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QUASICONCAVE UTILITY MAXIMIZATION FOR PORTFOLIO OPTIMIZATION

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Abstract. In this paper, we study quasiconcave utility maximization for portfolio optimization. We introduce portfolio optimization as quasiconcave utility maximization problem. We investigate solutions of the problem in terms of evenly quasiconcavity of the utility function. In addition, we show a numerical example as an application.

1. Introduction

Stocks are a typical example of risky assets, and it is uncertain what kind of returns they will generate in the future. Hence, returns are usually modeled as random variables. There are various methods for modeling investor behavior. The most important one is the following proposed by Markowitz [8, 9]:

(I) investors maximize the expected rate of return,
(II) investors minimize the variance of the rate of return.

In general, investors who want to higher returns will take on higher risks. There is a trade-off between return and risk, then we usually consider the following mean-variance model portfolio optimization problem:

$$\begin{array}{ll} \text{Minimize} & \displaystyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}, \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}, \text{ and } \sum_{i=1}^{n} w_i = 1. \end{array}$$

where w_i is the weight of asset i in a portfolio, σ_{ij} is the covariance of the return of the asset i with j, and $\bar{r}_i$ is the expected rate of return of i. In the problem, we fix the mean value at $\bar{r}$, and find the portfolio of minimum variance. There are various fruitful results on the mean-variance model, see [4, 7, 8, 9, 11, 23].

On the other hand, we study the following utility maximization problem:

$$\begin{array}{ll} \text{Maximize} & U(w) = u \circ g(w), \\ \text{subject to} & g(w) = (\sigma_w, \bar{r}_w), \\ & \sigma_w^2 = \displaystyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij},\ \bar{r}_w = \sum_{i=1}^{n} w_i \bar{r}_i, \text{ and } \sum_{i=1}^{n} w_i = 1. \end{array}$$

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By using the function g, a portfolio w is represented on mean-standard deviation diagram, then we consider maximization problem of quasiconcave utility function u on $\mathbb{R}^2$. Quasiconcave maximization problem is equivalent to quasiconvex minimization problem, and various researchers have investigated quasiconvex optimization, see [1, 2, 3, 6, 10, 12, 13, 14, 16, 17, 18, 19, 20, 21, 22, 24] and references therein. In particular, the authors study minimization problems whose objective functions are evenly quasiconvex in [19, 24]. We show optimality conditions and duality results by three types of evenly quasiconvex functions.

Based on the above previous research, we study quasiconcave utility maximization for portfolio optimization in this paper. We introduce portfolio optimization as quasiconcave utility maximization problem. We investigate solutions of the problem in terms of evenly quasiconcavity of the utility function u. In addition, we show a numerical example as an application.

The remainder of the present paper is organized as follows. In Section 2, we introduce some preliminaries and previous results. In Section 3, we study quasiconcave utility maximization for portfolio optimization. In Section 4, we show a numerical example.

2. Preliminaries

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space. The inner product of two vectors v and x in $\mathbb{R}^n$ is denoted by $\langle v, x\rangle$. We define the following families of open half spaces:

$$\begin{aligned} H &= \{\mathrm{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha \in \mathbb{R}\}, \\ H^+ &= \{\mathrm{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha > 0\}, \\ H^0 &= \{\mathrm{lev}(v, <, 0) : v \in \mathbb{R}^n\}, \\ H^- &= \{\mathrm{lev}(v, <, \alpha) : v \in \mathbb{R}^n, \alpha < 0\}, \end{aligned}$$

where $\mathrm{lev}(v, <, \alpha) = \{x \in \mathbb{R}^n : \langle v, x\rangle < \alpha\}$. A subset A of $\mathbb{R}^n$ is said to be evenly (H-evenly, O-evenly, and R-evenly) convex if it is the intersection of a subfamily of H (H^+, H^0, H^-, respectively). We define the whole space and the empty set is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex by convention. In [18], we show the following statements:

- if $t > 0$, $\mathrm{lev}(v, <, t)$ and $\mathrm{lev}(v, \leq, t)$ are H-evenly convex,
- $\mathrm{lev}(v, <, 0)$ is O-evenly convex and $\mathrm{lev}(v, \leq, 0)$ is H-evenly convex,
- if $t < 0$, $\mathrm{lev}(v, <, t)$ and $\mathrm{lev}(v, \leq, t)$ are R-evenly convex.

Let f be a function from $\mathbb{R}^n$ to $\overline{\mathbb{R}} = [-\infty, \infty]$. A function f is said to be quasiconvex if for each x, $y \in \mathbb{R}^n$ and $\alpha \in [0, 1]$,

$$f((1-\alpha)x + \alpha y) \leq \max\{f(x), f(y)\}.$$

Define the level sets of f with respect to a binary relation $\diamond$ on $\overline{\mathbb{R}}$ as

$$\mathrm{lev}(f, \diamond, \alpha) = \{x \in \mathbb{R}^n : f(x) \diamond \alpha\}$$

for each $\alpha \in \mathbb{R}$. It is well known that f is quasiconvex if and only if $\mathrm{lev}(f, \leq, \alpha)$ is convex for all $\alpha \in \mathbb{R}$. A function f is said to be evenly (H-evenly, O-evenly, and R-evenly) quasiconvex if $\mathrm{lev}(f, \leq, \alpha)$ is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex for all $\alpha \in \mathbb{R}$. We need the following statements for evenly quasiconvex functions, see [19, 24]:

- f is H-evenly quasiconvex if and only if f is evenly quasiconvex and $0 \in \mathbb{R}^n$ is a global minimizer of f in $\mathbb{R}^n$,
- if f is O-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t > 0$, $f(x) = f(tx)$,
- if f is R-evenly quasiconvex, then for each $x \in \mathbb{R}^n$ and $t \geq 1$, $f(x) \geq f(tx)$.

A function f is said to be quasiconcave if $-f$ is quasiconvex. Evenly, H-evenly, O-evenly, and R-evenly quasiconcavity of functions are defined by the same way. Various results for evenly convex sets and evenly quasiconvex functions have been investigated, see [2, 10, 3, 5, 16, 17, 18, 19, 20, 22, 24] and references therein.

Suppose that there are n assets with random rates of return $\{r_1, r_2, \ldots, r_n\}$, $\bar{r}_i$ denotes the expected value of r_i, and σ_{ij} is the covariance of the rate of return of the asset i with j. We denote Ω as the covariance matrix:

$$\Omega = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn} \end{pmatrix},$$

and $\mu = (\bar{r}_1, \ldots, \bar{r}_n)^T$ is the vector of the expected rate of returns. For each portfolio $w \in S = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i = 1\}$, the expected rate of return of w is

$$\bar{r}_w = \langle \mu, w \rangle = \sum_{i=1}^{n} w_i \bar{r}_i,$$

and the variance of the rate of return of w is

$$\sigma_w^2 = w^T \Omega w = \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}.$$

In [8, 9], Markowitz introduces the following mean-variance model portfolio optimization problem:

$$(P_{\bar{r}}) \quad \begin{array}{ll} \text{Minimize} & w^T \Omega w, \\ \text{subject to} & w \in S \text{ and } \langle \mu, w \rangle = \bar{r}. \end{array}$$

Let g be the following function from S to $\mathbb{R}^2$:

$$g(w) = (\sigma_w, \bar{r}_w).$$

By using the function g, a portfolio w is represented on mean-standard deviation diagram. The following set is called feasible region of the problem:

$$X = \{g(w) \in \mathbb{R}^2 \mid w \in S\}.$$

The left boundary of X is called the minimum-variance set. In other words, let $\mathrm{val}(P_{\bar{r}})$ be the optimal value of $(P_{\bar{r}})$, then

$$M = \{(\sigma, \bar{r}) \in X \mid \sigma = \mathrm{val}(P_{\bar{r}})\}$$

is the minimum-variance set. We denote the minimum-variance portfolio as $\bar{x} \in S$, that is,

$$g(\bar{x}) = (\sigma_{\bar{x}}, \bar{r}_{\bar{x}}) \in M, \text{ and } \sigma_{\bar{x}}^2 = \min_{w \in S} w^T \Omega w.$$

In mean-variance model,

(I) investors maximize the expected rate of return,
(II) investors minimize the variance of the rate of return.

Hence, only the upper part of M will be of interest to investors. The following set F, which is the upper part of M, is called the efficient frontier of X:

$$F = \{(\sigma, \bar{r}) \in M \mid \bar{r} \geq \bar{r}_{\bar{x}} = \langle \mu, \bar{x} \rangle\}$$

If Ω has the inverse matrix Ω^{-1}, then M is expressed in the following equation:

$$M = \left\{ (\sigma, \bar{r}) \in \mathbb{R}^2 \mid \sigma^2 = \frac{C\bar{r}^2 - 2A\bar{r} + B}{D} \right\},$$

where $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n$, $A = \mu^T \Omega^{-1} \mathbf{1}$, $B = \mu^T \Omega^{-1} \mu > 0$, $C = \mathbf{1}^T \Omega^{-1} \mathbf{1} > 0$, and $D = BC - A^2 > 0$. Hence, we can characterize F by a differentiable, monotone increasing concave function f on $[\sigma_{\bar{x}}, \infty)$ as follows:

$$F = \{(\sigma, \bar{r}) \in M \mid \bar{r} = f(\sigma)\}.$$

By the concavity of f, there exists $\bar{y} \in S$ such that

$$g(\bar{y}) = (\sigma_{\bar{y}}, \bar{r}_{\bar{y}}) \in F, \text{ and } \frac{\bar{r}_{\bar{y}}}{\sigma_{\bar{y}}} = \max_{w \in S} \frac{\bar{r}_w}{\sigma_w}.$$

For more details, see [7, 11, 15].

The following theorem is called the two-fund theorem, which has operational importance for investors:

Theorem 2.1. [7] *Let x, $y \in g^{-1}(M) = \{w \in S \mid g(w) \in M\}$, and $(\sigma, \bar{r}) \in M$. Then there exists $\alpha \in \mathbb{R}$ such that $(\sigma, \bar{r}) = g((1-\alpha)x + \alpha y)$.*

Especially,

$$M = \{g((1-\alpha)\bar{x} + \alpha\bar{y}) \in X \mid \alpha \in \mathbb{R}\}.$$

since $\bar{x}$, $\bar{y} \in g^{-1}(F) = \{w \in S \mid g(w) \in F\} \subset g^{-1}(M)$.

We can check easily that

$$F = \{g((1-\alpha)\bar{x} + \alpha\bar{y}) \in X \mid \alpha \geq 0\}.$$

By Theorem 2.1, all investors need only invest in linear combination of the minimum-variance portfolio $\bar{x}$ and the maximum-price of risk, $\frac{\bar{r}}{\sigma}$, portfolio $\bar{y}$. However, the weight α depends on investors' risk aversion and utility.

3. Main results

Let u be a utility function on $\mathbb{R}^2$, $\Omega \in \mathbb{R}^{n\times n}$ be the covariance matrix, and $\mu \in \mathbb{R}^n$ be the vector of the expected rate of returns. Assume that Ω has the inverse matrix Ω^{-1}, and $g(\bar{x}) > (0,0)$. In this section, we consider the following utility maximization problem:

$$(P_u) \quad \begin{array}{ll} \text{Maximize} & U(w) = u \circ g(w), \\ \text{subject to} & w \in S = \{x \in \mathbb{R}^n \mid \langle x, \mathbf{1} \rangle = 1\}. \end{array}$$

First of all, we introduce a notion of quasiconcavity which consistents with Markowitz's criteria:

definition 3.1. Let $K = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 \geq 0\}$. A function $u : \mathbb{R}^2 \to \overline{\mathbb{R}}$ is called M-quasiconcave if

(i) u is quasiconcave,
(ii) for each $\alpha \in \mathbb{R}$, $\text{lev}(u, \geq, \alpha) + K \subset \text{lev}(u, \geq, \alpha)$.

If a utility function u is M-quasiconcave, then an investor maximizes the expected rate of return, and minimizes the variance of the rate of return. A function u is said to be M-evenly (MH-evenly, MO-evenly, and MR-evenly) quasiconcave if u is M-quasiconcave and $\text{lev}(u, \geq, \alpha)$ is evenly (H-evenly, O-evenly, and R-evenly, respectively) convex for all $\alpha \in \mathbb{R}$.

In the following theorem, we show a characterization of F in terms of evenly quasiconcave utility functions:

Theorem 3.2. *Let f be a monotone increasing concave function on $[\sigma_{\bar{x}}, \infty)$, $F = \{(\sigma, \bar{r}) \in M \mid \bar{r} = f(\sigma)\}$, $\alpha \geq 0$, and $x = (1-\alpha)\bar{x} + \alpha\bar{y} \in g^{-1}(F)$. Then, the following statements hold:*

(i) *there exists M-evenly quasiconcave utility function u such that x is a global maximizer of (P_u),*
(ii) *if $\alpha < 1$, then there exists MH-evenly quasiconcave utility function u such that x is a global maximizer of (P_u),*
(iii) *there exists MO-evenly quasiconcave utility function u such that $\bar{y}$ is a global maximizer of (P_u),*
(iv) *if $\alpha > 1$, there exists MR-evenly quasiconcave utility function u such that $\bar{y}$ is a global maximizer of (P_u).*

Proof. Let $\alpha \geq 0$ and $x = (1-\alpha)\bar{x} + \alpha\bar{y} \in g^{-1}(F)$.

(i) The following function u is M-evenly quasiconcave:

$$u(\sigma, \bar{r}) = \left\langle (\sigma, \bar{r}), (-f'(\sigma_x), 1) \right\rangle.$$

Actually, u is linear and $f'(\sigma_x) > 0$. Hence for each $\alpha \in \mathbb{R}$,

$$\text{lev}(u, \geq, \alpha) + K = \text{lev}(u, \geq, \alpha).$$

Since f is concave, for each $(\sigma, \bar{r}) \in X$,

$$\bar{r} = f(\sigma) \leq f(\sigma_x) + f'(\sigma_x)(\sigma - \sigma_x)$$

$$\iff \quad u(\sigma,\bar{r}) = -f'(\sigma_x)\sigma + \bar{r} \le -f'(\sigma_x)\sigma_x + \bar{r}_x = u(\sigma_x,\bar{r}_x).$$

This shows that (i) holds.

(ii) Assume that $\alpha < 1$. Let u be the following function on $\mathbb{R}^2$:

$$u(\sigma,\bar{r}) = \begin{cases} 0 & \langle(\sigma,\bar{r}),(-f'(\sigma_x),1)\rangle \ge 0, \\ \langle(\sigma,\bar{r}),(-f'(\sigma_x),1)\rangle & \langle(\sigma,\bar{r}),(-f'(\sigma_x),1)\rangle < 0. \end{cases}$$

We show that u is MH-evenly quasiconcave. Actually, for each $\beta \in \mathbb{R}$,

$$\operatorname{lev}(u,\ge,\beta) = \begin{cases} \emptyset & \beta > 0, \\ \{(s,t) \in \mathbb{R}^2 \mid \langle(\sigma,\bar{r}),(-f'(\sigma_x),1)\rangle \ge \beta\} & \beta \le 0. \end{cases}$$

Hence, $\operatorname{lev}(u,\ge,\beta)$ is H-evenly convex, and

$$\operatorname{lev}(u,\ge,\alpha) + K = \operatorname{lev}(u,\ge,\alpha)$$

since $\emptyset + K = \emptyset$.

Let $L_{\bar{y}}$ be the following function on $\mathbb{R}$:

$$L_{\bar{y}}(\sigma) = f(\sigma_{\bar{y}}) + f'(\sigma_{\bar{y}})(\sigma - \sigma_{\bar{y}})$$

Since $\bar{y}$ is the maximum solution of $\frac{\bar{r}}{\sigma}$, we can check that

$$0 = L(0) = \left\langle(\sigma_{\bar{y}},\bar{r}_{\bar{y}}),(-f'(\sigma_{\bar{y}}),1)\right\rangle.$$

In other words,

$$(0,0),(\sigma_{\bar{y}},\bar{r}_{\bar{y}}) \in \{(s,t) \in \mathbb{R}^2 \mid \left\langle(\sigma,\bar{r}),(-f'(\sigma_{\bar{y}}),1)\right\rangle = 0\}.$$

Since $\alpha < 1$, $\frac{\bar{r}_x}{\sigma_x} < \frac{\bar{r}_{\bar{y}}}{\sigma_{\bar{y}}}$. Hence, we can show that

$$0 > \left\langle(\sigma_x,\bar{r}_x),(-f'(\sigma_x),1)\right\rangle = u(\sigma_x,\bar{r}_x).$$

By the concavity of f, for each $(\sigma,\bar{r}) \in X$,

$$\begin{aligned} & \bar{r} = f(\sigma) \le f(\sigma_x) + f'(\sigma_x)(\sigma - \sigma_x) \\ \iff \quad & -f'(\sigma_x)\sigma + \bar{r} \le -f'(\sigma_x)\sigma_x + \bar{r}_x = u(\sigma_x,\bar{r}_x) < 0. \end{aligned}$$

This shows that $u(\sigma,\bar{r}) < 0$ for each $(\sigma,\bar{r}) \in X$, and (ii) holds.

(iii) Let u be the following extended real-valued function on $\mathbb{R}^2$:

$$u(\sigma,\bar{r}) = \begin{cases} -\infty & \bar{r} \le 0, \\ \dfrac{\bar{r}}{\sigma} & \sigma > 0, \bar{r} > 0, \\ +\infty & \sigma \le 0, \bar{r} > 0. \end{cases}$$

By the definition, $\bar{y}$ is a global maximizer of $U = u \circ g$ on S. We show that u is MO-evenly quasiconcave. Actually, for each $\beta \in \mathbb{R}$,

$$\operatorname{lev}(u,\ge,\beta) = \begin{cases} \{(\sigma,\bar{r}) \in \mathbb{R}^2 \mid \bar{r} > 0\} & \beta \le 0, \\ \{(\sigma,\bar{r}) \in \mathbb{R}^n \mid \bar{r} > 0, \bar{r} \ge \beta\sigma\} & \beta > 0. \end{cases}$$

Hence, $\operatorname{lev}(u,\ge,\beta)$ is O-evenly convex, and

$$\operatorname{lev}(u,\ge,\alpha) + K = \operatorname{lev}(u,\ge,\alpha).$$

This shows that (iii) holds.

(iv) Assume that $\alpha > 1$. Let u be the following function on $\mathbb{R}^2$:

$$u(\sigma, \bar{r}) = \begin{cases} \langle(\sigma, \bar{r}), (-f'(\sigma_x), 1)\rangle & \langle(\sigma, \bar{r}), (-f'(\sigma_x), 1)\rangle > 0, \\ 0 & \langle(\sigma, \bar{r}), (-f'(\sigma_x), 1)\rangle \leq 0. \end{cases}$$

We show that u is MR-evenly quasiconcave. Actually, for each $\beta \in \mathbb{R}$,

$$\mathrm{lev}(u, \geq, \beta) = \begin{cases} \{(s,t) \in \mathbb{R}^2 \mid \langle(\sigma, \bar{r}), (-f'(\sigma_x), 1)\rangle \geq \beta\} & \beta > 0, \\ \mathbb{R}^2 & \beta \leq 0. \end{cases}$$

Hence, $\mathrm{lev}(u, \geq, \beta)$ is R-evenly convex, and

$$\mathrm{lev}(u, \geq, \alpha) + K = \mathrm{lev}(u, \geq, \alpha).$$

Since $\alpha > 1$, we can show that

$$0 < \big\langle(\sigma_x, \bar{r}_x), (-f'(\sigma_x), 1)\big\rangle = u(\sigma_x, \bar{r}_x).$$

By the concavity of f, for each $(\sigma, \bar{r}) \in X$,

$$\begin{aligned} & \bar{r} = f(\sigma) \leq f(\sigma_x) + f'(\sigma_x)(\sigma - \sigma_x) \\ \iff \quad & -f'(\sigma_x)\sigma + \bar{r} \leq -f'(\sigma_x)\sigma_x + \bar{r}_x = u(\sigma_x, \bar{r}_x). \end{aligned}$$

This shows that (iv) holds. □

By Theorem 3.2, each efficient portfolio $x \in g^{-1}(F)$ is characterized by M-evenly quasiconcave utility function u. In contrast, we show a characterization of solutions of quasiconcave utility maximization problems as follows.

Theorem 3.3. *Let u be an M-quasiconcave utility function. Assume that x_0 is a solution of (P_u). Then, the following statements hold:*

1. *if u is MH-evenly quasiconcave, then there exists $\alpha \in [0,1]$ such that $(1-\alpha)\bar{x} + \alpha\bar{y}$ is a solution of (P_u),*
2. *if u is MO-evenly quasiconcave, then $\bar{y}$ is a solution of (P_u),*
3. *if u is MR-evenly quasiconcave, then there exists $\alpha \geq 1$ such that $(1-\alpha)\bar{x} + \alpha\bar{y}$ is a solution of (P_u).*

Proof. Since u is M-quasiconcave and x_0 is a solution of (P_u), $x_0 \in g^{-1}(F)$. By Theorem 2.1, there exists $\alpha_0 \geq 0$ such that $x_0 = (1-\alpha_0)\bar{x} + \alpha_0\bar{y}$.

(i) Assume that u is MH-evenly quasiconcave and $\alpha_0 > 1$. Since u is H-evenly quasiconcave,

$$\begin{aligned} u(\lambda g(x_0)) &= u((1-\lambda)(0,0) + \lambda g(x_0)) \\ &\geq \min\{u(0,0), u(g(x_0))\} \\ &= u(g(x_0)) \end{aligned}$$

for each $\lambda \in [0,1]$. Let

$$\lambda_0 = \frac{\sigma_{\bar{y}}}{\sigma_{x_0}},$$

then we show that

$$\lambda_0 \in (0,1),\ \lambda_0 g(x_0) = (\sigma_{\bar{y}}, \lambda_0 \bar{r}_{x_0}),\text{ and } \lambda_0 \bar{r}_{x_0} < \bar{r}_{\bar{y}}.$$

Since $0 < \sigma_{\bar{x}} < \sigma_{\bar{y}}$ and $\alpha_0 > 1$, $\sigma_{\bar{y}} < \sigma_{x_0}$, that is, $\lambda_0 \in (0,1)$. By the definition of $\bar{y}$,

$$\frac{\bar{r}_{x_0}}{\sigma_{x_0}} < \frac{\bar{r}_{\bar{y}}}{\sigma_{\bar{y}}} \iff \lambda_0 \bar{r}_{x_0} = \frac{\sigma_{\bar{y}}}{\sigma_{x_0}} \bar{r}_{x_0} < \bar{r}_{\bar{y}}.$$

Since u is M-quasiconcave,

$$u(g(\bar{y})) \geq u(\lambda_0 g(x_0)) \geq u(g(x_0)).$$

This shows that $\bar{y} = (1-1)\bar{x} + 1\bar{y}$ is a solution of (P_u).

(ii) Assume that u is MO-evenly quasiconcave and $\alpha_0 \neq 1$. Since u is O-evenly quasiconcave,

$$u(\lambda g(x_0)) = u(g(x_0))$$

for each $\lambda > 0$. Let

$$\lambda_0 = \frac{\sigma_{\bar{y}}}{\sigma_{x_0}},$$

then we show that

$$\lambda_0 > 0,\ \lambda_0 g(x_0) = (\sigma_{\bar{y}}, \lambda_0 \bar{r}_{x_0}),\text{ and } \lambda_0 \bar{r}_{x_0} < \bar{r}_{\bar{y}}.$$

Since $0 < \sigma_{\bar{x}} < \sigma_{\bar{y}}$ and $\alpha_0 \geq 0$, $\sigma_{\bar{x}} \leq \sigma_{x_0}$, that is, $\lambda_0 > 0$. By the definition of $\bar{y}$,

$$\frac{\bar{r}_{x_0}}{\sigma_{x_0}} < \frac{\bar{r}_{\bar{y}}}{\sigma_{\bar{y}}} \iff \lambda_0 \bar{r}_{x_0} = \frac{\sigma_{\bar{y}}}{\sigma_{x_0}} \bar{r}_{x_0} < \bar{r}_{\bar{y}}.$$

Since u is M-quasiconcave,

$$u(g(\bar{y})) \geq u(\lambda_0 g(x_0)) = u(g(x_0)).$$

This shows that $\bar{y}$ is a solution of (P_u).

(iii) Assume that u is MR-evenly quasiconcave and $\alpha_0 < 1$. Since u is R-evenly quasiconcave,

$$u(\lambda g(x_0)) \geq u(g(x_0))$$

for each $\lambda \geq 1$. Let

$$\lambda_0 = \frac{\sigma_{\bar{y}}}{\sigma_{x_0}},$$

then we show that

$$\lambda_0 > 1,\ \lambda_0 g(x_0) = (\sigma_{\bar{y}}, \lambda_0 \bar{r}_{x_0}),\text{ and } \lambda_0 \bar{r}_{x_0} < \bar{r}_{\bar{y}}.$$

Since $0 < \sigma_{\bar{x}} < \sigma_{\bar{y}}$ and $\alpha_0 < 1$, $\sigma_{\bar{y}} > \sigma_{x_0} \geq \sigma_{\bar{x}}$, that is, $\lambda_0 > 1$. By the definition of $\bar{y}$,

$$\frac{\bar{r}_{x_0}}{\sigma_{x_0}} < \frac{\bar{r}_{\bar{y}}}{\sigma_{\bar{y}}} \iff \lambda_0 \bar{r}_{x_0} = \frac{\sigma_{\bar{y}}}{\sigma_{x_0}} \bar{r}_{x_0} < \bar{r}_{\bar{y}}.$$

Since u is M-quasiconcave,

$$u(g(\bar{y})) \geq u(\lambda_0 g(x_0)) \geq u(g(x_0)).$$

This shows that $\bar{y} = (1-1)\bar{x} + 1\bar{y}$ is a solution of (P_u). □

By Theorem 3.3, solutions of quasiconcave utility maximization problems are characterized by the weight α between $\bar{x}$ and $\bar{y}$. These two portfolio play an important role in quasiconcave utility maximization for portfolio optimization.

4. Numerical example

In this section, we show a numerical example as an application of our main results. As mentioned above, if the covariance matrix has the inverse matrix, then the minimum-variance set M is expressed in the equation

$$M = \left\{ (\sigma, \bar{r}) \in \mathbb{R}^2 \mid \sigma^2 = \frac{C\bar{r}^2 - 2A\bar{r} + B}{D} \right\},$$

where $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^n$, $A = \mu^T \Omega^{-1} \mathbf{1}$, $B = \mu^T \Omega^{-1} \mu$, $C = \mathbf{1}^T \Omega^{-1} \mathbf{1}$, and $D = BC - A^2 > 0$. Let f be the following function:

$$f(\sigma) = \frac{A}{C} + \frac{\sqrt{D(C\sigma^2 - 1)}}{C}.$$

Then, f is a differentiable, monotone increasing concave function f on $[\sigma_{\bar{x}}, \infty)$,

$$F = \{(\sigma, \bar{r}) \in M \mid \bar{r} = f(\sigma)\},$$

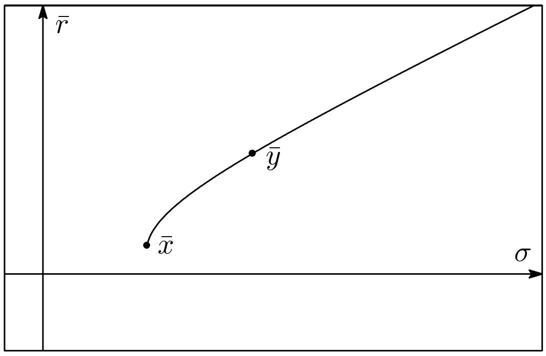
and

$$f'(\sigma) = \frac{\sigma\sqrt{D(C\sigma^2 - 1)}}{C\sigma^2 - 1}.$$

We obtain historical stock price data for the NASDAQ 100 stocks from January 1, 2023 to December 31, 2024 from Stooq (https://stooq.com/). By the price data, we can estimate the vector of the expected rate of returns μ, and the covariance matrix Ω over 20 trading days. In this case, Ω has the inverse matrix, hence the efficient frontier F is characterized by f. We can calculate them as follows:

$$\begin{aligned}
A &= 203.2333119, B = 25.825576, C = 16868.12872, D = 394325.3612, \\
f(\sigma) &= \frac{203.2333119}{16868.12872} + \frac{\sqrt{394325.3612(16868.12872\sigma^2 - 1)}}{16868.12872}, \\
f'(\sigma) &= \frac{\sigma\sqrt{394325.3612(16868.12872\sigma^2 - 1)}}{16868.12872\sigma^2 - 1}, \\
g(\bar{x}) &= (\sigma_{\bar{x}}, \bar{r}_{\bar{x}}) = (0.007699578, 0.012), \\
g(\bar{y}) &= (0.024990717, 0.127), \\
\frac{\bar{r}_{\bar{y}}}{\sigma_{\bar{y}}} &= 5.081886962.
\end{aligned}$$

We show a graph of F as follows:



By Theorem 2.1, all investors need only invest in linear combination of the minimum-variance portfolio $\bar{x}$ and the maximum-price of risk portfolio $\bar{y}$. However, the weight depends on the investor's utility function. By Theorem 3.2, each efficient portfolio $x = (1-\alpha)\bar{x} + \alpha\bar{y} \in g^{-1}(F)$ is characterized by M-evenly quasiconcave utility function u. Conversely, by Theorem 3.3, solutions of quasiconcave utility maximization problems are characterized by the weight α between $\bar{x}$ and $\bar{y}$. If $\alpha < 1$, then u can be considered MH-evenly quasiconcave. By the above numerical example, such a investor is risk-averse. If $\alpha > 1$, then u can be considered MR-evenly quasiconcave, and the investor seems to be risk-loving. If $\alpha = 0$, then the investor is only interested in the price of risk. These two portfolio $\bar{x}$, $\bar{y}$ is essential in quasiconcave utility maximization for portfolio optimization.

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References

[1] M. Avriel, W. E. Diewert, S. Schaible and I. Zang, *Generalized Concavity*, Math. Concepts Methods Sci. Engrg. Plenum Press, New York, 1988.
[2] A. Daniilidis and J. E. Martínez-Legaz, *Characterizations of evenly convex sets and evenly quasiconvex functions*, J. Math. Anal. Appl. **273** (2002), 58–66.
[3] M. D. Fajardo, M. A. Goberna, M. M. L. Rodríguez and J. Vicente-Pérez, *Even Convexity and Optimization*, EURO Advanced Tutorials on Operational Research Springer, 2020.
[4] M. S. Feldstein, *Mean-variance analysis in the theory of liquidity preference and portfolio selection*, The Review of Economic Studies **36** (1969), 5–12.
[5] W. Fenchel, *A remark on convex sets and polarity*, Comm. Sém. Math. Univ. Lund Tome Suppl. (1952), 82–89.
[6] H. J. Greenberg and W. P. Pierskalla, *Quasi-conjugate functions and surrogate duality*, Cah. Cent. Étud. Rech. Opér. **15** (1973), 437–448.
[7] D. G. Luenberger, *Investment Science*, 2nd ed., New York, Oxford University Press, 2014.
[8] H. M. Markowitz, *Portfolio selection*, The Journal of Finance **7** (1952), 77–91.
[9] H. M. Markowitz, *Foundations of portfolio theory*, The Journal of Finance **46** (1991), 469–477.

[10] J. E. Martínez-Legaz, *Characterization of R-evenly quasiconvex functions*, J. Optim. Theory Appl. **95** (1997), 717–722.
[11] R. C. Merton, *An analytic derivation of the efficient portfolio frontier*, J. Finan. Quant. Anal. **7** (1972), 1851–1872.
[12] U. Passy and E. Z. Prisman, *Conjugacy in quasi-convex programming*, Math. Program. **30** (1984), 121–146.
[13] U. Passy and E. Z. Prisman, *A convex-like duality scheme for quasi-convex programs*, Math. Program. **32** (1985), 278–300.
[14] J. P. Penot, *What is quasiconvex analysis?*, Optimization **47** (2000), 35–110.
[15] W. F. Sharpe, *Mutual fund performance*, The Journal of Business **39** (1966), 119–138.
[16] S. Suzuki, *Set containment characterization with strict and weak quasiconvex inequalities*, J. Global Optim. **47** (2010), 273–285.
[17] S. Suzuki, *Conjugate dual problem for quasiconvex programming*, J. Nonlinear Convex Anal. *23* (2022), 879–889.
[18] S. Suzuki, *Optimality conditions for quasiconvex programming in terms of quasiconjugate functions*, Minimax Theory Appl. **9** (2024), 407–420.
[19] S. Suzuki, *Quasiconjugate duality and optimality conditions for quasiconvex optimization*, J. Global Optim. **92** (2025), 279–293.
[20] S. Suzuki and D. Kuroiwa, *Set containment characterization for quasiconvex programming*, J. Global Optim. **45** (2009), 551–563.
[21] P. T. Thach, *Quasiconjugates of functions, duality relationship between quasiconvex minimization under a reverse convex constraint and quasiconvex maximization under a convex constraint, and applications*, J. Math. Anal. Appl. **159** (1991), 299–322.
[22] P. T. Thach, *Diewert-Crouzeix conjugation for general quasiconvex duality and applications*, J. Optim. Theory Appl. **86** (1995), 719–743.
[23] J. Tobin, *Liquidity preference as behavior towards risk*, The Review of Economic Studies **25** (1958), 65–86.
[24] H. Yasunaka and S. Suzuki, *Quasiconjugate dual problems for quasiconvex programming*, Linear Nonlinear Anal. **9** (2023), 103–113.

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