\Gamma_k(\vartheta - k)} \int_a^\xi \chi'(s) e^{-\frac{\aleph}{k}\phi(\xi,s)} \left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2} \mathfrak{f}(\tau,\mathfrak{z}(\tau)) d\tau \right) ds.$$

(2). If $\xi \in (a+(b-a)\mu_n, b]$, then

$$\mathcal{H}(\mathfrak{z}_n,\mu_n)(\xi) = \widehat{\mathfrak{u}}(\xi) = \mathcal{H}(\mathfrak{z},\mu)(\xi).$$

Thus

$$\|\mathcal{H}(\mathfrak{z}_n,\mu_n) - \mathcal{H}(\mathfrak{z},\mu)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Accordingly, $\mathcal{H}$ is continuous, yielding the contractibility of $\mathfrak{S}(\mathcal{J},\mathfrak{f})$ to the point $\widehat{\mathfrak{u}}$. □

3.2. **R_δ-property of $\mathfrak{S}(\mathcal{J},\mathfrak{f})$.** This subsection establishes the topological structure of $\mathfrak{S}(\mathcal{J},\mathfrak{f})$.

Theorem 3.3. *Assume that (H3)-(H4) and (H1) are fulfilled. Then $\mathfrak{S}(\mathcal{J},\mathfrak{f})$ is an R_δ-set.*

Proof. Introduce again the operator $\mathcal{W}$ represented by (3.4). For $N > 0$, we define

$$\mathbb{O}_N = \{\mathfrak{z} \in C(\mathcal{J},\mathbb{G}) \ : \ \|\mathfrak{z}\|_\infty < N\}.$$

The proof of Theorem 3.3 will proceed through several steps.

Step 1 : $\mathcal{W}$ maps bounded sets of $C(\mathcal{J},\mathbb{G})$ into itself.

By using (H3), for each $\xi \in \mathcal{J}$, we have

$$\|\mathfrak{f}(\tau,\mathfrak{z}(\tau))\| \leq G(\tau)\eta(\|\mathfrak{z}(\tau)\| + Q(\tau) \leq \eta(\|\mathfrak{z}\|_\infty)G^* + Q^*. \tag{3.7}$$

Let $\mathfrak{z} \in \mathbb{O}_N$. Then by (3.6) and (3.7) for each $\xi \in \mathcal{J}$, we have

$$\begin{aligned}
\|(\mathcal{W}\mathfrak{z})(\xi)\| &\leq \frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\|\mathfrak{f}(\tau,\mathfrak{z}(\tau))\|d\tau ds \\
&\leq \frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\Big[\eta(\|\mathfrak{z}\|_\infty)G^* + Q^*\Big]\int_a^\xi \chi'(s)\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}d\tau ds \\
&\leq \frac{e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\Big[\eta(\|\mathfrak{z}\|_\infty)G^* + Q^*\Big]\int_a^\xi \chi'(s)\phi(s,a)^{\frac{\vartheta}{k}-1}ds \\
&\leq \frac{e^{\frac{\aleph}{k}\phi(b,a)}}{\vartheta\Gamma_k(\vartheta-k)}\Big[\eta(\|\mathfrak{z}\|_\infty)G^* + Q^*\Big]\phi(b,a)^{\frac{\vartheta}{k}} := \widehat{N}.
\end{aligned}$$

This implies that:

$$\|\mathcal{W}\mathfrak{z}\|_\infty \leq \widehat{N}.$$

Step 2 : The continuity of the operator $\mathcal{W}$.

Let $\{\mathfrak{z}_n\}$ be a sequence in $\mathbb{O}_N$ such that $\mathfrak{z}_n \to \mathfrak{z}$ when $n \to \infty$. For each $\xi \in \mathfrak{J}$, making use of (H1), we easily have

$$\|\mathfrak{f}(s,\mathfrak{z}_n(s)) - \mathfrak{f}(s,\mathfrak{z}(s))\| \to 0, \quad \text{as } n \longrightarrow \infty.$$

Next, in view of (H3), one gets

$$\begin{aligned}
\|\mathfrak{f}(\tau,\mathfrak{z}_n(\tau)) - \mathfrak{f}(s,\mathfrak{z}(s))\| &\leq 2G(\xi)\eta(\|\mathfrak{z}_n(\tau)\|) + 2Q(\xi) \\
&\leq 2G^*\eta(N) + 2Q^*.
\end{aligned}$$

Since, the function $\tau \mapsto \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}$ is Lebesgue integrable over $[a,s]$ and the function $s \mapsto \chi'(s)\phi(s,a)^{\frac{\vartheta}{k}-1}$ is Lebesgue integrable over $[a,\xi]$. Then it follows from the Lebesgue dominated convergence theorem that

$$\begin{aligned}
&\|(\mathcal{W}\mathfrak{z}_n)(\xi) - (\mathcal{W}\mathfrak{z}_1)(\xi)\| \\
&\quad\leq \frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\|\mathfrak{f}(\tau,\mathfrak{z}_n(\tau)) - \mathfrak{f}(\tau,\mathfrak{z}(\tau))\|d\tau ds \\
&\quad\underset{n\to+\infty}{\longrightarrow} 0, \text{ for all } \xi \in \mathfrak{J}.
\end{aligned}$$

Therefore,

$$\|(\mathcal{W}\mathfrak{z}_n)(t) - (\mathcal{W}\mathfrak{z})(t)\| \to 0 \quad \text{as} \quad n \to \infty.$$

Then,

$$\|\mathcal{W}\mathfrak{z}_n - \mathcal{W}\mathfrak{z}\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, the operator $\mathcal{W}$ is continuous.

Step 3 : The family $\mathcal{W}(\mathbb{O}_N)$ is equicontinuous in $C(\mathcal{J},\mathbb{G})$.

For any $\xi_1,\xi_2\in\mathcal{J}$ with $\xi_1<\xi_2$ and $\mathfrak{z}\in\mathbb{O}_N$, we obtain

$$\|(\mathcal{W}\mathfrak{z})(\xi_2)-(\mathcal{W}\mathfrak{z})(\xi_1)\| \quad \le S_1+S_2,$$

where

$$S_1=\frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\int_{\xi_1}^{\xi_2}\chi'(s)\left(\int_a^s\chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\|\mathfrak{f}(\tau,\mathfrak{z}(\tau))\|d\tau\right)ds,$$

and

$$S_2=\frac{(\vartheta-k)}{k^2\Gamma_k(\vartheta-k)}\int_a^{\xi_1}\chi'(s)\left|e^{-\frac{\aleph}{k}\phi(\xi_2,s)}-e^{-\frac{\aleph}{k}\phi(\xi_1,s)}\right| \left\|\left(\int_a^s\chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}(\tau))d\tau\right)\right\|ds.$$

From (H3) and using (3.7), we get

$$\begin{aligned}S_1&\le\frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\Big[\eta(N)G^*+Q^*\Big]\int_{\xi_1}^{\xi_2}\chi'(s)\int_a^s\chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}d\tau ds\\&\le\frac{e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\Big[\eta(N)G^*+Q^*\Big]\int_{\xi_1}^{\xi_2}\chi'(s)\phi(s,a)^{\frac{\vartheta}{k}-1}ds\\&\le\frac{e^{\frac{\aleph}{k}\phi(b,a)}}{\vartheta\Gamma_k(\vartheta-k)}\Big[\eta(N)G^*+Q^*\Big]\Big[\phi(\xi_2,a)^{\frac{\vartheta}{k}}-\phi(\xi_1,a)^{\frac{\vartheta}{k}}\Big].\end{aligned}$$

Thus,

$$S_1\longrightarrow 0\quad\text{when}\quad \xi_2\longrightarrow\xi_1,\quad i=1,2. \tag{3.8}$$

On the other side,

$$S_2=\frac{(\vartheta-k)}{k^2\Gamma_k(\vartheta-k)}\left|e^{-\frac{\aleph}{k}\chi(\xi_2)}-e^{-\frac{\aleph}{k}\chi(\xi_1)}\right| \int_a^{\xi_1}e^{-\frac{\aleph}{k}\chi(s)}\chi'(s)\left\|\left(\int_a^s\chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}(\tau))d\tau\right)\right\|ds.$$

Thus,

$$S_2\longrightarrow 0\quad\text{when}\quad \xi_2\longrightarrow\xi_1. \tag{3.9}$$

From (3.8) and (3.9), it is clear that both inequalities are independent of $\mathfrak{z}$ and tends to zero as $\xi_2\to\xi_1$. Therefore, $\mathcal{W}(\mathbb{O}_N)$ is equicontinuous.

Step 4 : $\mathcal{W}$ is condensing.

For every bounded $\mathbb{A}\subset C(\mathcal{J},\mathbb{G})$, we define the MNC as

$$\widehat{\mathcal{O}}_\zeta(\mathbb{A})=\max_{\xi\in\mathcal{J}}e^{-\zeta\xi}\mathcal{O}(\mathbb{A}),\quad\text{for }\zeta>0. \tag{3.10}$$

Now, since $\chi'(\cdot)\chi(\cdot,a)^{\frac{\vartheta}{k}-1} \in L^1(\mathcal{J},\mathbb{R})$, we can choose ζ such that

$$\mathfrak{q}(\zeta) := \sup_{\xi\in\mathcal{J}} \frac{4\Theta^* e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)} \int_a^\xi \chi'(s)\phi(s,a)^{\frac{\vartheta}{k}-1} e^{-\zeta(\xi-s)} ds < \frac{1}{2}. \tag{3.11}$$

Next, let $\{Z^n\}_{n=1}^{+\infty} \subseteq \mathcal{W}(\mathbb{A})$ be the countable set and

$$\widehat{\mathcal{O}}_\zeta(\{Z^n\}_{n=1}^{+\infty}) = \widehat{\mathcal{O}}_\zeta\left(\mathcal{W}(\mathbb{A})\right). \tag{3.12}$$

Then, there exists $\{\mathfrak{z}_n\}_{n=1}^{+\infty} \subset \mathbb{A}$, such that

$$Z^n(\xi) = (\mathcal{W}\mathfrak{z}_n(\xi)), \quad \text{for } n \geq 1,\ \xi \in \mathcal{J}. \tag{3.13}$$

After that, from

$$(\mathcal{W}\mathfrak{z}_n)(\xi) \leq \frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)} \int_a^\xi \chi'(s) \int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}_n(\tau))d\tau ds, \tag{3.14}$$

we get

$$\begin{aligned}
&\widehat{\mathcal{O}}_\zeta(\{Z^n(\xi)\}_{n=1}^{+\infty}) = \widehat{\mathcal{O}}_\zeta\left(\{(\mathcal{W}\mathfrak{z}_n)(\xi)\}_{n=1}^{+\infty}\right) \\
&\leq \widehat{\mathcal{O}}_\zeta\left(\left\{\frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)} \int_a^\xi \chi'(s) \int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}_n(\tau))d\tau ds\right\}_{n=1}^{+\infty}\right).
\end{aligned} \tag{3.15}$$

Next, we estimate the left-hand side of the above inequality. Using (H4), for all $\tau \in [a,s]$, we have

$$\begin{aligned}
&\mathcal{O}\left(\left\{\chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}_n(\tau))\right\}_{n=1}^{+\infty}\right) \\
&\leq \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\Theta(\tau)\mathcal{O}(\{\mathfrak{z}_n(\tau)\}_{n=1}^{+\infty}) \\
&\leq \Theta(\tau)\chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}e^{\zeta\tau} \sup_{a\leq\tau\leq s} e^{-\zeta\tau}\mathcal{O}(\{\mathfrak{z}_n(\tau)\}_{n=1}^{+\infty}) \\
&\leq \Theta(\tau)\chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}e^{\zeta\tau}\widehat{\mathcal{O}}_\zeta(\{\mathfrak{z}_n(\tau)\}_{n=1}^{+\infty}).
\end{aligned}$$

Using Lemma 2.8, for all $\xi \in \mathcal{J}$, $s \in [a,\xi]$ and $\tau \leq s$, one gets

$$\begin{aligned}
&\mathcal{O}\left(\left\{\frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)} \int_a^\xi \chi'(s) \int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}_n(\tau))d\tau ds\right\}_{n=1}^{+\infty}\right) \\
&\leq \frac{4(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\Theta^*\widehat{\mathcal{O}}_\zeta(\{\mathfrak{z}_n(\xi)\}_{n=1}^{+\infty}) \int_a^\xi \chi'(s) \int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}e^{\zeta\tau}d\tau ds \\
&\leq \frac{4(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\Theta^*\widehat{\mathcal{O}}_\zeta(\{\mathfrak{z}_n(\xi)\}_{n=1}^{+\infty}) \int_a^\xi \chi'(s)e^{\zeta s} \int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}d\tau ds \\
&\leq \frac{4\Theta^* e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\widehat{\mathcal{O}}_\zeta(\{\mathfrak{z}_n(\xi)\}_{n=1}^{+\infty}) \int_a^\xi \chi'(s)\phi(s,a)^{\frac{\vartheta}{k}-1}e^{\zeta s}ds.
\end{aligned}$$

Multiplying both sides by $e^{-\zeta\xi}$, one obtains

(3.16)
$$\sup_{\xi\in\mathcal{J}} e^{-\zeta\xi}\mathcal{O}\left(\left\{\frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}_n(\tau))d\tau ds\right\}_{n=1}^{+\infty}\right)$$
$$\leq \sup_{\xi\in\mathcal{J}}\frac{4\Theta^* e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\widehat{\mathcal{O}}_\zeta(\{\mathfrak{z}_n(\xi)\}_{n=1}^{+\infty})\int_a^\xi \chi'(s)\phi(s,a)^{\frac{\vartheta}{k}-1}e^{-\zeta(\xi-s)}ds.$$

So, by (3.11), (3.15) and (3.16), we have

$$\widehat{\mathcal{O}}_\zeta\left(\{(\mathcal{W}\mathfrak{z}_n)(\xi)\}_{n=1}^{+\infty}\right) \leq \mathfrak{q}(\zeta)\widehat{\mathcal{O}}_\zeta(\{\mathfrak{z}_n(\xi)\}_{n=1}^{+\infty}). \tag{3.17}$$

Therefore, one has

$$\widehat{\mathcal{O}}_\zeta(\{\mathfrak{z}_n\}_{n=1}^{+\infty}) \leq \widehat{\mathcal{O}}_\zeta(\mathbb{A}) \leq \widehat{\mathcal{O}}_\zeta\left(\overline{\text{conv}}\left(\mathcal{W}(\mathbb{A})\cup\{(0)\}\right)\right) = \widehat{\mathcal{O}}_\zeta\left(\{Z^n\}_{n=1}^{+\infty}\right).$$

By (3.11) and the above inequality, it follows that

$$\widehat{\mathcal{O}}_\zeta\left(\{Z^n\}_{n=1}^{+\infty}\right) = 0.$$

Therefore, $\widehat{\mathcal{O}}_\zeta(\mathbb{A}) = 0$, which proves the compactness of the set $\overline{\mathbb{A}}$. Accordingly, $\mathcal{W}$ is condensing with respect to (3.10).

Step 5. The set $\mathbb{F}$ (see Theorem 2.15 (2)) is bounded.

Let $\mathfrak{z}\in C(\mathcal{J},\mathbb{G})$ and $\mathfrak{z} = \rho\mathcal{W}\mathfrak{z}$ for some $\rho\in(0,1)$. Then, for all $\xi\in\mathcal{J}$, one has

$$\mathfrak{z}(\xi) = \rho\left(\frac{\vartheta-k}{k^2\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)e^{-\frac{\aleph}{k}\phi(\xi,s)}\left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}(\tau,\mathfrak{z}(\tau))d\tau\right)ds\right).$$

From (3.6) and Dirichlet's formula [22], we can have

$$\begin{aligned}
\|\mathfrak{z}(\xi)\| &\leq \frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\|\mathfrak{f}(\tau,\mathfrak{z}(\tau))\|d\tau\right)ds\\
&\leq \frac{(\vartheta-k)e^{\frac{\aleph}{k}\phi(b,a)}}{k^2\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\|\mathfrak{f}(s,\mathfrak{z}(s))\|\int_s^\xi \chi'(\tau)\phi(\tau,s)^{\frac{\vartheta}{k}-2}d\tau ds\\
&= \frac{e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\|\mathfrak{f}(s,\mathfrak{z}(s))\|\phi(\xi,s)^{\frac{\vartheta}{k}-1}ds.
\end{aligned}$$

Recalling (H3), we get

$$\begin{aligned}
\|\mathfrak{z}(\xi)\| &\leq \frac{e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\phi(\xi,s)^{\frac{\vartheta}{k}-1}\Big[G(s)\eta(\|\mathfrak{z}(s)\|)+Q(s)\Big]ds\\
&\leq \frac{Q^* e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\phi(\xi,s)^{\frac{\vartheta}{k}-1}ds\\
&\quad + \frac{e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}\int_a^\xi \chi'(s)\phi(\xi,s)^{\frac{\vartheta}{k}-1}G(s)\eta(\|\mathfrak{z}(s)\|)ds\\
&\leq \mathcal{A}_0 + \mathcal{A}_1\int_a^\xi \chi'(s)\phi(\xi,s)^{\frac{\vartheta}{k}-1}G(s)\eta(\|\mathfrak{z}(s)\|)ds,
\end{aligned}$$

where $\mathcal{A}_0 = \frac{Q^* e^{\frac{\aleph}{k}\phi(b,a)}\phi(b,a)^{\frac{\vartheta}{k}}}{\vartheta\Gamma_k(\vartheta-k)}$ and $\mathcal{A}_1 = \frac{e^{\frac{\aleph}{k}\phi(b,a)}}{k\Gamma_k(\vartheta-k)}$. Using Lemma 2.16, we obtain

$$\|\mathfrak{z}(t)\| \leq \left[\Lambda^{-1}\left(\Lambda(\widehat{\mathcal{A}}_0) + \widehat{\mathcal{A}}_1 \int_a^t G(s)^q \chi'(s)ds\right)\right]^{1/q} := M, \quad \xi \in \mathcal{J},$$

where

$$\widehat{\mathcal{A}}_0 = 2^{q-1}\mathcal{A}_0^q, \quad \widehat{\mathcal{A}}_1 = 2^{q-1}\mathfrak{Z}_p\mathcal{A}_1^q, \quad \mathfrak{Z}_p = \left(\frac{\phi(b,a)^{p\left(\frac{\vartheta}{k}-1\right)+1}}{p\left(\frac{\vartheta}{k}-1\right)+1}\right)^{1/p}, \text{ and}$$

$\Lambda(x) = \int_{x_0}^{x} \frac{ds}{\left[\eta(x^{1/q})\right]^q}$, for $x_0, x > 0$ and Λ^{-1} is the inverse of Λ.

Hence, for any $\xi \in \mathcal{J}$, we obtain

$$\|\mathfrak{z}\|_\infty \leq M,$$

which implies that the set $\mathbb{F}$ is bounded. From Theorem 2.15, we deduce that $\mathfrak{S}(\mathcal{J}, \mathfrak{f})$ is non-empty and compact subset of $C(\mathcal{J}, \mathbb{G})$.

Step 6. $\mathfrak{S}(\mathcal{J}, \mathfrak{f})$ is an R_δ-set.

Given $\varepsilon_n \in (0,1)$ with $\varepsilon_n \to 0$ as $n \to \infty$. By (H1), according to Lemma 2.13, one can take a sequence $\{\mathfrak{f}_n\}$ of locally Lipschitz functions such that

$$\|\mathfrak{f}_n(\xi, y) - \mathfrak{f}(\xi, y)\| < \varepsilon_n, \quad \text{for all } \xi \in \mathcal{J} \text{ and } y \in \mathbb{G}. \tag{3.18}$$

Making use of (3.18) and (H3), we can assume that

$$\|\mathfrak{f}_n(\xi, y)\| \leq 1 + G(\xi)\eta(\|y\|) + Q(\xi), \quad n \geq 1.$$

We define the approximation operator $\mathcal{L}_n$ by

$$\begin{aligned}\mathcal{L}_n(\mathfrak{z})(\xi) = \frac{\vartheta - k}{k^2\Gamma_k(\vartheta-k)} \int_a^\xi \chi'(s) e^{-\frac{\aleph}{k}\phi(\xi,s)} \\ \left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2}\mathfrak{f}_n(\tau, \mathfrak{z}(\tau))d\tau\right) ds, \quad \xi \in \mathcal{J}.\end{aligned} \tag{3.19}$$

Since $\mathfrak{f}_n$ is locally Lipschitz, Theorem 3.2 implies that (3.19) is uniquely solvable.

Now, let

$$\mathcal{M}(\mathfrak{z}) = (I - \mathcal{L})(\mathfrak{z}).$$

By Step 4, one can show that $\mathcal{L}_n : C(\mathcal{J}, \mathbb{G}) \to C(\mathcal{J}, \mathbb{G})$ is condensing, which permits us to introduce the condensing perturbation of identity

$$\mathcal{M}_n(\mathfrak{z}) = (I - \mathcal{L}_n)(\mathfrak{z})$$

which are, by virtue of Lemma 2.9, proper maps.

On the other hand, the relation (3.18) allows the convergence of $\{\mathcal{M}_n\}$ to $\mathcal{M}$ uniformly in $C(\mathcal{J}, \mathbb{G})$.

Using (3.6), we can get

$$\begin{aligned}
&\|\mathcal{M}_n(\mathfrak{z})(\xi) - \mathcal{M}(\mathfrak{z})(\xi)\| \\
&\leq \frac{\vartheta - k}{k^2\Gamma_k(\vartheta - k)} \int_a^\xi \chi'(s) e^{-\frac{\aleph}{k}\phi(\xi,s)} \left(\int_a^s \chi'(\tau)\phi(s,\tau)^{\frac{\vartheta}{k}-2} \|\mathfrak{f}_n(\tau,\mathfrak{z}(\tau)) - \mathfrak{f}(\tau,\mathfrak{z}(\tau))\| d\tau \right) ds \\
&\leq \frac{e^{\frac{\aleph}{k}\phi(b,a)}}{\vartheta\Gamma_k(\vartheta - k)} \phi(b,a)^{\frac{\vartheta}{k}} \varepsilon_n, \quad \xi \in \mathcal{J},
\end{aligned}$$

and equation $\mathcal{M}_n(\mathfrak{z}) = y$ is uniquely solvable for each $y \in C(\mathcal{J}, \mathbb{R})$ as well as (3.19).

Accordingly, the conditions of Theorem 2.14 are fulfilled. Then, the solution set $\mathcal{M}^{-1}(0)$ is an R_δ-set.

□

4. Illustrative examples

In this section we introduce some examples that illustrate our theoretical results. Consider the Banach space

$$\mathbb{G} = c_0 = \{\mathfrak{z} = (\mathfrak{z}_1, \mathfrak{z}_2, \cdots, \mathfrak{z}_n, \cdots) \ : \ \mathfrak{z}_n \to 0 \text{ as } n \to \infty\}$$

endowed with

$$\|\mathfrak{z}\|_{c_0} = \sup_{n\geq 1} |\mathfrak{z}_n|.$$

(I) Illustration of Theorem 3.2.

Let us take $\chi(\xi) = \xi$. For $\xi \in \mathcal{J}$, consider the function $\mathfrak{f} : \mathcal{J} \times c_0 \to c_0$ defined by

$$\mathfrak{f}(t,\mathfrak{z}) = \left\{ \frac{e^{-\theta\xi - \frac{\aleph}{k}(b-a)}}{(b-a)^{\frac{\vartheta}{k}-\alpha}(1+e^\xi)} \left(13 + \tan^{-1}(|\mathfrak{z}_n|) + \frac{|\mathfrak{z}_n|}{1+|\mathfrak{z}_n|} \right) \right\}_{n\geq 1} \tag{4.1}$$

where $\theta > 0$ is a constant.

Obviously, the function $\mathfrak{f}$ is Carath迦odory and for any $u, v \in c_0$ and $\xi \in \mathcal{J}$, one can check that

$$\begin{aligned}
\|\mathfrak{f}(t,u) - \mathfrak{f}(t,v)\| &\leq \frac{e^{-\theta\xi - \frac{\aleph}{k}(b-a)}}{(b-a)^{\frac{\vartheta}{k}-\alpha}(1+e^\xi)} \left(\frac{\|u-v\|}{(1+\|u\|)(1+\|v\|)} + \left\| \tan^{-1}\left(\frac{|u|-|v|}{1+|uv|} \right) \right\| \right) \\
&\leq \frac{2e^{-\theta\xi - \frac{\aleph}{k}(b-a)}}{(b-a)^{\frac{\vartheta}{k}-\alpha}(1+e^\xi)} \|u-v\| \\
&\leq \frac{2e^{-\theta\xi - \frac{\aleph}{k}(b-a)}}{(b-a)^{\frac{\vartheta}{k}-\alpha}(1+e^a)} \|u-v\|.
\end{aligned}$$

So, the hypothesis (H2) holds with

$$K(\xi) = \frac{2e^{-\theta\xi - \frac{\aleph}{k}(b-a)}}{(b-a)^{\frac{\vartheta}{k}-\alpha}(1+e^a)} \quad \text{for} \quad \xi \in \mathcal{J}.$$

Next, for $0<\alpha<\frac{\vartheta}{k}-1$, let $K^*=\left\|\frac{2e^{-\theta\xi}}{1+e^{a}}\right\|_{L^{\frac{1}{\alpha}}}$. Choosing $\theta>0$ large enough and suitable $0<\alpha<\frac{\vartheta}{k}-1$, one can arrive at the following inequality

$$\mho_\alpha=\frac{(\vartheta-k)K^*}{k(\vartheta-k\alpha)\Gamma_k(\vartheta-k)}\left(\frac{k(1-\alpha)}{\vartheta-k(1+\alpha)}\right)^{1-\alpha}<1.$$

Therefore, all the conditions of Theorem 3.2 are satisfied. Then, the set of solutions of (1.1) with $\mathfrak{f}$ defined by (4.1) is a singleton.

(II) Illustration of Theorem 3.3.

For $\xi\in\mathcal{J}$ and $\mathfrak{z}=\{\mathfrak{z}_n\}_n\in c_0$, consider the nonlinear forcing terms,

$$\mathfrak{f}(t,\mathfrak{z})=\left\{\frac{1}{13+e^{\xi}}\left(3+\frac{1}{2^n}+4\tan^{-1}(|\mathfrak{z}_n|)\right)\right\}_{n\geq 1}. \tag{4.2}$$

Obviously, $\mathfrak{f}$ satisfies hypothesis (H1). To illustrate (H3), let $\xi\in\mathcal{J}$ and $\mathfrak{z}=\{\mathfrak{z}_n\}_n\in\mathbb{A}\subset c_0$. Then

$$\begin{aligned}\|\mathfrak{f}(\xi,\mathfrak{z})\| &\leq \tfrac{1}{13+e^{\xi}}(4+4\|\mathfrak{z}\|)\\ &\leq G(\xi)\eta(\|\mathfrak{z}\|)+Q(\xi).\end{aligned} \tag{4.3}$$

Therefore, (H3) is verified with

$$Q(\xi)=G(\xi)=\frac{4}{13+e^{\xi}}\ \text{for all } \xi\in\mathcal{J},\quad \eta(y)=y,\quad y\in[0,\infty).$$

Next, hypothesis (H4) is satisfied. Indeed, we recall that the Hausdorff MNC $\mathcal{Y}$ in $(c_0,\|\cdot\|_{c_0})$ can be computed by means of the formula

$$\mathcal{Y}(\mathbb{A})=\lim_{n\to\infty}\sup_{\mathfrak{z}\in\mathbb{A}}\|(\mathrm{I}-\mathrm{P}_n)\,\mathfrak{z}\|_\infty,$$

where $\mathbb{A}\in\mathcal{P}(c_0)$, P_n represents the projection onto the linear span of the first n vectors in the standard basis (see [5]). Using (4.3) (see also [25]) and from Hausdorff and Kuratowski MNCs which are related by the inequality

$$\Upsilon(\mathbb{A})\leq\mathcal{Y}(\mathbb{A})\leq 2\Upsilon(\mathbb{A}),$$

we get

$$\mathcal{O}\left(\mathfrak{f}(\xi,\mathbb{A})\right)\leq\Theta(\xi)\mathcal{O}(\mathbb{A}),\quad \text{for all } \xi\in\mathcal{J},$$

where

$$\Theta(\xi)=2G(\xi).$$

Since, all hypotheses of Theorem 3.3 are verified, the solution set of (1.1), with $\mathfrak{f}$ defined by (4.2), is an R_δ-set.

5. Conclusion

Under compactness and Nagumo type growth conditions, we establish new results regarding the topological characteristics of the solution set for nonlinear (k, χ)-Hilfer fractional Langevin equations in a Banach space. The employed strategy involves several steps. The first step is based on the classical contraction principle, and the second one relies on the nonlinear alternative for condensing maps with the Lasota-Opial approximation. These results contribute significantly to this emerging field.

Further applications, such as periodicity and approximate controllability, of the obtained findings will be discussed in a forthcoming manuscript.

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