Linear and Nonlinear Analysis
Yokohama Publishers
ISSN 2188-8167
Copyright 2025
Volume 11, Number 3, 2025, 135–144

SET-VALUED OPTIMIZATION BY USING TOTAL ORDER RELATION BETWEEN VECTORS

MASAMICHI KON

Abstract. In the present paper, we define a pseudo-order relation between sets by using a total order relation between vectors, and consider a set-valued optimization problem with respect to the pseudo-order. Two types of vector-valued optimization problems are derived from the set-valued problem by using two types of vectorization of sets. Then, we investigate relationships between optimal solutions of the vector-valued problems and non-dominated solutions of the set-valued problem.

1. Preliminaries

For $a, b \in \mathbb{R}$, we set $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ and $]a, b[= \{x \in \mathbb{R} : a < x < b\}$. Throughout the paper, let $K \subset \mathbb{R}^n$ be a total ordering cone. That is, we define $\leq_K$ as

$$\boldsymbol{x} \leq_K \boldsymbol{y} \overset{\text{def}}{\Leftrightarrow} \boldsymbol{y} - \boldsymbol{x} \in K$$

for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then $\leq_K$ is a total order on $\mathbb{R}^n$. We write also $\boldsymbol{y} \geq_K \boldsymbol{x}$ when $\boldsymbol{x} \leq_K \boldsymbol{y}$. We define

$$\boldsymbol{y} \leq_{-K} \boldsymbol{x} \overset{\text{def}}{\Leftrightarrow} \boldsymbol{x} - \boldsymbol{y} \in -K$$

for each $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, where $-K = \{-\boldsymbol{z} : \boldsymbol{z} \in K\}$. Then, it follows that

$$\boldsymbol{x} \leq_K \boldsymbol{y} \Leftrightarrow \boldsymbol{y} - \boldsymbol{x} \in K \Leftrightarrow \boldsymbol{x} - \boldsymbol{y} \in -K \Leftrightarrow \boldsymbol{y} \leq_{-K} \boldsymbol{x}. \tag{1.1}$$

It is known that K can be represented as

$$K = \left[\bigcup_{i=1}^{n} \{\boldsymbol{r} \in \mathbb{R}^n : \langle \boldsymbol{r}_j, \boldsymbol{r} \rangle = 0 (j < i), \langle \boldsymbol{r}_i, \boldsymbol{r} \rangle > 0\} \right] \cup \{\boldsymbol{0}\} \tag{1.2}$$

for some ordered orthogonal base $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n\} \subset \mathbb{R}^n$; [2], where $\langle \cdot, \cdot \rangle$ is a canonical inner product on $\mathbb{R}^n$. The ordered base $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n\}$ is distinguished from its permutations $\{\boldsymbol{r}_i, \boldsymbol{r}_j, \ldots, \boldsymbol{r}_k\}$ with $(i, j, \ldots, k) \neq (1, 2, \ldots, n)$.

Now, we recall the definition of lexicographic order.

Definition 1.1. The order relation $\leq_{lex}$ on $\mathbb{R}^n$ defined as

$$\boldsymbol{x} \leq_{lex} \boldsymbol{y} \overset{\text{def}}{\Leftrightarrow} \boldsymbol{x} = \boldsymbol{y} \text{ or } x_i < y_i \text{ for the first different } i\text{th coordinate of } \boldsymbol{x} \text{ and } \boldsymbol{y}$$

2020 *Mathematics Subject Classification.* 90C70.

Key words and phrases. Set-valued optimization, pseudo-order, non-dominated solution, vectorization, admissible order.

for each $\boldsymbol{x} = (x_1, x_2, \ldots, x_n), \boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ is called lexicographic order on $\mathbb{R}^n$.

The lexicographic order is a total order. If we choose $\boldsymbol{r}_1 = (1, 0, \ldots, 0)$, $\boldsymbol{r}_2 = (0, 1, 0, \ldots, 0), \ldots, \boldsymbol{r}_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$ in (1.2), then the total ordering cone K defines the lexicographic order.

The total order $\leq_K$ is the lexicographic order on $\mathbb{R}^n$ with the orthogonal base $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_n\}$; [2, Theorem 3]. That is, for $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, let

$$\boldsymbol{x} = a_1\boldsymbol{r}_1 + a_2\boldsymbol{r}_2 + \cdots + a_n\boldsymbol{r}_n, \quad \boldsymbol{y} = b_1\boldsymbol{r}_1 + b_2\boldsymbol{r}_2 + \cdots + b_n\boldsymbol{r}_n,$$

and let

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_n), \quad \boldsymbol{b} = (b_1, b_2, \ldots, b_n),$$

then we have

$$\boldsymbol{x} \leq_K \boldsymbol{y} \Leftrightarrow \boldsymbol{a} \leq_{lex} \boldsymbol{b}. \tag{1.3}$$

Definition 1.2. Let $S \subset \mathbb{R}^n$, and let $\boldsymbol{x} \in S$.

(i) The point $\boldsymbol{x}$ is said to be the *minimum point* of S with respect to $\leq_K$ if $\boldsymbol{x} \leq_K \boldsymbol{y}$ for any $\boldsymbol{y} \in S$. The minimum point of S with respect to $\leq_K$ is denoted by $\min S$ or $\min(S; K)$ if exists. Similarly, the minimum point of S with respect to $\leq_{-K}$ is defined. Whenever we omit K, $\min S = \min(S; K)$.

(ii) The point $\boldsymbol{x}$ is said to be the *maximum point* of S with respect to $\leq_K$ if $\boldsymbol{y} \leq_K \boldsymbol{x}$ for any $\boldsymbol{y} \in S$. The maximum point of S with respect to $\leq_K$ is denoted by $\max S$ or $\max(S; K)$ if exists. Similarly, the maximum point of S with respect to $\leq_{-K}$ is defined. Whenever we omit K, $\max S = \max(S; K)$.

Let $S \subset \mathbb{R}^n$. Then, $\min(S; K)$ exists if and only if $\max(S; -K)$ exists, and $\max(S; K)$ exists if and only if $\min(S; -K)$ exists. From (1.1), if $\min(S; K)$ and $\max(S; K)$ exist, then if follows that

$$\min(S; K) = \max(S; -K), \quad \max(S; K) = \min(S; -K). \tag{1.4}$$

2. Orderings and vectorization of sets

Let $\mathcal{C}(\mathbb{R}^n)$ be the set of all nonempty compact subsets of $\mathbb{R}^n$. For $A, B \in \mathcal{C}(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$, we define

$$A + B = \{\boldsymbol{x} + \boldsymbol{y} : \boldsymbol{x} \in A, \boldsymbol{y} \in B\}, \quad \lambda A = \{\lambda \boldsymbol{x} : \boldsymbol{x} \in A\}.$$

We define pseudo-orders (reflexive and transitive) $\preceq_K^\ell, \preceq_K^u, \preceq_K$ on $\mathcal{C}(\mathbb{R}^n)$ as

$$A \preceq_K^\ell B \overset{\text{def}}{\Leftrightarrow} B \subset A + K,$$

$$A \preceq_K^u B \overset{\text{def}}{\Leftrightarrow} A \subset B - K,$$

$$A \preceq_K B \overset{\text{def}}{\Leftrightarrow} B \subset A + K, A \subset B - K \Leftrightarrow A \preceq_K^\ell B, A \preceq_K^u B$$

for each $A, B \in \mathcal{C}(\mathbb{R}^n)$. Similarly, $\preceq_{-K}^\ell, \preceq_{-K}^u$, and $\preceq_{-K}$ are defined. Then, it follows that

$$A \preceq_K^u B \Leftrightarrow A \subset B - K \Leftrightarrow B \preceq_{-K}^\ell A \tag{2.1}$$

for $A, B \in \mathcal{C}(\mathbb{R}^n)$. It also follows that

$$A \preceq^{\ell}_K B \quad \text{or} \quad B \preceq^{\ell}_K A,$$

$$A \preceq^{u}_K B \quad \text{or} \quad B \preceq^{u}_K A,$$

and

$$A \preceq^{\ell}_K B \Leftrightarrow \min(A;K) \leq_K \min(B;K) \tag{2.2}$$

for $A, B \in \mathcal{C}(\mathbb{R}^n)$; [3, Lemma 4.6 and Corollary 4.8]. (2.2) in which K is replace by $-K$ holds. For $A, B \in \mathcal{C}(\mathbb{R}^n)$, since

$$\begin{aligned} A \preceq^{u}_K B &\Leftrightarrow B \preceq^{\ell}_{-K} A \quad \text{(from (2.1))} \\ &\Leftrightarrow \min(B;-K) \leq_{-K} \min(A;-K) \quad \text{(from (2.2))} \\ &\Leftrightarrow \max(B;K) \leq_{-K} \max(A;K) \quad \text{(from (1.4))} \\ &\Leftrightarrow \max(A;K) \leq_K \max(B;K) \quad \text{(from (1.1))}, \end{aligned}$$

we have

$$A \preceq_K B \Leftrightarrow \min(A;K) \leq_K \min(B;K), \max(A;K) \leq_K \max(B;K). \tag{2.3}$$

In the present paper, we investigate a minimization problem with respect to $\preceq_K$. In [3], a minimization problem with respect to $\preceq^{\ell}_K$ is considered.

Definition 2.1. Let $\mathcal{S} \subset \mathcal{C}(\mathbb{R}^n)$, and let $A \in \mathcal{S}$. Then, A is said to be a *non-dominated element* of $\mathcal{S}$ if $B \in \mathcal{S}$ and $B \preceq_K A$ imply $A \preceq_K B$.

We define two types of *vectorization* $V_2(A)$ and $V_\lambda(A)$ for each $A \in \mathcal{C}(\mathbb{R}^n)$ as

$$V_2(A) = (\min A, \max A) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{2.4}$$

$$V_\lambda(A) = \lambda \min A + (1-\lambda) \max A \in \mathbb{R}^n \tag{2.5}$$

where $\lambda \in [0,1]$. Throughout the rest of the paper, we set

$$C = K \times K.$$

Then, C is a partial ordering cone in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. That is, we define $\leq_C$ as

$$\begin{aligned} (\boldsymbol{x}, \boldsymbol{y}) \leq_C (\boldsymbol{z}, \boldsymbol{w}) &\overset{\text{def}}{\Leftrightarrow} (\boldsymbol{z}, \boldsymbol{w}) - (\boldsymbol{x}, \boldsymbol{y}) \in C \\ &\Leftrightarrow (\boldsymbol{z}-\boldsymbol{x}, \boldsymbol{w}-\boldsymbol{y}) \in K \times K \\ &\Leftrightarrow \boldsymbol{z}-\boldsymbol{x} \in K, \boldsymbol{w}-\boldsymbol{y} \in K \\ &\Leftrightarrow \boldsymbol{x} \leq_K \boldsymbol{z}, \boldsymbol{y} \leq_K \boldsymbol{w} \end{aligned} \tag{2.6}$$

for each $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{R}^n \times \mathbb{R}^n$, then $\leq_C$ is a partial order on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$.

Definition 2.2. A total order $\leq_{adm}$ on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ defined by a total ordering cone is said to be an *admissible order* if $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{R}^n \times \mathbb{R}^n$ and $(\boldsymbol{x}, \boldsymbol{y}) \leq_C (\boldsymbol{z}, \boldsymbol{w})$ imply $(\boldsymbol{x}, \boldsymbol{y}) \leq_{adm} (\boldsymbol{z}, \boldsymbol{w})$.

Definition 2.2 is a natural extension of an admissible order for intervals or on $\mathbb{R}^2$ in [1, 4, 5]. Let $\overline{K} \subset \mathbb{R}^{2n}$ be a total ordering cone, and we define a total order $\leq_{\overline{K}}$ on $\mathbb{R}^{2n}$ as

$$(\boldsymbol{x}, \boldsymbol{y}) \leq_{\overline{K}} (\boldsymbol{z}, \boldsymbol{w}) \overset{\text{def}}{\Leftrightarrow} (\boldsymbol{z}, \boldsymbol{w}) - (\boldsymbol{x}, \boldsymbol{y}) \in \overline{K}$$

for each $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{z}, \boldsymbol{w}) \in \mathbb{R}^n \times \mathbb{R}^n$. Then, $\leq_{\overline{K}}$ is an admissible order if and only if $C \subset \overline{K}$. See section 4 for construction of $\overline{K}$ such that $\leq_{\overline{K}}$ is an admissible order. Let $A, B \in \mathcal{C}(\mathbb{R}^n)$, and let $\lambda \in [0, 1]$. Since

$$\begin{aligned} A \preceq_K B \quad &\Leftrightarrow \quad \min(A; K) \leq_K \min(B; K), \max(A; K) \leq_K \max(B; K) \\ &\Rightarrow \quad \lambda \min(A; K) + (1 - \lambda) \max(A; K) \\ &\qquad \leq_K \lambda \min(B; K) + (1 - \lambda) \max(B; K) \end{aligned}$$

from (2.3), we have

$$A \preceq_K B \Leftrightarrow V_2(A) \leq_C V_2(B) \Rightarrow V_\lambda(A) \leq_K V_\lambda(B) \tag{2.7}$$

from (2.4), (2.5), and (2.6).

We present an example such that $V_\lambda(A) \leq_K V_\lambda(B)$ but not $V_2(A) \leq_C V_2(B)$ in (2.7).

Example 2.3. In $\mathbb{R}^2$, let $\leq_K$ be the lexicographic order $\leq_{lex}$. We set $A = \{0\} \times [-1, 1] \in \mathcal{C}(\mathbb{R}^2)$, $B = \{0\} \times [-2, 2] \in \mathcal{C}(\mathbb{R}^2)$, and $\lambda = \frac{1}{2}$. Then, it follows that $\min(A; K) = (0, -1)$, $\max(A; K) = (0, 1)$, $\min(B; K) = (0, -2)$, $\max(B; K) = (0, 2)$, and that $V_\lambda(A) = \frac{1}{2}(0, -1) + \left(1 - \frac{1}{2}\right)(0, 1) = (0, 0)$, $V_\lambda(B) = \frac{1}{2}(0, -2) + \left(1 - \frac{1}{2}\right)(0, 2) = (0, 0)$. Thus, we have $V_\lambda(A) = V_\lambda(B)$. On the other hand, it follows that $\min(B; K) \leq_K \min(A; K)$, $\min(B; K) \neq \min(A; K)$, $\max(A; K) \leq_K \max(B; K)$, $\max(A; K) \neq \max(B; K)$, and that $\min(A; K) \not\leq_K \min(B; K)$, $\max(A; K) \leq_K \max(B; K)$. Thus, we have $V_2(A) \not\leq_C V_2(B)$.

3. Set-valued optimization

Throughout this section, let X be a nonempty set, and let $F : X \to \mathcal{C}(\mathbb{R}^n)$ be a set-valued mapping. Our main problem is the following set-valued optimization problem (P) with respect to $\preceq_K$:

$$\text{(P)} \qquad \left| \begin{array}{ll} \min & F(x) \\ \text{s.t.} & x \in X. \end{array} \right.$$

Definition 3.1. An element $\overline{x} \in X$ is said to be a *non-dominated solution* of the problem (P) if $F(\overline{x})$ is a non-dominated element of $F(X) = \{F(x) : x \in X\}$.

For the set-valued problem (P), we consider the following vector-valued optimization problem (VP) with respect to $\leq_C$:

$$\text{(VP)} \qquad \left| \begin{array}{ll} \min & V_2(F(x)) \\ \text{s.t.} & x \in X. \end{array} \right.$$

Definition 3.2. An element $\overline{x} \in X$ is said to be a *non-dominated solution* of the problem (VP) if

$$(\{V_2(F(\overline{x}))\} - C) \cap V_2(F(X)) = \{V_2(F(\overline{x}))\}.$$

We also consider the following vector-valued optimization problem ($\overline{\text{VP}}$) with respect to $\leq_{adm}$:

$$(\overline{\text{VP}}) \qquad \left| \begin{array}{ll} \min & V_2(F(x)) \\ \text{s.t.} & x \in X. \end{array} \right.$$

The difference between the vector-valued problems (VP) and ($\overline{\text{VP}}$) is only their orders of vectors. The problem (VP) is a minimization problem with respect to the partial order $\leq_C$, and the problem ($\overline{\text{VP}}$) is a minimization problem with respect to the total order $\leq_{adm}$. In this section, let $\overline{K}$ be a total ordering cone for the admissible order $\leq_{adm}$, and we use $\leq_{\overline{K}}$ instead of $\leq_{adm}$. Since $\leq_{\overline{K}}$ is an admissible order, we have $C \subset \overline{K}$. Therefore, if $x^* \in X$ an optimal solution of the problem ($\overline{\text{VP}}$), then x^* is a non-dominated solution of the problem (VP); [2, Lemma 4].

Moreover, we consider the following vector-valued optimization problem (VP_λ) with respect to $\leq_K$:

$$(\text{VP}_\lambda) \qquad \left| \begin{array}{ll} \min & V_\lambda(F(x)) \\ \text{s.t.} & x \in X \end{array} \right.$$

where $\lambda \in [0, 1]$.

Theorem 3.3. *If $x^* \in X$ is a non-dominated solution of the vector-valued problem (VP), then x^* is a non-dominated solution of the set-valued problem (P).*

Proof. Let $x^* \in X$ be a non-dominated solution of the problem (VP). Then, it follows that

$$\begin{aligned}
& x \in X, F(x) \preceq_K F(x^*) \\
& \Rightarrow \min(F(x); K) \leq_K \min(F(x^*); K), \\
& \qquad \max(F(x); K) \leq_K \max(F(x^*); K) \quad \text{(from (2.3))} \\
& \Rightarrow (\min(F(x); K), \max(F(x); K)) \\
& \qquad \leq_C (\min(F(x^*); K), \max(F(x^*); K)) \quad \text{(from (2.6))} \\
& \Rightarrow V_2(F(x)) \leq_C V_2(F(x^*)) \quad \text{(from (2.4))} \\
& \Rightarrow V_2(F(x^*)) - V_2(F(x)) \in C \quad \text{(from (2.6))}
\end{aligned}$$

$$\begin{aligned}
& \Rightarrow \exists \boldsymbol{c} \in C \text{ s.t. } V_2(F(x^*)) - V_2(F(x)) = \boldsymbol{c} \\
& \Rightarrow V_2(F(x)) = V_2(F(x^*)) - \boldsymbol{c} \\
& \qquad \in (\{V_2(F(x^*))\} - C) \cap V_2(F(X)) = \{V_2(F(x^*))\} \\
& \qquad \text{(since } x^* \text{ is a non-nondominated solution of the problem (VP))} \\
& \Rightarrow V_2(F(x)) = V_2(F(x^*)) \\
& \Rightarrow (\min(F(x^*); K), \max(F(x^*); K))
\end{aligned}$$

$$
\begin{aligned}
&= (\min(F(x);K), \max(F(x);K)) \quad \text{(from (2.4))}\\
\Rightarrow\ & (\min(F(x^*);K), \max(F(x^*);K))\\
&\quad \leq_C (\min(F(x);K), \max(F(x);K))\\
\Rightarrow\ & \min(F(x^*);K) \leq_K \min(F(x);K),\\
&\quad \max(F(x^*);K) \leq_K \max(F(x);K) \quad \text{(from (2.6))}\\
\Rightarrow\ & F(x^*) \preceq_K F(x) \quad \text{(from (2.3))}.
\end{aligned}
$$

Therefore, x^* is a non-dominated solution of the problem (P). □

Corollary 3.4. *If $x^* \in X$ is an optimal solution of the vector-valued problem $(\overline{VP})$, then x^* is a non-dominated solution of the set-valued problem (P).*

Theorem 3.5. *For any $\lambda \in]0,1[$, if $x^* \in X$ is an optimal solution of the vector-valued problem (VP_λ), then x^* is a non-dominated solution of the set-valued problem (P).*

Proof. Fix any $\lambda \in]0,1[$, and let $x^* \in X$. Suppose that x^* is not a non-dominated solution of the problem (P). We show that x^* is not an optimal solution of the problem (VP_λ). Since x^* is not a non-dominated solution of the problem (P), it follows that

$$\exists x \in X \text{ s.t. } F(x) \preceq_K F(x^*), F(x^*) \not\preceq_K F(x).$$

Since $F(x) \preceq_K F(x^*)$ and $F(x^*) \not\preceq_K F(x)$, it follows that

$$\min(F(x);K) \leq_K \min(F(x^*);K),\ \max(F(x);K) \leq_K \max(F(x^*);K)$$

and

$$\min(F(x^*);K) \not\leq_K \min(F(x);K) \text{ or } \max(F(x^*);K) \not\leq_K \max(F(x);K)$$

from (2.3). Thus, we have

$$\min(F(x);K) \leq_K \min(F(x^*);K),\ \min(F(x);K) \neq \min(F(x^*);K)$$

or

$$\max(F(x);K) \leq_K \max(F(x^*);K),\ \max(F(x);K) \neq \max(F(x^*);K).$$

In the sense of (1.3), the total order $\leq_K$ on $\mathbb{R}^n$ is the lexicographic order $\leq_{lex}$ on $\mathbb{R}^n$ with the orthogonal base $\{\boldsymbol{r}_1, \boldsymbol{r}_2, \dots, \boldsymbol{r}_n\}$. We set

$$
\begin{aligned}
\min(F(x);K) &= a_1\boldsymbol{r}_1 + a_2\boldsymbol{r}_2 + \cdots + a_n\boldsymbol{r}_n,\\
\min(F(x^*);K) &= a_1^*\boldsymbol{r}_1 + a_2^*\boldsymbol{r}_2 + \cdots + a_n^*\boldsymbol{r}_n,\\
\max(F(x);K) &= b_1\boldsymbol{r}_1 + b_2\boldsymbol{r}_2 + \cdots + b_n\boldsymbol{r}_n,\\
\max(F(x^*);K) &= b_1^*\boldsymbol{r}_1 + b_2^*\boldsymbol{r}_2 + \cdots + b_n^*\boldsymbol{r}_n.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
&\lambda \min(F(x);K) + (1-\lambda)\max(F(x);K)\\
&= (\lambda a_1 + (1-\lambda)b_1)\boldsymbol{r}_1 + (\lambda a_2 + (1-\lambda)b_2)\boldsymbol{r}_2 +\\
&\qquad \cdots + (\lambda a_n + (1-\lambda)b_n)\boldsymbol{r}_n,
\end{aligned}
$$

$$\begin{aligned}&\lambda \min(F(x^*);K) + (1-\lambda)\max(F(x^*);K)\\&= (\lambda a_1^* + (1-\lambda)b_1^*)\boldsymbol{r}_1 + (\lambda a_2^* + (1-\lambda)b_2^*)\boldsymbol{r}_2 +\\&\quad \cdots + (\lambda a_n^* + (1-\lambda)b_n^*)\boldsymbol{r}_n.\end{aligned}$$

If $\min(F(x);K) \leq_K \min(F(x^*);K)$ and $\min(F(x);K) \neq \min(F(x^*);K)$, then it follows that

$$\exists i_{\min} \in \{1,2,\ldots,n\} \text{ s.t. } a_j = a_j^*(j < i_{\min}), a_{i_{\min}} < a^*_{i_{\min}}.$$

If $\max(F(x);K) \leq_K \max(F(x^*);K)$ and $\max(F(x);K) \neq \max(F(x^*);K)$, then it follows that

$$\exists i_{\max} \in \{1,2,\ldots,n\} \text{ s.t. } b_j = b_j^*(j < i_{\max}), b_{i_{\max}} < b^*_{i_{\max}}.$$

(i) Suppose that

$$\begin{aligned}&\min(F(x);K) \leq_K \min(F(x^*);K),\ \max(F(x);K) \leq_K \max(F(x^*);K),\\&\min(F(x^*);K) \not\leq_K \min(F(x);K),\ \max(F(x^*);K) \not\leq_K \max(F(x);K).\end{aligned}$$

We set

$$i_0 = \min\{i_{\min}, i_{\max}\}.$$

Then, it follows that

$$\begin{gathered}a_j = a_j^*,\ b_j = b_j^*, \quad \forall j < i_0,\\a_{i_0} < a^*_{i_0}, b_{i_0} < b^*_{i_0} \text{ or } a_{i_0} < a^*_{i_0}, b_{i_0} = b^*_{i_0} \text{ or } a_{i_0} = a^*_{i_0}, b_{i_0} < b^*_{i_0},\end{gathered}$$

and we have

$$\begin{gathered}\lambda a_j + (1-\lambda)b_j = \lambda a_j^* + (1-\lambda)b_j^*, \quad \forall j < i_0,\\\lambda a_{i_0} + (1-\lambda)b_{i_0} < \lambda a^*_{i_0} + (1-\lambda)b^*_{i_0}.\end{gathered}$$

(ii) Suppose that

$$\begin{aligned}&\min(F(x);K) \leq_K \min(F(x^*);K),\ \max(F(x);K) \leq_K \max(F(x^*);K),\\&\min(F(x^*);K) \not\leq_K \min(F(x);K),\ \max(F(x^*);K) \leq_K \max(F(x);K).\end{aligned}$$

It follows that

$$\begin{gathered}a_j = a_j^*,\ b_j = b_j^*, \quad \forall j < i_{\min},\\a_{i_{\min}} < a^*_{i_{\min}}, b_{i_{\min}} = b^*_{i_{\min}},\end{gathered}$$

and we have

$$\begin{gathered}\lambda a_j + (1-\lambda)b_j = \lambda a_j^* + (1-\lambda)b_j^*, \quad \forall j < i_{\min},\\\lambda a_{i_{\min}} + (1-\lambda)b_{i_{\min}} < \lambda a^*_{i_{\min}} + (1-\lambda)b^*_{i_{\min}}.\end{gathered}$$

(iii) Suppose that

$$\begin{aligned}&\min(F(x);K) \leq_K \min(F(x^*);K),\ \max(F(x);K) \leq_K \max(F(x^*);K),\\&\min(F(x^*);K) \leq_K \min(F(x);K),\ \max(F(x^*);K) \not\leq_K \max(F(x);K).\end{aligned}$$

It follows that

$$\begin{gathered}a_j = a_j^*,\ b_j = b_j^*, \quad \forall j < i_{\max},\\a_{i_{\max}} = a^*_{i_{\max}}, b_{i_{\max}} < b^*_{i_{\max}},\end{gathered}$$

and we have

$$\lambda a_j + (1-\lambda) b_j = \lambda a_j^* + (1-\lambda) b_j^*, \quad \forall j < i_{\max},$$
$$\lambda a_{i_{\max}} + (1-\lambda) b_{i_{\max}} < \lambda a_{i_{\max}}^* + (1-\lambda) b_{i_{\max}}^*.$$

From (i), (ii), and (iii), we have

$$\begin{aligned} V_\lambda(F(x)) &= \lambda \min(F(x);K) + (1-\lambda)\max(F(x);K) \\ &\leq_K \lambda \min(F(x^*);K) + (1-\lambda)\max(F(x^*);K) \\ &= V_\lambda(F(x^*)), \end{aligned}$$

$$\begin{aligned} V_\lambda(F(x)) &= \lambda \min(F(x);K) + (1-\lambda)\max(F(x);K) \\ &\neq \lambda \min(F(x^*);K) + (1-\lambda)\max(F(x^*);K) \\ &= V_\lambda(F(x^*)). \end{aligned}$$

Therefore, x^* is not an optimal solution of the problem (VP$_\lambda$). □

4. Example of construction of $\overline{K}$ for admissible order

In this section, we consider the orthogonal base

$$\{(\boldsymbol{r}_1,\boldsymbol{0}),(\boldsymbol{0},\boldsymbol{r}_1),(\boldsymbol{r}_2,\boldsymbol{0}),(\boldsymbol{0},\boldsymbol{r}_2),\ldots,(\boldsymbol{r}_n,\boldsymbol{0}),(\boldsymbol{0},\boldsymbol{r}_n)\} \subset \mathbb{R}^n \times \mathbb{R}^n.$$

We set

$$(\boldsymbol{r}'_j,\boldsymbol{s}'_j) = \begin{cases} (\boldsymbol{r}_k,\boldsymbol{0}) & \text{if } j = 2k-1, k \in \{1,2,\ldots,n\}, \\ (\boldsymbol{0},\boldsymbol{r}_k) & \text{if } j = 2k, k \in \{1,2,\ldots,n\} \end{cases}$$

and define a total ordering cone $\overline{K}$ in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ as follows:

$$\overline{K} = \left[\bigcup_{i=1}^{2n}\{(\boldsymbol{r},\boldsymbol{s}) \in \mathbb{R}^n \times \mathbb{R}^n : \langle(\boldsymbol{r}'_j,\boldsymbol{r}'_j),(\boldsymbol{r},\boldsymbol{s})\rangle = 0 (j<i), \langle(\boldsymbol{r}'_i,\boldsymbol{r}'_i),(\boldsymbol{r},\boldsymbol{s})\rangle > 0\}\right] \cup \{(\boldsymbol{0},\boldsymbol{0})\}.$$

The following theorem shows that the total order $\leq_{\overline{K}}$ on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ is an admissible order.

Theorem 4.1. $C \subset \overline{K}$.

Proof. Let $(\boldsymbol{r},\boldsymbol{s}) \in C = K \times K$, where K is represented as (1.2). If $\boldsymbol{r} = \boldsymbol{0}$ and $\boldsymbol{s} = \boldsymbol{0}$, then $(\boldsymbol{r},\boldsymbol{s}) \in \overline{K}$.

Suppose that $\boldsymbol{r} \neq \boldsymbol{0}$ and $\boldsymbol{s} = \boldsymbol{0}$. Since $\boldsymbol{r} \neq \boldsymbol{0}$ and $\boldsymbol{s} = \boldsymbol{0}$, it follows that

$$\exists i_r \in \{1,2,\ldots,n\} \text{ s.t. } \langle \boldsymbol{r}_j,\boldsymbol{r}\rangle = 0 (j<i_r), \langle \boldsymbol{r}_{i_r},\boldsymbol{r}\rangle > 0,$$
$$\langle \boldsymbol{r}_j,\boldsymbol{s}\rangle = 0, \quad \forall j \in \{1,2,\ldots,n\}.$$

It follows that

$$\langle(\boldsymbol{r}'_j,\boldsymbol{s}'_j),(\boldsymbol{r},\boldsymbol{s})\rangle = \langle \boldsymbol{r}'_j,\boldsymbol{r}\rangle + \langle \boldsymbol{s}'_j,\boldsymbol{s}\rangle = 0+0 = 0,$$

for $j < 2i_r - 1$, and that

$$\begin{aligned}\langle(\boldsymbol{r}'_{2i_r-1}, \boldsymbol{s}'_{2i_r-1}), (\boldsymbol{r}, \boldsymbol{s})\rangle &= \langle \boldsymbol{r}'_{2i_r-1}, \boldsymbol{r}\rangle + \langle \boldsymbol{s}'_{2i_r-1}, \boldsymbol{s}\rangle \\ &= \langle \boldsymbol{r}'_{2i_r-1}, \boldsymbol{r}\rangle + 0 \\ &= \langle \boldsymbol{r}'_{2i_r-1}, \boldsymbol{r}\rangle \\ &> 0.\end{aligned}$$

Therefore, we have $(\boldsymbol{r}, \boldsymbol{s}) \in \overline{K}$.

Similarly, if $\boldsymbol{r} = \boldsymbol{0}$ and $\boldsymbol{s} \neq \boldsymbol{0}$, then we have $(\boldsymbol{r}, \boldsymbol{s}) \in \overline{K}$.

Suppose that $\boldsymbol{r} \neq \boldsymbol{0}$ and $\boldsymbol{s} \neq \boldsymbol{0}$. Since $\boldsymbol{r} \neq \boldsymbol{0}$ and $\boldsymbol{s} \neq \boldsymbol{0}$, it follows that

$$\exists i_r \in \{1, 2, \ldots, n\} \text{ s.t. } \langle \boldsymbol{r}_j, \boldsymbol{r}\rangle = 0 (j < i_r), \langle \boldsymbol{r}_{i_r}, \boldsymbol{r}\rangle > 0,$$

$$\exists i_s \in \{1, 2, \ldots, n\} \text{ s.t. } \langle \boldsymbol{r}_j, \boldsymbol{s}\rangle = 0 (j < i_s), \langle \boldsymbol{r}_{i_s}, \boldsymbol{s}\rangle > 0.$$

We set

$$i_0 = \min\{i_r, i_s\}.$$

If $i_0 = i_r$ and $i_r \leq i_s$, then

$$\langle(\boldsymbol{r}'_j, \boldsymbol{s}'_j), (\boldsymbol{r}, \boldsymbol{s})\rangle = \langle \boldsymbol{r}'_j, \boldsymbol{r}\rangle + \langle \boldsymbol{s}'_j, \boldsymbol{s}\rangle = 0 + 0 = 0$$

for $j < 2i_0 - 1$, and then

$$\begin{aligned}\langle(\boldsymbol{r}'_{2i_0-1}, \boldsymbol{s}'_{2i_0-1}), (\boldsymbol{r}, \boldsymbol{s})\rangle &= \langle \boldsymbol{r}'_{2i_0-1}, \boldsymbol{r}\rangle + \langle \boldsymbol{s}'_{2i_0-1}, \boldsymbol{s}\rangle \\ &= \langle \boldsymbol{r}'_{2i_0-1}, \boldsymbol{r}\rangle + 0 \\ &= \langle \boldsymbol{r}'_{2i_0-1}, \boldsymbol{r}\rangle \\ &> 0.\end{aligned}$$

If $i_0 = i_s$ and $i_s < i_r$, then

$$\langle(\boldsymbol{r}'_j, \boldsymbol{s}'_j), (\boldsymbol{r}, \boldsymbol{s})\rangle = \langle \boldsymbol{r}'_j, \boldsymbol{r}\rangle + \langle \boldsymbol{s}'_j, \boldsymbol{s}\rangle = 0 + 0 = 0$$

for $j < 2i_0$, and then

$$\begin{aligned}\langle(\boldsymbol{r}'_{2i_0}, \boldsymbol{s}'_{2i_0}), (\boldsymbol{r}, \boldsymbol{s})\rangle &= \langle \boldsymbol{r}'_{2i_0}, \boldsymbol{r}\rangle + \langle \boldsymbol{s}'_{2i_0}, \boldsymbol{s}\rangle \\ &= 0 + \langle \boldsymbol{s}'_{2i_0}, \boldsymbol{s}\rangle \\ &= \langle \boldsymbol{s}'_{2i_0}, \boldsymbol{s}\rangle \\ &> 0.\end{aligned}$$

Therefore, we have $(\boldsymbol{r}, \boldsymbol{s}) \in \overline{K}$. □

For the last of this section, if we consider the orthogonal base

$$\{(\boldsymbol{0}, \boldsymbol{r}_1), (\boldsymbol{r}_1, \boldsymbol{0}), (\boldsymbol{0}, \boldsymbol{r}_2), (\boldsymbol{r}_2, \boldsymbol{0}), \ldots, (\boldsymbol{0}, \boldsymbol{r}_n), (\boldsymbol{r}_n, \boldsymbol{0})\} \subset \mathbb{R}^n \times \mathbb{R}^n$$

and set

$$(\boldsymbol{r}'_j, \boldsymbol{s}'_j) = \begin{cases} (\boldsymbol{0}, \boldsymbol{r}_k) & \text{if } j = 2k - 1, k \in \{1, 2, \ldots, n\}, \\ (\boldsymbol{r}_k, \boldsymbol{0}) & \text{if } j = 2k, k \in \{1, 2, \ldots, n\} \end{cases}$$

and define a total ordering cone $\overline{K}$ in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$ as

$$\overline{K} = \left[\bigcup_{i=1}^{2n} \{(\boldsymbol{r}, \boldsymbol{s}) \in \mathbb{R}^n \times \mathbb{R}^n : \langle (\boldsymbol{r}'_j, \boldsymbol{r}'_j), (\boldsymbol{r}, \boldsymbol{s}) \rangle = 0 (j < i), \langle (\boldsymbol{r}'_i, \boldsymbol{r}'_i), (\boldsymbol{r}, \boldsymbol{s}) \rangle > 0\} \right] \cup \{(\boldsymbol{0}, \boldsymbol{0})\},$$

then Theorem 4.1 also holds.

5. Conclusion

In the present paper, we defined a pseudo-order relation between sets by using a total order relation between vectors, and considered a set-valued optimization problem with respect to the pseudo-order. Two types of vector-valued optimization problems were derived from the set-valued problem by using two types of vectorization of sets. Then, we showed that any optimal solution of the derived vector-valued problems is a non-dominated solution of the original set-valued problem.

References

[1] H. Bustince, J. Fernandeza, A. Kolesárová and R. Mesiar, *Generation of linear orders for intervals by means of aggregation functions*, Fuzzy Sets and Systems **220** (2013), 69–77.

[2] M. Küçük, M. Soyertem and Y. Küçük, *On constructing total orders and solving vector optimization problems with total orders*, J. Global Optim. **50** (2011), 235–247.

[3] M. Küçük, M. Soyertem, Y. Küçük and İ. Atasever, *Vectorization of set-valued maps with respect to total ordering cones and its applications to set-valued optimization problems*, J. Math. Anal. Appl. **385** (2012), 285–292.

[4] D. Li, Y. Leung and W. Wu, *Multiobjective interval linear programming in admissible-order vector space*, Information Sci. **486** (2019), 1–19.

[5] L. Li, *Optimality conditions for nonlinear optimization problems with interval-valued objective function in admissible orders*, Fuzzy Optim. Decis. Mak. **22** (2023), 247–265.

Manuscript received 3 December 2025
revised 29 December 2025

M. Kon
Graduate School of Science and Technology, Hirosaki University, 3 Bunkyo, Hirosaki, Aomori 036-8561, Japan
E-mail address: `masakon@hirosaki-u.ac.jp`