



Volume 10, Number 5, 2024, 155 144

WEAK EXTENSION OF THE HAM-SANDWICH THEOREM: SOME RATIO NOT IN HALF

HIDEFUMI KAWASAKI

ABSTRACT. The ham-sandwich theorem is famous as an application of Borsuk's antipodal theorem. It states that given n measurable sets A_1, \ldots, A_n with positive Lebesgue measure in \mathbb{R}^n , it is possible to divide each one of them in half by a hyperplane. Recently the author applied Borsuk's antipodal theorem to optimization theory in [3]. In this paper we extend the ham-sandwich theorem to some ratio not in half by using the technique of [3].

1. INTRODUCTION

Borsuk's antipodal theorem [1] is an important theorem of algebraic topology. It states that for any continuous mapping φ from the *n*-sphere S^n to the Euclidean space \mathbb{R}^n , there exists a point $u \in S^n$ such that $\varphi(u) = \varphi(-u)$. As for its applications, the ham-sandwich theorem, the necklace problem, and coloring of Kneser graph by Lovász [5] are well-known, see e.g. Matoušek [6].

The author [2, 3, 4] applied Borsuk's antipodal theorem to an *n*-tuple of parametric optimization problems with parameter $u \in S^n$, and presented antipodal theorems for them.

Now we explain our notations. For any $\boldsymbol{u} = (u_1, \ldots, u_{n+1}) \in S^n$, we set $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\boldsymbol{u} = (u, u_{n+1})$. Then u is an element of the *n*-disk D^n . We assign to $\boldsymbol{u} \in S^n$ a hyperplane $H_{\boldsymbol{u}} = \{x \in \mathbb{R}^n \mid \langle u, x \rangle = u_{n+1}\}$ and two closed half-spaces:

$$H_{\boldsymbol{u}}^+ = \{ x \in \mathbb{R}^n \mid \langle u, x \rangle \ge u_{n+1} \}, \ H_{\boldsymbol{u}}^- = \{ x \in \mathbb{R}^n \mid \langle u, x \rangle \le u_{n+1} \},$$

where $\langle u, x \rangle$ denotes the inner product $u_1 x_1 + \cdots + u_n x_n$.

Next, let us review the ham-sandwich theorem. Let $A_i \subset \mathbb{R}^n$ (i = 1, ..., n) be measurable sets with positive Lebesgue measure. Define $\varphi_i(\boldsymbol{u}) = \mu(A_i \cap H_{\boldsymbol{u}}^+)$. Then $\varphi = (\varphi_1, \ldots, \varphi_n)$ is a continuous mapping from S^n to \mathbb{R}^n . By applying Borsuk's antipodal theorem to φ , there exists $\boldsymbol{u} \in S^n$ such that $\mu(A_i \cap H_{\boldsymbol{u}}^+) = \mu(A_i \cap H_{-\boldsymbol{u}}^+)$. Since $H_{-\boldsymbol{u}}^+ = H_{\boldsymbol{u}}^-$, we have

(1.1)
$$\mu(A_i \cap H_{\boldsymbol{u}}^+) = \mu(A_i \cap H_{\boldsymbol{u}}^-) \quad (i = 1, \dots, n),$$

which is the ham-sandwich theorem. If we denote the ratio of the division by

²⁰²⁰ Mathematics Subject Classification. 90C31, 54C05.

Key words and phrases. Borsuk's antipodal theorem, ham-sandwich theorem.

This research is supported by JSPS KAKENHI Grant Number 20K03751 and the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.



FIGURE 1. Place A_1, \ldots, A_n on the hyperplane $u_{n+1} = -1$ and divide them in half by a hyperplane passing through the origin of \mathbb{R}^{n+1} whose normal vector is $\boldsymbol{u} = (u, u_{n+1}) \in S^n$. Then the intersection of two hyperplanes is $\{(x, -1) \mid \langle u, x \rangle = u_{n+1}\}$.

(1.2)
$$\rho_i(\boldsymbol{u}) := \frac{\mu(A_i \cap H_{\boldsymbol{u}}^+)}{\mu(A_i)},$$

then (1.1) implies $\rho_i(\boldsymbol{u}) = 1/2$. In this paper we show that there exists $\boldsymbol{u} \in S^n$ such that

$$\frac{1}{2} < \rho_1(\boldsymbol{u}) = \cdots = \rho_n(\boldsymbol{u}) < 1.$$

2. Weak extension of the ham-sandwich theorem

In this section, we will extend the ham-sandwich theorem. We start with a general setting. Let $f_i(x, u)$ (i = 1, ..., n) be real-valued continuous functions defined on $\mathbb{R}^n \times D^n$. We assume that $f_i(x, u) = f_i(x, -u)$ for any $(x, u) \in \mathbb{R}^n \times D^n$. For any $\gamma_i \in \mathbb{R}$ and $u = (u, u_{n+1}) \in S^n$, we define

(2.1)
$$\nu_i(\boldsymbol{u}) := \int_{A_i \cap H_{\boldsymbol{u}}^+} f_i(x, \boldsymbol{u}) dx - \gamma_i u_{n+1}.$$

Then $\nu := (\nu_1, \ldots, \nu_n) : S^n \to \mathbb{R}^n$ is continuous. Taking $f_i(x, u) = 1$ and $\gamma_i = 0$ in (2.1), we see that $\nu_i(u) = \mu(A_i \cap H_u^+)$. The term $-\gamma_i u_{n+1}$ is important in this paper. It comes from [3], where we used

$$\varphi_i(\boldsymbol{u}) := \max_{x \in A_i} f_i(x, u) - \gamma_i u_{n+1}.$$

By applying Borsuk's antipodal theorem to ν , we obtain the following.

Theorem 2.1. (a) For any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, there exists $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that

(2.2)
$$\int_{A_i \cap H_{\boldsymbol{u}}^+} f_i(x, u) dx - \int_{A_i \cap H_{\boldsymbol{u}}^-} f_i(x, u) dx = 2\gamma_i u_{n+1} \ (i = 1, \dots, n).$$

- (b) There exists $\mathbf{u} \in S^n$ such that $\int_{A_i \cap H_{\mathbf{u}}^+} f_i(x, u) dx = \int_{A_i \cap H_{\mathbf{u}}^-} f_i(x, u) dx$ for any $i = 1, \dots, n$.
- (c) If there is no $\boldsymbol{v} = (v, 0) \in S^n$ such that

(2.3)
$$\int_{A_i \cap H_{\boldsymbol{v}}^+} f_i(x, v) dx = \int_{A_i \cap H_{\boldsymbol{v}}^-} f_i(x, v) dx \ (i = 1, \dots, n).$$

then for any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, there exists $u \in S^n$ such that $u_{n+1} > 0$ and

(2.4)
$$\int_{A_i \cap H_u^+} f_i(x, u) dx - \int_{A_i \cap H_u^-} f_i(x, u) dx \ (i = 1, \dots, n)$$

are proportionate to γ_i (i = 1, ..., n).

Proof. (a) By Borsuk's antipodal theorem, there exists $\boldsymbol{u} \in S^n$ such that $\nu(\boldsymbol{u}) = \nu(-\boldsymbol{u})$. Hence

$$\int_{A_{i}\cap H_{u}^{+}} f_{i}(x,u)dx - \gamma_{i}u_{n+1} = \int_{A_{i}\cap H_{-u}^{+}} f_{i}(x,-u)dx + \gamma_{i}u_{n+1}$$
$$= \int_{A_{i}\cap H_{u}^{-}} f_{i}(x,u)dx + \gamma_{i}u_{n+1}.$$

(b) is a direct consequence of (a) for $\gamma_i = 0$.

By the assumption of (c), we have $u_{n+1} \neq 0$. Therefore (c) follows from (a). When u_{n+1} is negative, it suffices to take $-\boldsymbol{u}$ instead of \boldsymbol{u} .

By taking $f_i(x, u) = 1/\mu(A_i)$ in Theorem 2.1, we obtain the following. (b) is nothing but the ham-sandwich theorem. (a) and (c) are new results.

Theorem 2.2. (a) For any $\gamma_1, \ldots, \gamma_n \in \mathbb{R}$, there exists $\boldsymbol{u} = (u, u_{n+1}) \in S^n$ such that

(2.5)
$$\rho_i(\boldsymbol{u}) - \frac{1}{2} = \gamma_i u_{n+1}.$$

In particular when $\gamma_i \neq -1/2$, it holds that $-1 < u_{n+1} < 1$.

- (b) There exists a hyperplane that divides each A_i in half.
- (c) If there is no hyperplane passing the origin of \mathbb{R}^n that divides each A_i in half, then for any $\gamma_i \neq -1/2$, there exists $\mathbf{u} \in S^n$ such that $0 < u_{n+1} < 1$ and

(2.6)
$$\rho_1(\boldsymbol{u}) - \frac{1}{2} : \dots : \rho_n(\boldsymbol{u}) - \frac{1}{2} = \gamma_1 : \dots : \gamma_n$$

In particular, for any $\gamma_1 = \cdots = \gamma_n = c$ with $0 < c \le 1/2$, there exists $u \in S^n$ such that $0 < u_{n+1} < 1$ and

(2.7)
$$\frac{1}{2} < \rho_1(\boldsymbol{u}) = \dots = \rho_n(\boldsymbol{u}) < \frac{1}{2} + c \ (\leq 1).$$

Proof. When $f_i(x, u) = 1/\mu(A_i)$, (2.2) reduces to

$$2\gamma_i u_{n+1} = \frac{\mu(A_i \cap H_{\boldsymbol{u}}^+)}{\mu(A_i)} - \frac{\mu(A_i \cap H_{\boldsymbol{u}}^-)}{\mu(A_i)} = 2\rho_i(\boldsymbol{u}) - 1,$$

HIDEFUMI KAWASAKI

which implies (2.5). If $u_{n+1} = 1$, then since $A_i \cap H_{\boldsymbol{u}}^+$ is empty, we have $\rho_i(\boldsymbol{u}) = 0$. Hence, we see from (2.5) that $\gamma_i = -1/2$, which contradicts the assumption of (a). If $u_{n+1} = -1$, then since $A_i \cap H_{\boldsymbol{u}}^+ = A_i$, we have $\rho_i(\boldsymbol{u}) = 1$. Hence, we see from (2.5) that $\gamma_i = -1/2$, which contradicts the assumption of (a). Therefore we have $-1 < u_{n+1} < 1$.

(b) follows from (a) by taking $\gamma_i = 0$. (c) If $u_{n+1} = 0$ in (2.5), then hyperplane $H_{\boldsymbol{u}}$ passes through the origin, and $\rho_1(\boldsymbol{u}) = \cdots = \rho_n(\boldsymbol{u}) = 1/2$, which contradicts the assumption of (c). Therefore $u_{n+1} \neq 0$. So (2.5) implies (2.6). When u_{n+1} is negative, it suffices to take $-\boldsymbol{u}$ instead of \boldsymbol{u} . In particular, if we take $\gamma_1 = \cdots = \gamma_n = c$ with $0 < c \leq 1/2$, we get (2.7) from $0 < u_{n+1} < 1$.

Remark 2.3. We see from (2.5) that

$$|\gamma_i u_{n+1}| = \left| \rho_i(\boldsymbol{u}) - \frac{1}{2} \right| \le \frac{1}{2}$$

for any *i*. Hence, if some $|\gamma_j|$ is exceptionally large, $|u_{n+1}|$ becomes small. So $\rho_i(\boldsymbol{u})$ $(i \neq j)$ are approximately equal to 1/2.

When we take $\gamma_1 = \cdots = \gamma_n = 1/2$, we see from (2.5) that

$$\rho_i(\boldsymbol{u}) = \frac{u_{n+1} + 1}{2}$$

Since u_{n+1} is unknown, $\rho_1(u) = \cdots = \rho_n(u)$ are also unknown. This is the reason why the title is considered "weak extension".

Example 2.4. This example indicates the possibility of extending the ham-sandwich theorem. Given ratio $0 \le \rho \le 1$, we compute $\ell : \langle u, x \rangle = u_3$ that divides two triangles with $\rho_1(u) = \rho_2(u) = \rho$. Line ℓ passing through (-2, a) and (2, b) is



FIGURE 2. $2\mu(S_1) = 2\mu(S_2) = \rho$.

 $(a-b)x_1 + 4x_2 = 2(a+b)$. Let P and Q be the intersections of ℓ and hypotenuses $x_2 = x_1 + 2$ and $x_2 = x_1 - 1$, respectively. Then their x_1 -coordinates are

$$x_1 = \frac{4a}{a-b+4} - 2, \ \frac{4-4b}{a-b+4},$$

respectively. Hence the areas of triangle S_1 and S_2 are given by

$$\mu(S_1) = \frac{1}{2} - T_1 = \frac{1}{2} \left(1 - \frac{4a^2}{a - b + 4} \right), \quad \mu(S_2) = \frac{1}{2} (1 - b) \frac{4 - 4b}{a - b + 4},$$

respectively. Therefore $\mu(S_1) = \mu(S_2)$ if and only if $4a^2 + 4b^2 - a - 7b = 0$, which is equivalent to

$$\left(a - \frac{1}{8}\right)^2 + \left(b - \frac{7}{8}\right)^2 = \frac{25}{32}.$$

Hence, we may put

$$a - \frac{1}{8} = \frac{5}{4\sqrt{2}}\cos\theta, \quad b - \frac{7}{8} = \frac{5}{4\sqrt{2}}\sin\theta.$$

Therefore, the equation of ℓ is as follows.

$$\left\{\frac{5}{4\sqrt{2}}(\cos\theta - \sin\theta) - \frac{3}{4}\right\}x_1 + 4x_2 = \frac{5}{2\sqrt{2}}(\sin\theta + \cos\theta) + 2.$$

Normalizing the coefficients, we obtain $u \in S^2$. Since the area of the triangles is 1/2, the ratio of the division is

$$\rho_1(\boldsymbol{u}) = 2S_1 = 1 - \frac{\left(\frac{5}{4\sqrt{2}}\cos\theta + \frac{1}{8}\right)^2}{\frac{5}{4\sqrt{2}}\left(\cos\theta - \sin\theta\right) + \frac{3}{4}} = 1 - \frac{1}{8} \cdot \frac{(5\sqrt{2}\cos\theta + 1)^2}{5\sqrt{2}(\cos\theta - \sin\theta) + 6}$$

3. Concluding Remarks

In the ham-sandwich theorem, we placed A_1, \ldots, A_n on the hyperplane $u_{n+1} = -1$ and divided them by a hyperplane passing through the origin of \mathbb{R}^{n+1} whose normal vector is $\mathbf{u} = (u, u_{n+1}) \in S^n$ (Figure 3 Left). On the other hand, in Theorem 2.2, we place A_i on the hyperplane $u_{n+1} = -\gamma_i - 1$ (Figure 3 Right). This is why Theorem 2.2 is richer than the ham-sandwich theorem.



FIGURE 3. Left: $\gamma_1 = \gamma_2 = 0$. Right: $\gamma_1 < \gamma_2$.

4. Acknowledgements

The author would like to thank the referee for careful reading.

HIDEFUMI KAWASAKI

References

- K. Borsuk, Drei Sätze üver die n-dimensionale euklidische Sphäre, Fundamenta Mathematicae 20 (1933), 177–190.
- [2] H. Kawasaki, An application of Borsuk-Ulam's theorem to parametric optimization, Linear and Nonlinear Analysis 9 (2023) 245–252.
- [3] H. Kawasaki, An antipodal theorem for parametric optimization problems, Bulletin of informatics and cybernetics 56 (2024) 1-8 (https://doi.org/10.5109/7178785).
- [4] H. Kawasaki, An application of Borsuk-Ulam's theorem to nonlinear programming, J. Operations Research Society of Japan 68 (2025) 21-29 (https://doi.org/10.15807/jorsj.68.21).
- [5] L. Lovász, Kneser's conjecture, chromatic number and homotopy, Journal of Combinatorial Theory, Ser. A 25 (1978), 319–324.
- [6] J. Matoušek, Using the Borsuk-Ulam Theorem, Springer, Berlin Heidelberg, 2008.

Manuscript received 28 December 2024 revised 18 March 2025

H. KAWASAKI

Motooka 744, Nishi-ku, Fukuoka 819-0395, Japan *E-mail address:* kawasaki.hidefumi.245@m.kyushu-u.ac.jp