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HÖLDER REGULARITY OF THE WEAK SOLUTIONS TO PARABOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

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Abstract. We consider the quasilinear parabolic partial differential equation in the divergent form under the p-growth condition. Assuming the existence of a weak bounded solution to the parabolic problem we establish its Hölder regularity by showing its belonging to a certain De Giorgi class $B_p(D_T,\ M)$.

1. Introduction

Let Ω be a bounded, open, strictly convex domain in R^l for $l \geq 3$ with the boundary $\partial\Omega$ is at least $C^{2+\alpha}$, $\alpha \in (0,\ 1)$ of smoothness. This article deals with the Hölder regularity of the weak solution to quasilinear parabolic problem

$$\frac{\partial u}{\partial t} = div\left(a_i\left(x,\ t,\ \nabla u\right)\right) + a\left(x,\ t,\ u,\ \nabla u\right), \tag{1.1}$$

$$u|_{\Gamma(T)} = \varphi|_{\Gamma(T)}, \tag{1.2}$$

where $\Gamma(T) = \{(x,\ t)\ :\ x \in \partial\Omega,\ t \in [0,\ T]\} \cup \{(x,\ t)\ :\ x \in \Omega,\ t = 0\}$, and an unknown function is defined $u(x,\ t)$ in $clos(D_T)$, where $clos(D_T)$ is the closure of the domain $D(T) = \Omega \times (0,\ T)$. The equation (1.2) contains in itself initial and boundary conditions. We assume that given function φ is smooth enough in $clos(D_T)$.

The parabolic form-boundary class is a subset of $L^2_{loc}(\Omega \times (0,\ T))$ for elements of which the inequality

$$\begin{aligned}&\int_{[0,\ T]} \|f(\cdot,\ t)\,\phi(\cdot,\ t)\|^p_{L^p}\,dt \leq \\ &\leq \beta \int_{[0,\ T]} \|\nabla\phi(\cdot,\ t)\|^2_{L^2}\,dt + \int_{[0,\ T]} c_\beta(t)\,\|\phi(\cdot,\ t)\|^2_{L^2}\,dt\end{aligned}$$

for all $\phi \in W^2_1(\Omega \times [0,\ T])$ and some strictly positive β, and $c_\beta(\cdot) \in L^1_{loc}(\Omega \times [0,\ T])$, where $f \in L^2_{loc}(\Omega \times (0,\ T))$.

As a simple example of a form-boundary function we consider $f(x) = \frac{x}{|x|^2}$. By Hardy's inequality, we have

$$\left\|\frac{\phi}{x}\right\|^p_{L^p} \leq \left(\frac{p}{l-p}\right)^p \|\nabla\phi\|^p_{L^p}$$

for all $\phi \in W^p_1(\Omega)$, therefore, for $p = 2$, we obtain exact form-boundary $\beta = \left(\frac{2}{l-2}\right)^2$ and $c_\beta(t) = 0$.

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The Hölder space $H^{\alpha,\,\frac{\alpha}{p}}\left(clos\left(D\left(T\right)\right)\right)$ consists of all functions $u\ :\quad D\left(T\right)\to R$ for which the value

$$\|u\|_{H^{\alpha,\,\frac{\alpha}{p}}}=\sup_{(x,\,t),\,(y,\,t)\in D}\frac{\left|u\left(x,\ t\right)-u\left(y,\ t\right)\right|}{\left|x-y\right|^{\alpha}}+\sup_{(x,\,t),\,(x,\,\tau)\in D}\frac{\left|u\left(x,\ t\right)-u\left(x,\ \tau\right)\right|}{\left|t-\tau\right|^{\frac{\alpha}{p}}}+\|u\|_{L^{\infty}}\,.$$

is bounded.

Although the literature on parabolic partial differential equations is too extensive to be reviewed in this paper, we will mention a few classical authors and describe some recent results. The regularity properties of solutions to parabolic problems were studied by De Giorgi [6], J. Moser [16, 17] and reviewed by O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uralceva [14] and extensive references therein; we mention recent works of Z. Chen and J. Wang [4] where the existence and uniqueness of the fundamental solution to non-local operators with perturbations and its applications to stochastic differential equations; Z. Chen and X. Zhang [5] investigated the Hölder continuity of fundamental solutions to the non-local Levy-type operator and showed the solvability of the parabolic equation by application of the maximum principle; in [7], M. Ding, C. Zhang, and S. Zhou studied properties of weak solution to nonhomogeneous equation with symmetric kernel and fractional p-Laplacian, the boundedness of these solutions was shown by the De Giorgi-Nash-Moser iteration method; in [**?**, **?**], H. Dong, T. Jin, and H. Zhang studied the Dini-Schauder estimates for concave, nonlinear global parabolic equations with rough kernels and drift terms; H. Dong, X. Pan, and Q. S. Zhang [8, 11], authors consider the analyticity in time for solutions to incompressible Navier-Stokes equations in the whole space, and in the half-space with the Dirichlet boundary conditions by proving integrability properties of certain heat kernels; in [24], C. Zeng investigated the pointwise time analyticity of the heat equations with perturbations of potential type and power nonlinearity, a necessary and sufficient conditions of analyticity in time at zero time were obtained; in [23], Y. G. Zhao and M. X. Wang studied the Lotka-Volterra equation that describes dynamical competition system invasive and native species, authors described long time behavior of successful development of invaders. Some modern applications to problems of physics and mathematical modeling can be found in [1, 2, 3, 13, 15, 19, 21, 22].

The classes $B_2\left(D_T,\ M\right)$ were introduced by De Giorgi [6] and further studied by O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Uralceva [14]. In [14], authors showed that classes $B_2\left(D_T,\ M\right)$ are embedded into the Hölder space $H^{\alpha,\,\frac{\alpha}{2}}\left(D_T\right)$ for some appropriate positive parameter α so that each function $u\in B_2\left(D_T,\ M\right)$ is Hölder continuous and the Hölder norm can be estimated by parameters of the class $B_2\left(D_T,\ M\right)$. Weak bounded solutions of parabolic quasilinear equations in the divergent form belong to appropriate classes $B_p\left(D_T,\ M\right)$ under certain assumptions on their coefficients.

The main goal of this paper is to establish the Hölder regularity of weak solutions to the problem for the parabolic equation in the divergent form under the form-boundary conditions on its coefficients. Section 2 is dedicated to the introduction and description of the functional classes $B_p(D_T,\ M)$. In Section 3, we present the proof of the Hölder regularity of a weak bounded solution to the problem (1.1) - (1.2) under conditions (3.1) - (3.3).

2. Preliminaries on functional classes $B_p(D_T,\ M)$

The Banach space $V_{1,0}^p(D_T)$ is defined as the subset of the Sobolev space $W_{1,0}^p(D_T)$, of all elements $\upsilon(x,\ t)$ which are continuous at t in the topology of norm $L^p(\Omega)$ and the norm $V_{1,0}^p(D_T)$ is given by

$$\|u\|_{V_{1,0}^p(D_T)} = \max_{t\in[0,\ T]} \|u(\cdot,\ t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^p(D_T)}\,.$$

Definition 2.1. The class $B_p(D_T,\ M)$ consists of all functions $u\ :\ \ \Omega\times[0,\ T]\to R$ such that

i) $u\in V_{1,0}^p(D_T)$ and $ess\max_{D_T}|u|\le M$;

ii) let $w(x,\ t)=\pm u(x,\ t)$ and $w_k(x,\ t)=\max\{w(x,\ t)-k,\ 0\}$ then

$$\max_{t_1\le t\le t_1+\tau}\|w_k(x,\ t)\|^2_{2,\ B(\rho-\sigma_1\rho)}\le\|w_k(x,\ t_1)\|^2_{2,\ B(\rho)}$$
$$+\delta\left((\sigma_1\rho)^{-p}\|w_k\|^p_{p,\ D(\rho,\ \tau)}+\eta^{\frac{p}{r}(1+\chi)}(k,\ \rho,\ \tau)\right)$$

and

$$\|w_k\|^p_{V_{1,0}^p(D(\rho-\sigma_1\rho,\ \tau-\sigma_2\tau))}$$
$$\le\delta\left(\left((\sigma_1\rho)^{-p}+(\sigma_2\tau)^{-1}\right)\|w_k\|^p_{p,\ D(\rho,\ \tau)}+\eta^{\frac{p}{r}(1+\chi)}(k,\ \rho,\ \tau)\right),$$

where $D(\rho,\ \tau)=\{B(\rho)\times(t_1,\ t_1+\tau)\}$, and $\rho,\ \tau$ are a pair of positive numbers; and $\sigma_1,\ \sigma_2\in(0,\ 1)$, and $\eta(k,\ \rho,\ \tau)=\int_{(t_1,\ t_1+\tau)} meas^{\frac{r}{s}}(E_k(\rho)(t))\,dt$; $M,\ \delta,\ s,\ r,\ c_3,\ \chi$ are fixed positive numbers such that number k is chosen so that

$$ess\max_{D(\rho,\ \tau)} w(x,\ t)-k\le c_3$$

and $\frac{p}{2}\frac{1}{r}+\frac{l}{2s}=\frac{l}{4}$, $s\in\left(2,\ \frac{2l}{l-2}\right]$, $r\in[2,\ \infty)$, $l\ge3$.

For additional information, readers can see [14].

The sufficient condition that function u belongs to class $B_p(D_T,\ M)$ is that function u satisfies the following inequality

$$\left\|\theta^{\frac{p}{2}}(x,\ t_1+\tau)\,w_k(x,\ t_1+\tau)\right\|^2_{2,\ B(\rho)}+\nu\|\theta\nabla w_k\|^p_{p,\ D(\rho,\ \tau)}$$
$$\le\left\|\theta^{\frac{p}{2}}(x,\ t_1)\,w_k(x,\ t_1)\right\|^2_{2,\ B(\rho)}$$
$$+c\int_{D(\rho,\ \tau)}\left({w_k}^p|\nabla\theta|^p+{w_k}^2\theta^{p-1}\partial_t\theta\right)dxdt$$

$$+c_1\left(\int_{[t_1,\ t_1+\tau]}\left(\int_{E_k(\rho)(t)}\theta dx\right)^{\frac{r}{s}}dt\right)^{\frac{p}{r}(1+\chi)},$$

where $w(x,\ t)=\pm u(x,\ t)$, $w_k(x,\ t)=\max\{w(x,\ t)-k,\ 0\}$, and $\theta(x,\ t)$ is an arbitrary continuous, twice smooth function such that $0\le\theta(x,\ t)\le 1$ and equals zero on sides of the cylinder $D(\rho,\ \tau)$ and its gradient vanishes on $\partial\Omega\times[0,\ T]$.

We recall the following embedment theorem [14].

Theorem 2.2. *There exists an embedment of class $B_p(D_T,\ M)$ into the Hölder space $H^{\alpha,\ \frac{\alpha}{p}}(D_T)$ for some positive number α.*

3. A nonlinear parabolic partial differential equation involving p-Laplacian

We are going to investigate the regularity of the solution to the parabolic problem (1.1) - (1.2) under the following assumptions. Functions $a_i(x,\ t,\ u,\ \xi)$ and $a(x,\ t,\ u,\ \xi)$ are defined for all $(x,\ t)\in clos(D_T)$ and $u\in R,\ \xi\in R^l$; $a_i(x,\ t,\ u,\ \xi)$ and $a(x,\ t,\ u,\ \xi)$ are continuous at u and ξ. The coefficients satisfy the assumptions

$$\sum_i a_i(x,\ t,\ \xi)\,\xi_i\ge\nu\,|\xi|^p,\tag{3.1}$$

$$|a_i(x,\ t,\ \xi)|\le\mu\,|\xi|^{p-1}+\gamma_1(x,\ t),\tag{3.2}$$

$$|a(x,\ t,\ u,\ \xi)|\le\mu_1(|u|)\,|\xi|^p+\mu_2(x)\,|u|^p+\gamma_2(x,\ t),\tag{3.3}$$

with a positive continuous and monotone decreasing function ν, and a positive continuous and monotone increasing function μ_1. We assume form-boundary condition ${\gamma_1}^{\frac{q}{p}},\ {\gamma_2}^{\frac{1}{p}}\in PK(\beta)$ and $\mu_2\in L^\infty$.

For all $s\ge 1$ and $r\ge 1$, we define a norm

$$\|u\|_{s,r,D(T)}=\left(\int_{[0,\ T]}\left(\int_\Omega|u(x,\ t)|^s\,dx\right)^{\frac{r}{s}}dt\right)^{\frac{1}{r}}.\tag{3.4}$$

Definition 3.1. The class $PK(\beta)$ consists of all functions $f\in L^1_{loc}(\Omega\times(0,\ T))$ such that

$$\begin{aligned}\int_{[0,\ T]}\|f(\cdot,\ t)\,\phi(\cdot,\ t)\|^p_{L^p}\,dt\le\beta\frac{p^2}{4}\int_{[0,\ T]}\|\phi(\cdot,\ t)\|^{p-2}_{L^p}\,\|\nabla\phi(\cdot,\ t)\|^2_{L^p}\,dt\\+\int_{[0,\ T]}c_\beta(t)\,\|\phi(\cdot,\ t)\|^p_{L^p}\,dt\end{aligned}\tag{3.5}$$

for all $\phi\in W^p_1(\Omega\times(0,\ T))$.

Also, readers can be interesting in [20, 22, 23].

Definition 3.2. The function $u(x,\ t)$ is said to be a weak solution to the equation (1.1) if $u \in L^1_{loc}(\Omega \times (0,\ T))$, $|\nabla u| \in L^1_{loc}(\Omega \times (0,\ T))$ and the identity

$$\int_{\Omega} u(x,\ t)\,\phi(x,\ t)\,dxdt\Big|_0^T - \int_{[0,\ T]}\int_{\Omega} u\partial_t\phi dxdt + \int_{[0,\ T]}\int_{\Omega} a_i(x,\ t,\ \nabla u)\,\nabla_i\phi dxdt + \int_{[0,\ T]}\int_{\Omega} a(x,\ t,\ u,\ \nabla u)\,\phi dxdt = 0 \tag{3.6}$$

holds for $\phi \in C_0^\infty(\Omega \times (0,\ T))$. The solution u is said to be a bounded weak solution to the equation (1.1) if $\underset{D_T}{ess\max}|u| < \infty$.

For additional information see [14].

Straightforwardly, we conclude the inequality

$$\begin{aligned}\int_{\Omega} u(x,\ t)\,\phi(x,\ t)\,dxdt\Big|_0^T &- \int_{[0,\ T]}\int_{\Omega} u\partial_t\phi dxdt \\ &+ \int_{[0,\ T]}\int_{\Omega} a_i(x,\ t,\ \nabla u)\,\nabla_i\phi dxdt \\ &\le \int_{[0,\ T]}\int_{\Omega} (\mu_1|\nabla u|^p + \mu_2|u|^p + \gamma_2)\,|\phi|\,dxdt,\end{aligned} \tag{3.7}$$

for all $\phi \in C_0^\infty(\Omega \times (0,\ T))$, where $\mu_1 = \mu_1\left(\underset{D_T}{ess\max}|u|\right)$.

Theorem 3.3. *We assume that functions a_i and a satisfy conditions* (3.1) - (3.3)*, and u is a bounded weak solution to the problem* (1.1) - (1.2) *with $\underset{D_T}{ess\max}|u| = M < \infty$. Then, the solution u belongs to the $H^{\alpha,\ \frac{\alpha}{p}}(D_T)$- class, where the positive constant αdepends only on $\nu,\ \mu,\ \mu_0,\ l,\ r,\ M,\ p,\ \beta$.*

Proof. Let $u \in V^p_{1,0}(D(T))\,,\ D(T) \in \Omega \times (0,\ T)$ be a weak solution to the parabolic problem (1.1), (1.2). We define the functions

$$u_h(x,\ t) = \frac{1}{h}\int_{[t,\ t+h]} u(x,\ s)\,ds$$

and

$$u_{\bar{h}}(x,\ t) = \frac{1}{h}\int_{[t-h,\ t]} u(x,\ s)\,ds.$$

For $h \le t_1 \le t_2 \le T - h$, we obtain the identity

$$\int_{[t_1,\ t_2]}\int_{\Omega} \eta\partial_t u_h dxdt + \int_{[t_1,\ t_2]}\int_{\Omega} a_i(x,\ t,\ \nabla u)\,\nabla_i\eta_{\bar{h}} dxdt + \int_{[0,\ T]}\int_{\Omega} a(x,\ t,\ u,\ \nabla u)\,\eta_{\bar{h}} dxdt = 0,$$

for all $u \in V^p_{1,0}(D(T-h))$. Let θ be an arbitrary non-negative continuous, twice smooth function, which vanishes on $\partial\Omega \times [0,\ T]$ with its gradient, and such that

$0 \le \theta(x,\ t) \le 1$. Then, we take

$$\eta(x,\ t) = \theta^p(x,\ t)\max\{u_h(x,\ t) - k,\ 0\} = \theta^p u_{h,\ k}$$

as a test function, and obtain

$$\int_{[t_1,\ t_2]}\int_\Omega \theta^p u_{h,\ k}\frac{\partial u_h}{\partial t}dxdt = \frac{1}{2}\int_\Omega \theta^p(x,\ t)\, {u_{h,\ k}}^2(x,\ t)\Big|_{t_1}^{t_2} - \frac{p}{2}\int_{[t_1,\ t_2]}\int_\Omega \theta^{p-1}{u_{h,\ k}}^2\frac{\partial\theta}{\partial t}dxdt.$$

We pass to the limit as $h \to 0$ and deduce

$$\frac{1}{2}\left\|\theta^{\frac{p}{2}}(x,\ t)\,u_k(x,\ t)\right\|_{2,\ \Omega}^2\Big|_{t_1}^{t_2} - \frac{p}{2}\int_{[t_1,\ t_2]}\int_\Omega {u_k}^2\theta^{p-1}\frac{\partial\theta}{\partial t}dxdt$$
$$+\int_{[t_1,\ t_2]}\int_\Omega a_i(x,\ t,\ \nabla u)\nabla_i(\theta^p u_k)\,dxdt$$
$$+\int_{[0,\ T]}\int_\Omega a(x,\ t,\ u,\ \nabla u)\,\theta^p u_k dxdt = 0.$$

Let the function θ vanish outside the ball $B(\rho)$ then we estimate

$$\frac{1}{2}\left\|\theta^{\frac{p}{2}}(x,\ t)\,u_k(x,\ t)\right\|_{2,\ B(\rho)}^2\Big|_{t_1}^{t_2} + \nu\int_{[t_1,\ t_2]}\int_{B(\rho)}|\nabla u_k|^p\theta^p dxdt$$
$$\le p\int_{[t_1,\ t_2]}\int_{E_k(\rho)}\left(\mu|\nabla u|^{p-1} + \gamma_1\right)(u-k)\,\theta^{p-1}|\nabla\theta|\,dxdt$$
$$+\int_{[t_1,\ t_2]}\int_{E_k(\rho)}(\mu_1|\nabla u|^p + \mu_2|u|^p + \gamma_2)\,\theta^p(u-k)\,dxdt$$
$$+\frac{p}{2}\int_{[t_1,\ t_2]}\int_{E_k(\rho)}(u-k)^2\theta^{p-1}\frac{\partial\theta}{\partial t}dxdt$$

where $E_k(\rho) = \{x \in B(\rho)\ :\quad u(x,\ t) > k\}$.

By the Young inequality, we have

$$\tfrac{1}{2}\left\|\theta^{\frac{p}{2}}(x,\ t)\,u_k(x,\ t)\right\|_{2,\ B(\rho)}^2\Big|_{t_1}^{t_2} + \nu\int_{D(\rho,\ t_2-t_1)}|\nabla u_k|^p\theta^p dxdt \le$$
$$\le p\tfrac{\varepsilon_1}{q}\mu\int_{[t_1,\ t_2]}\int_{E_k(\rho)}|\nabla u|^p\theta^p dxdt+$$
$$+p\tfrac{1}{\varepsilon_1 p}\mu\int_{[t_1,\ t_2]}\int_{E_k(\rho)}(u-k)^p|\nabla\theta|^p dxdt+$$
$$+p\tfrac{1}{\varepsilon_2 p}\int_{[t_1,\ t_2]}\int_{E_k(\rho)}|\nabla\theta|^p(u-k)^p dxdt + p\tfrac{\varepsilon_2}{q}\int_{[t_1,\ t_2]}\int_{E_k(\rho)}{\gamma_1}^q\theta^p dxdt+$$
$$+\mu_1\int_{[t_1,\ t_2]}\int_{E_k(\rho)}|\nabla u|^p\theta^p\max_{D(\rho,\ t_2-t_1)}(u-k)\,dxdt+$$
$$+\int_{[t_1,\ t_2]}\int_{E_k(\rho)}(\mu_2|u|^p + \gamma_2)\,\theta^p\max_{D(\rho,\ t_2-t_1)}(u-k)\,dxdt+$$
$$+\tfrac{p}{2}\int_{[t_1,\ t_2]}\int_{E_k(\rho)}{u_k}^2\theta^{p-1}\tfrac{\partial\theta}{\partial t}dxdt,$$

where we denote $D(\rho,\ t_2-t_1) = \{B(\rho)\times(t_2-t_1)\}$, and $\varepsilon_1,\ \varepsilon_2$ are arbitrary positive numbers. Taking into consideration $ess\max_{D_T}|u| = M < \infty$ and choosing k so that

$$\max_{D(\rho,\, t_2-t_1)} \left(u\left(x,\ t\right)-k\right) \le c_3 \frac{\nu}{8\mu_0},$$

we obtain

$$\frac{1}{2}\left\|\theta^{\frac{p}{2}}\left(x,\ t_2\right)u_k\left(x,\ t_2\right)\right\|_{2,\, B(\rho)}^2 + \nu\int_{D(\rho,\, t_2-t_1)} \left|\nabla u_k\right|^p \theta^p dxdt$$
$$\le \frac{1}{2}\left\|\theta^{\frac{p}{2}}\left(x,\ t_1\right)u_k\left(x,\ t_1\right)\right\|_{2,\, B(\rho)}^2 + c_1\int_{[t_1,\, t_2]}\int_{E_k(\rho)} \left({u_k}^p\left|\nabla\theta\right|^p + {u_k}^2\theta^{p-1}\frac{\partial\theta}{\partial t}\right) dxdt$$
$$+c_2\int_{[t_1,\, t_2]}\int_{E_k(\rho)} \left({\gamma_1}^q + c_4\gamma_2\right)\theta^p dxdt.$$

We denote

$$\Xi = \left({\gamma_1}^q + c_4\gamma_2\right)\theta^p.$$

Applying form-boundary condition, we have

$$\int_{[t_1,\, t_2]}\int_{E_k(\rho)} \left({\gamma_1}^{\frac{q}{p}}\theta\right)^p dxdt \le \beta\frac{p^2}{4}\int_{[0,\, T]} \left\|\phi\left(\cdot,\ t\right)\right\|_{L^p}^{p-2}\left\|\nabla\phi\left(\cdot,\ t\right)\right\|_{L^p}^2 dt$$
$$+\int_{[0,\, T]} c_\beta\left(t\right)\left\|\phi\left(\cdot,\ t\right)\right\|_{L^p}^p dt$$
$$\le \beta\frac{p^2}{4}\int_{[0,\, T]}\left(\frac{2\sigma^{\frac{p}{2}}}{p}\left\|\nabla\phi\left(\cdot,\ t\right)\right\|_{L^p}^p + \frac{(p-2)}{\sigma^{\frac{p}{p-2}}p}\left\|\phi\left(\cdot,\ t\right)\right\|_{L^p}^p\right) dt$$
$$+\int_{[0,\, T]} c_\beta\left(t\right)\left\|\phi\left(\cdot,\ t\right)\right\|_{L^p}^p dt,$$

similar arguments deal with the terms containing γ_2. Therefore, we have

$$\int_{[t_1,\, t_2]}\int_{E_k(\rho)(t)} \Xi\left(x,\ t\right)dxdt \le \left\|\Xi\right\|\left\|1\right\|_{E_k(\rho)\times[t_1,\, t_2]}^p$$
$$\le \tilde{c}\left(\int_{[t_1,\, t_1+\tau]} meas\left(E_k\left(\rho\right)\left(t\right)\right)^{\frac{r}{s}} dt\right)^{\frac{p}{r}(1+\chi)}.$$

Thus, we conclude

$$\frac{1}{2}\left\|\theta^{\frac{p}{2}}\left(x,\ t_2\right)u_k\left(x,\ t_2\right)\right\|_{2,\, B(\rho)}^2 + \nu\left\|\theta\nabla u_k\right\|_{L^p,\, D(\rho,\, t_2-t_1)}^p$$
$$\le \frac{1}{2}\left\|\theta^{\frac{p}{2}}\left(x,\ t_1\right)u_k\left(x,\ t_1\right)\right\|_{2,\, B(\rho)}^2$$
$$+c_1\int_{[t_1,\, t_2]}\int_{E_k(\rho)}\left({u_k}^p\left|\nabla\theta\right|^p + {u_k}^2\theta^{p-1}\frac{\partial\theta}{\partial t}\right) dxdt$$
$$+\tilde{c}\left(\int_{[t_1,\, t_1+\tau]} meas\left(E_k\left(\rho\right)\left(t\right)\right)^{\frac{r}{s}} dt\right)^{\frac{p}{r}(1+\chi)}.$$

A similar argument holds for the function $-u(x,\ t)$. Thus, the function u belongs to the class $B_p(D_T,\ M)$. Therefore, by Theorem 2.2, there exists a number α such that $u \in H^{\alpha,\ \frac{\alpha}{p}}(D_T)$. The theorem has been proven. $\square$

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