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COMMON FIXED POINT RESULTS IN TOPOLOGICAL SPACES

MAHER BERZIG

Dedicated to Professor Sehie Park

Abstract. We study the existence of common fixed points for self-mappings in topological spaces. Under some suitable conditions, we show that if a mapping f is contractive relative to g or it is g-contractive, then f and g have a unique common fixed point. Some of the upshots are presented in compact, sequentially compact and metric spaces. As consequences, we extend some Park's [26, 27] theorems and generalizes the main result of Liepiņš [22].

1. Introduction

Let X be a nonempty set, $Y \subseteq X$ and $f, g\colon X \to X$ be given mappings. We say that f *is contractive relative to* g, or g *is non-contractive relative to* f, on Y if there exists a function $\varphi\colon X \times X \to \mathbb{R}_+$ such that

$$\varphi(fx, fy) < \varphi(gx, gy) \quad \text{for all } x, y \in Y \text{ such that } gx \neq gy. \tag{1.1}$$

We simply call f (resp. g) contractive (resp. non-contractive) on Y if $g = id$ (resp. $f = id$), where id is the identity mapping on X. We say that f *is* g*-nonexpansive*, or g *is* f*-expansive*, on Y if the strict inequality of (1.1) becomes a large one even for $gx = gy$. Moreover, f (resp. g) is simply called nonexpansive (resp. expansive) on Y if $g = id$ (resp. $f = id$).

In 1936, Nemytskii [24] showed that a contractive mapping in a compact metric space has a fixed point. After forty years, Jungck [16] generalized Nemytskii's result, and studied the existence of common fixed point in compact metric spaces for relative contractive mappings. Edelstein [9] showed the same result as Nemytskii but in metric space containing a nonempty set of limit points of certain sequence. For more references on the development of fixed and common fixed point theorems for contractive mappings, we refer the reader to [1–8, 12, 14, 15, 17–21, 25–34].

A mapping f is called g*-contractive* on $Y \subseteq X$ if there exists a function $\varphi\colon X \times X \to \mathbb{R}_+$ such that

$$\varphi(fx, fy) < \varphi(gx, gy) \quad \text{for all } x, y \in Y \text{ such that } fx \neq fy. \tag{1.2}$$

In both contractive conditions (1.1) and (1.2), if $Y = X$ we will omit the term 'on Y' in the sequel. Park [26, 27] studied the g-contractive mappings and showed the existence of common fixed points when a specific sequence $\{gx_i\}$ has a convergent subsequence. This sequence is called g-iteration of $x_0 \in X$ under f and it is given

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by $gx_{i+1} = fx_i$ for all integer $i \geq 0$ whenever $f(X) \subseteq g(X)$. The following result is derived from Park [26, Theorem 2.2].

Theorem A. *Let (X, φ) be a compact metric space and $f, g \colon X \to X$ be commuting mappings such that g is continuous and $f(X) \subseteq g(X)$. Assume that f is g-contractive. If for some $x_0 \in X$ the g-iteration of x_0 under f has a convergent subsequence to $y \in X$, then gy is the unique common point of f and g.*

It is interesting to observe that if we replace g by id in Theorem A it generalizes the fixed point theorem of Edelstein [9, Theorem 1], and this indeed because in metric spaces every contractive mapping is id-contractive. However, this observation does not necessarily hold true in any topological space. A fixed point theorem generalizing that of Edelstein in topological spaces was established by Liepiņš [22] in 1980. In order to state this theorem, I will quickly recall some of the basic elements.

Let X be a set such that $x \in X$ and $f \colon X \to X$ be a given mapping. Denote by $\mathcal{O}(x; f)$ the set of all the terms of the sequence $\{f^k x\}_{k \geq 0}$ and denote by $\mathrm{lp}\{\mathcal{O}(x; f)\} := \bigcap_{n \in \mathbb{N}} \mathrm{cl}(\{f^k x \colon k \geq n\})$ the set of the limit points of $\mathcal{O}(x; f)$, where $\mathrm{cl}(A)$ denotes the closure of a set A and $\mathbb{N}$ the set of positive integers. Here and in the rest of this paper, we assume that φ is continuous with respect to the topology of X. The fixed point theorem of Liepiņš [22] is the following.

Theorem B. *Let X be a topological space and $f \colon X \to X$ be a continuous mapping. Assume that f is contractive. Let $x \in X$, then $\mathrm{lp}\{\mathcal{O}(x; f)\}$ is empty or it contains only the unique fixed point of f.*

In this paper, we intend to show the existence of common fixed points for mappings satisfying some contractive conditions in topological spaces. We start by adding some fixed point theorems in topological spaces, from which we derive some common fixed point theorems when a single map is contractive. Then we introduce the concepts of primary and secondary sequences to investigate the existence of common fixed points for mappings satisfying the contractive conditions (1.1) and (1.2). We also exploit the continuity of certain single valued mapping to establish more results of common fixed point. Finally, we provide some sufficient conditions for the existence and uniqueness of common fixed point in compact, sequentially compact and metric spaces.

2. Fixed and common fixed point in topological spaces

Although the proof of the next theorem is very similar to that of Theorem B, we provide it here, for the sake of completeness.

Theorem 2.1. *Let X be a topological space and $f \colon X \to X$ be a continuous mapping. Assume that f is contractive (or id-contractive) on $f(X)$. Let $x \in X$, then $\mathrm{lp}\{\mathcal{O}(x; f)\}$ is empty or it contains only the unique fixed point of f.*

Proof. Let $x \in X$ and let $z \in \mathrm{lp}\{\mathcal{O}(x; f)\}$. Because of the continuity of f, we have $fz \in \mathrm{lp}\{f(\mathcal{O}(x; f))\}$ and $f^2 z \in \mathrm{lp}\{f^2(\mathcal{O}(x; f))\}$, and thus $fz, f^2 z \in \mathrm{lp}\{\mathcal{O}(x; f)\}$.

Define now $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(x, fx)$ for all $x \in X$. The continuity of f and φ implies this of ψ. As consequences, we get $\psi(fz), \psi(f^2z) \in \mathrm{lp}\{\psi(\mathcal{O}\,(x; f)\}$.

Next, assume that $f^nz \neq f^{n+1}z$ for all $n \in \mathbb{N}$. By the contractive condition, it follows that

$$\psi(f^{n+1}z) = \varphi(f^{n+1}z, ff^{n+1}z) < \varphi(f^nz, f^{n+1}z) = \psi(f^nz).$$

We deduce that $\{\psi(\mathcal{O}\,(x; f)\}$ is decreasing in $\mathbb{R}_+$, hence converges to a unique point in $\mathrm{lp}\{\psi(\mathcal{O}\,(x; f)\}$. Thus, $\psi(fz) = \psi(f^2z)$ or equivalently $\varphi(fz, f^2z) = \varphi(f^2z, f^3z)$, which implies that $fz = f^2z$ (or $f^2z = f^3z$) according to the contractive condition. Thus $y = fz$ (or $y = f^2z$) is a fixed point of f.

If we assume that $f^nz = f^{n+1}z$ for some order $n \in \mathbb{N}$, we deduce that the sequence $\{f^nz\}$ becomes constant from this order and therefore the same conclusion holds. Finally, a fixed point $y' \neq y$ of f in X should satisfies $\varphi(y, y') = \varphi(fy, fy') < \varphi(y, y')$, which is a contradictory inequality, so the fixed point is unique in X. □

The following result follows from Theorem 2.1.

Theorem 2.2. *Let X be a topological space and $f, g\colon X \to X$ be two commuting mappings such that $f(X) \subseteq g(X)$. Assume that f is continuous and contractive (or id-contractive) on $f(X)$. Let $x \in X$, then $\mathrm{lp}\{\mathcal{O}\,(x; f)\}$ is empty or it contains only the unique common fixed point of f and g.*

Proof. All the assumptions of Theorem 2.1 are fulfilled. Thus $\mathrm{lp}\{\mathcal{O}\,(x; f)\}$ is empty or contains only y the unique fixed point of f. We claim that y is a fixed point of g. Assume that $y \neq gy$, then by using the contractive condition and the commutativity of f and g, we get

$$\varphi(y, gy) = \varphi(fy, gfy) = \varphi(fy, fgy) < \varphi(y, gy),$$

which is a contradiction. We conclude that $y = fy = gy$. The uniqueness of the common fixed point of f and g follows from the contractive condition. □

Remark 2.3. *The commutativity on X of Theorem 2.2 can be relaxed to the commutativity on the set of fixed points of f.*

Let X be a set, $x \in X$ and $g\colon X \to X$ be given mapping such that $X \subseteq g(X)$. The g-iteration of x under id in the sens of Park [26] is the sequence $S_x := \{gx_i\}$ given by $gx_i = x_{i-1}$ for all $i \in \mathbb{N}$ where $x_0 = x$. Denote by $\mathcal{O}\,(x; id, g)$ the set of all the terms of the sequence $\{gx_i\}_{i\in\mathbb{N}}$. The set of limit points of $\mathcal{O}\,(x; id, g)$ will be denoted by $\mathrm{lp}\{\mathcal{O}\,(x; id, g)\} := \bigcap_{n\in\mathbb{N}} \mathrm{cl}\{gx_i\colon i \geq n\}$.

Theorem 2.4. *Let X be a topological space and $g\colon X \to X$ be a continuous mapping such that $X \subseteq g(X)$. Assume that g is non-contractive (or id is g-contractive). Let $x \in X$, then $\mathrm{lp}\{\mathcal{O}\,(S_x; id, g)\}$ is empty or it contains only the unique fixed point of g.*

Proof. Let $x \in X$, and construct a sequence $S_x := \{gx_i\}$ such that $gx_i = x_{i-1}$ for all $i \in \mathbb{N}$ and $x_0 = x$. Let $z \in \mathrm{lp}\{\mathcal{O}\,(S_x; id, g)\}$. Since g is continuous,

$gz \in \mathrm{lp}\{g(\mathcal{O}(S_x; id, g))\}$. Define now $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(gx, x)$ for all $x \in X$. It is clear that ψ is continuous, since so are g and φ. Thus, $\psi(z), \psi(gz) \in \mathrm{lp}\{\psi(\mathcal{O}(S_x; id, g))\}$.

Assume next that gx_i is not a fixed point of g for all $i \in \mathbb{N}$. Then, we have

$$\psi(gx_i) = \varphi(g^2x_i, gx_i) = \varphi(gx_{i-1}, x_{i-1}) < \varphi(g^2x_{i-1}, gx_{i-1}) = \psi(gx_{i-1}).$$

This proves that $\{\psi(gx_i)\}$ is decreasing, and since it is bounded below, it converges to a single point in $\mathrm{lp}\{\psi(\mathcal{O}(S_x; id, g))\}$. Therefore, $\psi(z) = \psi(gz)$, which is equivalent to $\varphi(gz, z) = \varphi(g^2z, gz)$, which is impossible by the contractive condition unless $g^2z = gz$ (or $gz = z$). The case gx_i is a fixed point of g for some $i \in \mathbb{N}$ leads to the same conclusion. The uniqueness of the fixed point is obvious. □

The following theorem follows from Theorem 2.4.

Theorem 2.5. *Let X be a topological space and $f, g\colon X \to X$ be two commuting and surjective mappings such that g is continuous. Assume that g is non-contractive (or the id is g-contractive) on $g(X)$. Let $x \in X$, then $\mathrm{lp}\{\mathcal{O}(x; id, g)\}$ is empty or contains only the unique common fixed point of f and g.*

Proof. All the assumptions of Theorem B are fulfilled. Then $\mathrm{lp}\{\mathcal{O}(x; id, g)\}$ is empty or contains only y the unique fixed point of g. We claim that y is a fixed point of f. Assume that $y \neq fy$, then by using the contractive condition and the commutativity of f and g, we get

$$\varphi(y, fy) < \varphi(gy, gfy) = \varphi(gy, fgy) = \varphi(y, fy),$$

which is a contradiction. We conclude that $y = fy = gy$. The uniqueness of the fixed point is obvious. □

Remark 2.6. *The commutativity on X of Theorem 2.5 can be relaxed to the commutativity on the set of fixed points of g.*

3. Common fixed point in topological spaces

We start this section by presenting the concepts of primary and secondary sequences. Let X be a topological space and $f, g\colon X \to X$ be given mappings such that $f(X) \subseteq g(X)$. Let $x \in X$ and the g-iteration of x under f in the sens of Park [26] is called the *primary sequence*, that is, $S_x := \{gx_i\}$ which is given by $gx_{i+1} = fx_i$ for all $i \geq 0$ where $x_0 = x$. The *secondary sequences* are constructed over the primary sequence by $\{g^kx_i\}_{k,i\in\mathbb{N}}$. It is interesting to note that if f and g commute then $g^kx_i \in f(X)$ since $g^kx_i = fg^{k-1}x_{i-1}$, where $k, i \in \mathbb{N}$. Let $\mathcal{O}(S_x; f, g)$ denote the double-indexed sequence $\{g^kx_i\}_{k,i\in\mathbb{N}}$. The notation $\phi(\mathcal{O}(S_x; f, g))$ means $\{\phi(g^kx_i)\}_{k,i\in\mathbb{N}}$. The *set of limit points* of $\mathcal{O}(S_x; f, g)$ will be denoted by $\mathrm{lp}\{\mathcal{O}(S_x; f, g)\} := \bigcap_{n\in\mathbb{N}} \mathrm{cl}\{g^kx_i\colon k, i \geq n\}$. Observe also that $\mathcal{O}(S_x; f, id) = \mathcal{O}(x; f, id) = \mathcal{O}(x; f)$. It is not difficult to see that if f and g are continuous and commuting mappings on X, then $fy \in \mathrm{lp}\{f(\mathcal{O}(S_x; f, g))\}$ whenever $y \in \mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$. A double-indexed sequence $\{u_{i,j}\}_{i,j\in\mathbb{N}}$ is called decreasing (resp. increasing), if $u_{i,j} \geq u_{p,q}$ (resp. $u_{i,j} \leq u_{p,q}$) for all $(i,j) \leq (p,q)$ ([13, 4.1]). Recall now the monotone convergence theorem for a double-indexed sequence of real numbers.

Theorem C (Monotone Convergence Theorem [13, 4.2])**.** *A double-indexed sequence* $\{u_{i,j}\}_{i,j\in\mathbb{N}}$ *of real numbers is convergent iff it is bounded. Further, if it is decreasing and bounded below (resp. increasing and bounded above), then*

$$\lim_{i\to\infty}(\lim_{j\to\infty} u_{i,j}) = \lim_{j\to\infty}(\lim_{i\to\infty} u_{i,j}) = \lim_{i,j\to\infty} u_{i,j},$$

is also equal to $\inf_{i,j\in\mathbb{N}}\{u_{i,j}\}$ *(resp.* $\sup_{i,j\in\mathbb{N}}\{u_{i,j}\}$*).*

Now we state the common fixed point theorems in topological spaces for contractive conditions of types (1.1) and (1.2).

Theorem 3.1. *Let X be a topological space and $f, g\colon X \to X$ be two continuous and commuting mappings such that and $f(X) \subseteq g(X)$. Assume that g is nonexpansive on $f(X)$ and f is contractive relative to g on $f(X)$. Let $x \in X$, then* $\mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$ *is empty or contains only the unique common fixed point of f and g.*

Proof. Let $x \in X$, and construct a sequence $S_x \coloneqq \{gx_n\}_{n\geq 0}$ such that $gx_{n+1} = fx_n$ for all $n \geq 0$ and $x_0 = x$. Let $z \in \mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$. Since f is continuous, $fz \in \mathrm{lp}\{f(\mathcal{O}(S_x; f, g))\}$ and $f^2z \in \mathrm{lp}\{f^2(\mathcal{O}(S_x; f, g))\}$. Then by using the commutativity of f and g and the definition of the sequence, we deduce that $fz \in \mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$ and $f^2z \in \mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$. Define now $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(gx, fx)$ for all $x \in X$. It is clear that ψ is continuous, since so are f, g and φ. Thus, $\psi(fz), \psi(f^2z) \in \mathrm{lp}\{\psi(\mathcal{O}(S_x; f, g))\}$.

Assume next that g^kx_i is not a coincidence point of f and g for all $i, k \in \mathbb{N}$. Then, we have

$$\begin{aligned}\psi(g^kx_{i+1}) &= \varphi(gg^kx_{i+1}, fg^kx_{i+1})\\ &= \varphi(fg^kx_i, fg^kx_{i+1})\\ &< \varphi(g^{k+1}x_i, g^{k+1}x_{i+1})\\ &= \varphi(gg^kx_i, fg^kx_i)\\ &= \psi(g^kx_i),\end{aligned}$$

and by using the nonexpansiveness of g on $f(X)$, we obtain

$$\begin{aligned}\psi(g^{k+1}x_i) &= \varphi(gg^{k+1}x_i, fg^{k+1}x_i)\\ &= \varphi(g^{k+2}x_i, g^{k+2}x_{i+1})\\ &\leq \varphi(g^{k+1}x_i, g^{k+1}x_{i+1})\\ &= \varphi(gg^kx_i, fg^kx_i)\\ &= \psi(g^kx_i).\end{aligned}$$

This proves that the double-indexed sequence $\{\psi(g^k x_i)\}$ is decreasing, and since it is bounded below it converges, according to the monotone convergence theorem, to a single point in $\mathrm{lp}\{\psi(\mathcal{O}(S_x; f, g))\}$. Therefore, $\psi(fz) = \psi(f^2z)$, and by commutativity of f and g it is equivalent to $\varphi(fgz, f^2z) = \varphi(f^2gz, f^3z)$, which is impossible because of the contractive condition unless $fgz = f^2z$. Let now $y = fgz = f^2z$. If $y \neq gy$, then

$$\varphi(y, gy) = \varphi(f^2z, gf^2z) = \varphi(f^2z, f^2gz) < \varphi(gfz, gfgz) = \varphi(y, gy),$$

which absurd, so $y = gy = gf^2z = fgfz = fy$, that is, y is a common fixed of f and g in X. The case $g^k x_i$ is a fixed point of g for some $i \in \mathbb{N}$ leads to the same conclusion. The uniqueness of the common fixed point is obvious. □

Theorem 3.2. *Let X be a topological space and $f, g\colon X \to X$ be two continuous, commuting mappings and $f(X) \subseteq g(X)$. Assume that g is nonexpansive on $f(X)$ and f is g-contractive on $f(X)$. Let $x \in X$, then $\mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$ is empty or contains only the unique common fixed point of f and g.*

Proof. Let $x \in X$, and construct a sequence $S_x \coloneqq \{gx_i\}_{i\in\mathbb{N}}$ such that $gx_i = fx_{i-1}$ for all $i \in \mathbb{N}$ and $x_0 = x$. Let $z \in \mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$. Since f is continuous, $fz \in \mathrm{lp}\{f(\mathcal{O}(S_x; f, g))\}$ and $f^2z \in \mathrm{lp}\{f^2(\mathcal{O}(S_x; f, g))\}$. Then by using the commutativity of f and g, we deduce that $fz \in \mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$ and $f^2z \in \mathrm{lp}\{\mathcal{O}(S_x; f, g)\}$. Define now $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(gx, fx)$ for all $x \in X$. It is clear that the continuity of ψ follows from that of f, g and φ. Thus, $\psi(fz), \psi(f^2z) \in \mathrm{lp}\{\psi(\mathcal{O}(S_x; f, g))\}$.

Assume next that $g^k x_i$ is not a coincidence point of f and g for all $i, k \in \mathbb{N}$. Then, we have

$$\begin{aligned}
\psi(g^{k+1}x_{i+1}) &= \varphi(gg^{k+1}x_{i+1}, fg^{k+1}x_{i+1}) \\
&= \varphi(fg^{k+1}x_i, fg^{k+1}x_{i+1}) \\
&< \varphi(gg^{k+1}x_i, gg^{k+1}x_{i+1}) \\
&= \varphi(gg^{k+1}x_i, fg^{k+1}x_i) \\
&= \psi(g^{k+1}x_i),
\end{aligned}$$

and by using the nonexpansiveness of g, we obtain

$$\begin{aligned}
\psi(g^{k+1}x_{i+1}) &= \varphi(gg^{k+1}x_{i+1}, fg^{k+1}x_{i+1}) \\
&= \varphi(g^{k+2}x_{i+1}, g^{k+2}x_{i+1}) \\
&\leq \varphi(g^{k+1}x_{i+1}, g^{k+1}x_{i+1}) \\
&= \varphi(gg^k x_{i+1}, fg^k x_{i+1}) \\
&= \psi(g^k x_{i+1}).
\end{aligned}$$

The rest of the proof is exactly the same as this of Theorem 3.1. □

We say that a function $\varphi\colon X \times X \to \mathbb{R}_+$ vanishes only on the diagonal if

$$\varphi(x, y) = 0 \iff x = y.$$

Remark 3.3. *Let X be a nonempty set and $f, g\colon X \to X$ be two mappings. Assume that f is g-contractive and $f(X) \subseteq g(X)$. Then $fg^{-1}\colon g(X) \to g(X)$ is a single valued mapping. To see this, let $y \in g(X)$ and x_1, x_2 be two distinct points of $fg^{-1}y$. Let $z_1, z_2 \in g^{-1}y$ such that $x_1 = fz_1$ and $x_2 = fz_2$. Thus, if φ vanishes only on the diagonal, $\varphi(fz_1, fz_2) < \varphi(gz_1, gz_2) = 0$ implies a contradiction. This observation is due to Gabeleh et al. [11] for the case where φ is a metric.*

Based on the previous observation, we do not need the non-expansiveness of g.

Theorem 3.4. *Let X be a topological space and $f, g\colon X \to X$ be two commuting mappings such that $f(X) \subseteq g(X)$ and f is g-contractive with φ vanishes only on the diagonal. Assume that fg^{-1} is continuous on $g(X)$, then for all $x \in X$, the set $\mathrm{lp}\{\mathcal{O}(gx; fg^{-1})\}$ is empty or contains only the unique common fixed point of f and g.*

Proof. Let $x \in X$ and let $z \in \mathrm{lp}\{\mathcal{O}(gx; fg^{-1})\}$. Because of the continuity of fg^{-1}, we have $fg^{-1}z \in \mathrm{lp}\{fg^{-1}(\mathcal{O}(gx; fg^{-1}))\}$, and thus $fg^{-1}z \in \mathrm{lp}\{\mathcal{O}(gx; fg^{-1})\}$. Define now $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(fg^{-1}x, x)$ for all $x \in X$. The continuity of fg^{-1} and φ implies this of ψ. As consequence, we get $\psi(z), \psi(fg^{-1}z) \in \mathrm{lp}\{\psi(\mathcal{O}(gx; fg^{-1}))\}$.

Next, assume that $(fg^{-1})^n z \neq (fg^{-1})^{n+1} z$ for all $n \in \mathbb{N}$. By using the contractive condition, it follows that

$$\begin{aligned}\psi((fg^{-1})^{n+1}z) &= \varphi(fg^{-1}(fg^{-1})^{n+1}z, (fg^{-1})^{n+1}z)\\ &< \varphi((fg^{-1})^{n+1}z, (fg^{-1})^{n}z)\\ &= \psi((fg^{-1})^{n}z).\end{aligned}$$

We deduce that $\{\psi((fg^{-1})^n z)\}$ is decreasing in $\mathbb{R}_+$, hence converges to a unique point in $\mathrm{lp}\{\psi(\mathcal{O}(gx; fg^{-1}))\}$. So, $\psi(z) = \psi(fg^{-1}z)$ or equivalently $\varphi(fg^{-1}z, z) = \varphi((fg^{-1})^2 z, fg^{-1}z)$, which implies that $z = fg^{-1}z$. To see this, assume that $z \neq fg^{-1}z$, and since φ vanishes only on the diagonal, it is equivalent to $\varphi(fg^{-1}z, z) \neq 0$. We also deduce that $\varphi((fg^{-1})^2 z, fg^{-1}z) \neq 0$, that is, $(fg^{-1})^2 \neq fg^{-1}z$ and by the g-contractive condition it follows that

$$\varphi((fg^{-1})^2 z, fg^{-1}z) < \varphi(fg^{-1}z, z),$$

which is a contradiction, unless $(fg^{-1})^2 z = fg^{-1}z$, and this also means that $\varphi(fg^{-1}z, z) = \varphi((fg^{-1})^2 z, fg^{-1}z) = 0$, that is, $fg^{-1}z = z$.

Next, by using the commutativity of f and g it follows that

$$fz = fgg^{-1}z = gfg^{-1}z = gz.$$

Let $y = fz = gz$ and assume that $y \neq gy$, then

$$\varphi(y, gy) = \varphi(fz, gfz) = \varphi(fz, fgz) < \varphi(gz, ggz) = \varphi(y, gy),$$

which absurd, so $y = gy = gfz = fgz = fy$, that is, y is a common fixed of f and g. If we assume that $(fg^{-1})^n z = (fg^{-1})^{n+1} z$ for some order $n \in \mathbb{N}$, we deduce that the sequence $\{(fg^{-1})^n z\}$ becomes constant from this order and thus the same conclusion holds. The uniqueness of the common fixed point is obvious. □

Remark 3.5. *We have:*

(i) *Theorems A and B follows from Theorem 3.1.*

(ii) *Assume that f is contractive relative to g (or f is g-contractive).*

  (a) *If g is nonexpansive, then the unique common fixed point of f and g is the unique fixed point of f and clearly it is in $f(X) \cap g(X)$.*

  (b) *If y is the unique fixed point of f and g in X, then it is also their unique coincidence point in X.*

4. Common fixed point in compact and sequentially compact spaces

Compact and sequentially compact are generally different, but in metric spaces they are equivalent. Recall now the extreme value theorem in compact and sequentially compact spaces.

Theorem D. *Let X be a compact or sequentially compact topological space. If a function $\psi\colon X \to \mathbb{R}$ is continuous, then it attains its minimum and maximum values, i.e. there are $x, y \in X$ such that $\psi(x) \leq \psi(z)$ and $\psi(y) \geq \psi(z)$ for all $z \in X$.*

Proof. Since ψ is continuous, then $\psi(X)$ is compact or sequentially compact in $\mathbb{R}$, which equivalent to $\psi(X)$ is closed and bounded (see for instance [10, Theorem 2.42]. Let α and β be respectively the greatest lower bound and the least upper bound of $\psi(X)$. The greatest lower bound and the least upper bound are limit points of $\psi(X)$, and since $\psi(X)$ is closed, $\alpha, \beta \in \psi(X)$, which implies that there exist $x, y \in X$ such that $\psi(x) = \alpha$ and $\psi(y) = \beta$. □

Theorem 4.1. *Let X be a compact or sequentially compact topological space and $f, g\colon X \to X$ be two commuting mappings such that $f(X) \subseteq g(X)$. Assume that f is continuous, contractive (or id-contractive) on $f(X)$. Then f and g have a unique common fixed point.*

Proof. We first show the existence of fixed point for f. Define $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(x, fx)$ for all $x \in X$. It is clear that ψ is continuous, since so are f and φ. Moreover it bounded below which proves that ψ attains its minimum at some point $z \in X$. If $z \neq fz$ (or $fz \neq f^2 z$), then

$$\psi(fz) = \varphi(fz, f^2 z) < \varphi(z, fz) = \psi(z),$$

which contradict that ψ attains its minimum at z unless $z = fz$ (or $fz \neq f^2 z$). We claim that z is a common fixed point of f and g. Assume that $z \neq gz$, then by using the contractive condition and the commutativity of f and g, we get

$$\varphi(z, gz) = \varphi(fz, gfz) = \varphi(fz, fgz) < \varphi(z, gz),$$

which is a contradiction. We conclude that $z = fz = gz$. The uniqueness of the common fixed point of f and g follows easily from the contractive condition. □

Theorem 4.2. *Let X be a compact or sequentially compact topological space and $f, g\colon X \to X$ be two commuting and surjective mappings such that g is continuous. Assume that g is non-contractive (or the id is g-contractive) on $g(X)$. Then f and g have a unique common fixed point.*

Proof. We first show the existence of fixed point for g. Define $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(gx, x)$ for all $x \in X$. It is clear that ψ is continuous, since so are g and φ. Moreover it bounded below which proves that ψ attains its maximum at some point $z \in X$. If $g^2z \neq gz$ (or $gz \neq z$), then

$$\psi(z) = \varphi(gz, z) < \varphi(g^2z, gz) = \psi(gz),$$

which contradict that ψ attains its maximum at z unless $g^2z = gz$ (or $gz = z$). We claim that z is a common fixed point of f and g. Assume that $z \neq fz$, then by using the contractive condition and the commutativity of f and g, we get

$$\varphi(z, fz) < \varphi(gz, gfz) = \varphi(z, fz)$$

which is a contradiction. We conclude that $z = fz = gz$. The uniqueness of the common fixed point of f and g follows from the contractive condition. □

Theorem 4.3. *Let X be a compact or sequentially compact topological space and $f, g\colon X \to X$ be two continuous and commuting mappings such that g is nonexpansive. Assume that f is contractive relative to g where φ vanishes only on the diagonal. Then f and g have a unique common fixed point.*

Proof. Define $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(gx, fx)$. Since f, g and φ are continuous on X, then so is ψ. Now, by compactness of X, we deduce by the extreme value theorem that ψ attains its minimum at some point $z \in X$. We claim that $gfz = g^2z$. Assume that $gfz \neq g^2z$, then by the commutativity, the contractive condition and the nonexpansiveness assumptions, we obtain

$$\psi(fz) = \varphi(gfz, f^2z) < \varphi(g^2z, gfz) \leq \varphi(gz, fz) = \psi(z),$$

which is a contradiction with the fact that ψ attains its minimum at z. Hence, our claim holds. Further, we have

$$\psi(gz) = \varphi(g^2z, fgz) = \varphi(g^2z, gfz) = 0,$$

thus $\psi(gz) = 0$, which indicate that the minimum value 0 is reached by ψ at gz. However, we also know that this minimum value must be attained at z as well. Therefore, $\psi(z) = 0$ which leads to the conclusion that $fz = gz$. Let $y = fz = gz$ and assume that $y \neq gy$, then

$$\varphi(y, gy) = \varphi(fz, gfz) = \varphi(fz, fgz) < \varphi(gz, g^2z) = \varphi(y, gy),$$

which absurd, so $y = gy = gfz = fgz = fy$, that is, z is a common fixed of f and g. The uniqueness of the common fixed point is obvious. □

Theorem 4.4. *Let X be a compact or sequentially compact topological space and $f, g\colon X \to X$ be continuous and commuting mappings. Assume that f is g-contractive with φ vanishes only on the diagonal. Then f and g have a unique common fixed point.*

Proof. Define $\psi\colon X \to \mathbb{R}_+$ by $\psi(x) = \varphi(gx, fx)$. Since f, g and φ are continuous on $g(X)$, then so is ψ. Now, by compactness of X, we deduce by the extreme value theorem that ψ attains its minimum at some point $z \in X$. We claim that $fgz = f^2z$. Assume that $fgz \neq f^2z$, then by the commutativity, the contractive condition and the nonexpansiveness assumptions, we obtain

$$\psi(fz) = \varphi(gfz, f^2z) = \varphi(fgz, f^2z) < \varphi(g^2z, gfz) \leq \varphi(gz, fz) = \psi(z),$$

which contradict the fact that ψ attains its minimum at z. Thus our claim holds. Further, we have

$$\psi(f^2z) = \varphi(gf^2z, f^3z) = \varphi(f^3z, f^3z) = 0,$$

thus $\psi(f^2z) = 0 = \psi(z)$ as above, which implies that $fgz = f^2z$. Let $y = fgz = f^2z$ and assume that $y \neq gy$, then

$$\varphi(y, gy) = \varphi(f^2z, gf^2z) = \varphi(f^2z, f^2gz) < \varphi(gfz, gfgz) = \varphi(y, gy),$$

which absurd, so $y = gy = gfz = fgz = fy$, that is, z is a common fixed of f and g. The uniqueness of the common fixed point is obvious. □

In the following result, we use Remark 3.3.

Theorem 4.5. *Let X be a compact or sequentially compact topological space and $f, g\colon X \to X$ be two commuting mappings such that $f(X) \subseteq g(X)$ and f is g-contractive with φ vanishes only on the diagonal. Assume that fg^{-1} is continuous on $g(X)$. Then f and g have a unique common fixed point.*

Proof. Define $\psi\colon g(X) \to \mathbb{R}_+$ by $\psi(x) = \varphi(fg^{-1}x, x)$. Since fg^{-1} and φ are continuous on $g(X)$, then so is ψ. Now, by compactness of $g(X)$, we deduce by the extreme value theorem that ψ attains its minimum at some point $z \in g(X)$. We claim that $fg^{-1}fg^{-1}z = fg^{-1}z$. Assume that $fg^{-1}fg^{-1}z \neq fg^{-1}z$, then by the contractive condition, we obtain

$$\psi(fg^{-1}z) = \varphi(fg^{-1}fg^{-1}z, fg^{-1}z) < \varphi(fg^{-1}z, z) = \psi(z),$$

which contradict the fact that ψ attains its minimum at z, and our claim holds. Next, by using the commutativity of f and g it follows that

$$f^2g^{-1}z = fgg^{-1}fg^{-1}z = gfg^{-1}fg^{-1}z = gfg^{-1}z.$$

Let $y = fg^{-1}fg^{-1}z = fg^{-1}z$ and assume that $y \neq gy$, then

$$\varphi(y, gy) = \varphi(fg^{-1}fg^{-1}z, gfg^{-1}fg^{-1}z) < \varphi(fg^{-1}z, gfg^{-1}z) = \varphi(y, gy),$$

which absurd, so $y = gy = gfg^{-1}fg^{-1}z = f^2g^{-1}z = fy$, so y is a common fixed of f and g. The uniqueness of the common fixed point is obvious. □

5. Common fixed point in metric spaces

The following corollary is an immediate consequence of Theorem 3.4.

Corollary 5.1. *Let (X, φ) be a metric space and $f, g\colon X \to X$ be commuting mappings such that $f(X) \subseteq g(X)$ and f is g-contractive. Assume that fg^{-1} is continuous on $g(X)$. If there exists $x \in X$ such that the sequence $\{(fg^{-1})^n gx\}$ has a convergent subsequence to some $y \in X$, then y is the unique common fixed point of f and g.*

Corollary 5.2. *Let X be a compact metric space and $f, g\colon X \to X$ be two commuting mappings such that $f(X) \subseteq g(X)$. Assume that f is continuous and contractive (or id-contractive) on $f(X)$. Then f and g have a unique common fixed point y. Moreover, for all $x \in X$ the sequence $\{f^n x\}$ converges to y.*

Proof. The existence of the unique common fixed point y follows from Theorem 4.1. It remains to show the convergence of the sequence $\{f^n x\}$ to this y. Now, let $x \in X$ and consider the sequence $\{f^n x\}$. If $f^n x \neq y$ (or $f^{n+1}x \neq y$),

$$\varphi(f^{n+1}x, y) = \varphi(ff^n x, fy) < \varphi(f^n x, y), \tag{5.1}$$

thus, $\{\varphi(f^n x, y)\}$ is strictly decreasing and converges to some $\alpha \geq 0$. Next, by compactness of X, there is a convergent subsequence $\{f^{n(k)}x\}$ to some $q \in X$, so

$$\begin{aligned} \alpha = \varphi(q, y) &= \lim_{k\to\infty} \varphi(f^{n(k)}x, y) \\ &= \lim_{k\to\infty} \varphi(f^{n(k)+1}x, y) && \text{by (5.1)} \\ &= \varphi(fq, y). \end{aligned}$$

If $fq \neq y$ (or $q \neq y$) then

$$\varphi(fq, y) = \varphi(fq, fy) < \varphi(q, y).$$

Therefore, any subsequence of the sequence $\{f^n gx\}$ converges to y, so the sequence itself also converges to y. □

Corollary 5.3. *Let X be a compact metric space and $f, g\colon X \to X$ be two commuting and surjective mappings such that g is continuous. Assume that g is non-contractive (or the id is g-contractive) on $g(X)$. Then f and g have a unique common fixed point z. Moreover, for all $x \in X$ the sequence $\{g^n x\}$ converges to z.*

Proof. The existence of the unique common fixed point z follows from Theorem 4.2, where the function $\psi(x) = \varphi(gx, x)$ attains its maximum at z. It remains to show the convergence of the sequence $\{g^n x\}$ to z. Now, let $x \in X$ and consider the sequence $\{g^n x\}$. If $g^{n+1}x \neq gz$ (or $g^n x \neq gz$),

$$\varphi(g^n x, z) < \varphi(gg^n x, gz) = \varphi(g^{n+1}x, z), \tag{5.2}$$

thus, $\{\varphi(g^n x, z)\}$ is strictly increasing, so converges to some $\alpha \geq 0$. Next, by compactness of X, there is a convergent subsequence $\{f^{n(k)}x\}$ to some $q \in X$, so

$$\alpha = \varphi(q, y) = \lim_{k\to\infty} \varphi(f^{n(k)}x, z)$$

$$= \lim_{k\to\infty} \varphi(f^{n(k)+1}x, z) \qquad \text{by (5.2)}$$
$$= \varphi(fq, z).$$

If $q \neq z$ (or $fq \neq z$) then

$$\varphi(q, z) < \varphi(fq, fz) = \varphi(fq, z).$$

Therefore, any subsequence of $\{f^n gx\}$ converges to z, so the sequence itself also converges to z. □

Since the continuity of a bijective mapping in a compact space implies the continuity of its inverse [23, Theorem 26.6], we get the following result.

Corollary 5.4. *Let (X, φ) be a compact metric space and $f, g\colon X \to X$ be commuting mappings such that g is continuous and bijective on X. Assume that f is g-contractive. Then f and g have a unique common fixed point. Moreover, for all $x \in X$ the sequence $\{(fg^{-1})^n gx\}$ converges to this common fixed point.*

Proof. Firstly, observe that the continuity of f follows from continuity of g and the contractive condition. The continuity of g^{-1} follows from the compactness of X and the continuity of g. This proves the continuty of fg^{-1} on X. By Theorem 4.5, we deduce that f and g have a unique common fixed point y. Now, let $x \in X$ and consider the sequence $\{(fg^{-1})^n gx\}$. If $(fg^{-1})^{n+1}gx \neq y$,

$$\varphi((fg^{-1})^{n+1}gx, y) = \varphi(fg^{-1}(fg^{-1})^n gx, fg^{-1}y) < \varphi((fg^{-1})^n gx, y), \tag{5.3}$$

thus, $\{\varphi((fg^{-1})^n gx, y)\}$ is strictly decreasing and converges to some $\alpha \geq 0$. Next, since X is compact, there is a convergent subsequence $\{(fg^{-1})^{n(k)}gx\}$ to some $q \in X$, so

$$\alpha = \varphi(q, y) = \lim_{k\to\infty} \varphi((fg^{-1})^{n(k)}gx, y)$$
$$= \lim_{k\to\infty} \varphi((fg^{-1})^{n(k)+1}gx, y) \qquad \text{by (5.3)}$$
$$= \varphi(fg^{-1}q, y).$$

If $fg^{-1}q \neq y$ then

$$\varphi(fg^{-1}q, y) = \varphi(fg^{-1}q, fg^{-1}y) < \varphi(q, y).$$

Therefore, any subsequence of the sequence $\{(fg^{-1})^n gx\}$ converges to y, so the sequence itself also converges to y. □

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M. Berzig
Université de Tunis, École Nationale Supérieure d'Ingénieurs de Tunis,
Département de Mathématiques, 5 avenue Taha Hussein, 1008 Montfleury, Tunisie
E-mail address: maher.berzig@gmail.com