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FIXED POINT PRINCIPLES FOR WEAKLY CONTRACTIVE MAPS

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Abstract. Our aim in this article is to show some well-known theorems on compact metric spaces also hold for quasi-metric spaces (without symmetry) from the beginning. For a quasi-metric space (X,δ), a selfmap $f : X \to X$ is called a weakly contractive map if

$$\delta(fx, f^2x) < \delta(x, fx) \text{ for } x \in X \text{ with } x \neq fx.$$

We generalize the Edelstein fixed point theorem and others to weakly contractive maps on quasi-metric spaces. Consequently, we obtain equivalent or improved forms of known theorems in the metric fixed point theory.

1. Introduction

The metric fixed point theory is originated from the Banach contraction principle in 1922. It was extended to multimaps by Nadler [18] in 1969 and Covitz-Nadler [7] in 1970. Moreover, the Banach contraction was extended to hundreds of contractive type conditions and to almost one thousand artificial spaces; for classical examples for traditional metric spaces, see Rhoades [29], Park [22] and others.

A large number of generalizations, modifications, improvements and extensions of the metric concept have appeared in different fields due to its fundamental role in analytic sciences and applications. Among such ones in the literature, the quasi-metric is the simplest one not necessarily symmetric. In fact, a quasi-metric δ satisfies all axioms of a metric except the symmetry $\delta(x, y) = \delta(y, x)$ for all x, y in the space.

Let (X, δ) be a quasi-metric space. A Rus-Hicks-Rhoades (RHR) map $f : X \to X$ due to Rus [31] and Hicks-Rhoades [10] is the one satisfying $\delta(fx, f^2x) \leq \alpha\, \delta(x, fx)$ for every $x \in X$, where $0 < \alpha < 1$. In [25], we began to collect, extend or improve fixed point theorems on RHR maps. It is natural to think that the RHR theorem extends the Banach contraction principle, but we found that it is also close to the multi-valued versions of the Banach principle due to Nadler [18] and Covitz-Nadler [7].

Recently we began to trace the history and to investigate the realm of the RHR maps in several papers [25],[26] and others, which are rich sources to improve the metric fixed point theory.

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For a quasi-metric space (X, δ), a map $f : X \to X$ is called a (Edelstein) *contractive map* if

$$\delta(fx, fy) < \delta(x, y) \text{ holds for all } x,\ y \in X \text{ with } x \neq y.$$

In the present article, we define a *weakly contractive map* $f : X \to X$ is the one satisfying

$$\delta(fx, f^2x) < \delta(x, fx) \text{ holds for all } x \in X \text{ with } x \neq fx.$$

It is not well-known that the Banach contraction principle holds for quasi-metric spaces. Our aim in this article is to show some well-known theorems on metric spaces also hold for quasi-metric spaces from the beginning; for references, see [26].

This article is organized as follows: Section 2 introduces basic concepts on quasi-metric spaces. It is known that almost all metric fixed point theorems hold for quasi-metric spaces. In Section 3, we derive our earlier basic fixed point theorems in topological spaces as basis of the present paper.

Section 4 is to introduce our 2023 Metatheorem in the ordered fixed point theory as a basis of various types of fixed point theorems for quasi-metric spaces. In Section 5, we apply the Metatheorem to compact quasi-metric spaces and obtain new results on various types of maps or multimaps.

Section 6 is to introduce or improve historically well-known theorems on the weakly contractive maps or generalized contractive maps by applying basic principles on topological spaces. Moreover, our Metatheorem can be applied to such examples and hence we can obtain many new theorems on them.

Finally, Section 7 is to give final comments or some conclusion.

2. Preliminaries

Recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $\delta : X \times X \to \mathbb{R}^+ = [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $\delta(x, y) = \delta(y, x) = 0 \Longleftrightarrow x = y$;
(b) (triangle inequality) $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$.

A *metric* on a set X is a quasi-metric δ satisfying that for all $x, y \in X$,

(c) (symmetry) $\delta(x, y) = \delta(y, x)$.

Definition 2.2. Let (X, δ) be a quasi-metric space and $f : X \to X$ a selfmap. The *orbit* of f at $x \in X$ is the set

$$O_f(x) = \{x, fx, \cdots, f^n x, \cdots\}.$$

The space X is said to be *f-orbitally complete* if every Cauchy sequence in $O_f(x)$ is convergent in X. A selfmap f of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} x_n = x_0 \implies \lim_{n\to\infty} fx_n = fx_0$$

for any sequence $\{x_n\}$ of X.

Remark 2.3. Definition 2.2 also works for a topological space X and a function $\delta : X \times X \to \mathbb{R}^+ = [0, \infty)$ such that $\delta(x, y) = 0$ implies $x = y$ for $x, y \in X$.

Every quasi-metric induces a metric, that is, if (X, δ) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = \max\{\delta(x, y), \delta(y, x)\}$$

is a metric on X; see Jleli-Samet [12].

We borrow the following from Alsulami et al. [1]:

Definition 2.4. We will say that a self-mapping $f : X \to X$ on a quasi-metric space (X, δ) is

(i) asymptotically right-regular at a point $x \in X$ if $\lim_{n\to\infty} \delta(f^n x, f^{n+1}x) = 0$;
(ii) asymptotically left-regular at a point $x \in X$ if $\lim_{n\to\infty} \delta(f^{n+1}x, f^n x) = 0$;
(iii) asymptotically regular if it is both asymptotically right-regular and asymptotically left-regular.

3. Some basic fixed point theorems

In this section, we introduce some basic fixed point theorems in topological spaces.

The following basic principle improves the one given in our earlier work [21] in 1980:

Basic Principle. *Let f be a selfmap of a topological space X and $\delta : X \times X \to [0, \infty)$ be a a lower semicontinuous function such that $\delta(x, y) = 0$ implies $x = y$. If there exists a $u \in X$ such that $\lim_i \delta(f^i u, f^{i+1}u) = 0$ and if $\{f^i u\}$ has a convergent subsequence with a limit $\xi \in X$, then ξ is a fixed point of f if and only if f is orbitally continuous at ξ.*

For a quasi-metric space (X, δ), this principle implies: If a selfmap f is asymptotically regular at $u \in X$ and $\{f^i u\}$ has a cluster point $\xi \in X$, then $\xi = f\xi$ iff f is orbitally continuous at ξ.

In [21], we showed that the above principle implies a large number of metric fixed point theorems for contractive type maps satisfying many contractive type conditions in the well-known list of Billy E. Rhoades [29] and many others shown in Park [22].

We give a basic principle for weakly contractive maps. The original version of the following was given for a metric space as a main result in [21]:

Theorem A. *Let f be a selfmap of a topological space X and $\delta : X \times X \to \mathbb{R}^+ = [0, \infty)$ a function such that $\delta(x, y) = 0$ implies $x = y$ for $x, y \in X$. If*

(i) *there exists a point $u \in X$ such that $O_f(u)$ has a cluster point $\xi \in X$,*
(ii) *f satisfies*

$$\delta(fx, f^2x) < \delta(x, fx)$$

for all $x \in \overline{O_f(u)}$ with $x \neq fx$.

Then ξ is a fixed point of f if and only if f is orbitally continuous at ξ and $f\xi$.

Proof. Setting $c_i = \delta(f^i u, f^{i+1}u)$, we have $c_{i+1} \leq c_i$. Therefore, $\{c_i\}$ is monotone decreasing and bounded also. Then $c_i \to l$ as $i \to \infty$, where $l = \inf\{c_i\}$.

Suppose that f is orbitally continuous at ξ and $f\xi$. Since a subsequence $\{f^{i_k}u\}$ converges to $\xi \in X$ by (i), we have

$$f^{i_k+1}u = f(f^{i_k}u) \to f\xi \quad \text{and} \quad f^{i_k+2}u = f(f^{i_k+1}u) \to f^2\xi$$

as $k \to \infty$. Thus we have

$$l = \lim_k \delta(f^{i_k}u, f^{i_k+1}u) = \delta(\xi, f\xi),$$
$$l = \lim_k \delta(f^{i_k+1}u, f^{i_k+2}u) = \delta(f\xi, f^2\xi).$$

Suppose $\xi \neq f\xi$. Then we have

$$\delta(f\xi, f^2\xi) < \delta(\xi, f\xi),$$

which is impossible. Hence we have $\xi = f\xi$.

Conversely, suppose that $\xi = f\xi$. Since $f^{i_k}u \to \xi$ for some $\{i_k\}$, we have

$$f(f^{i_k}u) \to f\xi = \xi \quad \text{and} \quad f^2(f^{i_k}u) \to f^2\xi = f\xi = \xi.$$

Therefore, f is orbitally continuous at ξ. □

Remark 3.1. (1) The sufficiency of Theorem A was originally given as [21, Theorem 1] for metric spaces without using the symmetry. For a contractive map f, the condition (i) is needed in order to ensure that every such f possesses a fixed point (Rhoades [29], Theorem 2).

(2) The requirement (i) is implied by the compactness of X. In fact, if $\overline{O_f(u)}$ or X is compact, the condition (i) is not necessary.
(3) In Theorem A, if f is contractive, then f has a unique fixed point. Hence, we obtain Edelstein's theorem on contractive maps [8].
(4) In [21], we listed a large number of historically well-known consequences of Theorem A for metric spaces due to Pal-Maiti, Pal-Maiti-Achari, Rhoades, Wong, Meir-Keeler, Ćirić, Husain-Sehgal, and Taskovitz.
(5) Billy E. Rhoades [31] in 2007 noted that the original form of Theorem A (in Park [21]) and "other one contain as special cases a number of papers involving contractive conditions not covered by my Transactions paper."

4. From our metatheorem

In order to find some equivalents of the Ekeland variational principle, we obtained a metatheorem in 1985-1986. Later we found more additional equivalents and, consequently, we obtained several extended versions of the metatheorem in 2022 and 2023; see [23], [24].

At last, we obtained the following form called the new 2023 Metatheorem. For its history, see [24]:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y \in X$. Then the following are equivalent:*

(α) *There exists an element $v \in A$ such that the negation of $G(v,w)$ holds for any $w \in X\backslash\{v\}$.*

($\beta 1$) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then f has a fixed element $v \in A$, that is, $v = fv$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq fx$, there exists a $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\gamma 1$) *If $f : A \to X$ is a map such that $G(x,fx)$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = fv$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $G(x,fx)$ for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*

($\delta 1$) *If $F : A \multimap X$ is a multimap such that, for any $x \in A\backslash Fx$ there exists $y \in X\backslash\{x\}$ satisfying $G(x,y)$, then F has a fixed element $v \in A$, that is, $v \in Fv$.*

($\delta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A\backslash Fx$ there exists $y \in X\backslash\{x\}$ satisfying $G(x,y)$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*

($\epsilon 1$) *If $F : A \multimap X$ is a multimap satisfying $G(x,y)$ for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then F has a stationary element $v \in A$, that is, $\{v\} = Fv$.*

($\epsilon 2$) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $G(x,y)$ holds for any $x \in A$ and any $y \in Fx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $G(x,z)$, then there exists a $v \in A \cap Y$.*

Here all multimaps have non-empty values. For the proof, see Park [23, 24]. Note that, when $\mathfrak{F}$ is a singleton, ($\beta 2$)-($\epsilon 2$) reduce to ($\beta 1$)-($\epsilon 1$), respectively.

5. Metatheorem for compact quasi-metric spaces

From Basic Principle or Theorem A, we immediately have the following:

Theorem 5.1. *Let (X,δ) be a compact quasi-metric space and $f : X \to X$ be a weakly contractive map, that is,*

$$\delta(fx, f^2x) < \delta(x, fx) \quad \text{for any } x \in X \text{ with } x \neq fx.$$

Then f has a fixed point $\xi \in X$ if and only if f is orbitally continuous at ξ.

Example 5.2. Let $X = \{0\} \cup \{1/n\}_{n=1}^{\infty} \cup \{2\} \subset \mathbb{R}$ and $f : X \to X$ be a continuous map such that

$$f(0) = 0, \ f(\frac{1}{n}) = \frac{1}{n+1}, \ f(2) = 2.$$

Then X is a compact metric space and it is easy to check that $f : X \to X$ is a weakly contractive map and has two fixed points. Note that f is not a contractive map.

Let (X, δ) be a quasi-metric space. In this section, we apply Metatheorem to a particular situation when $f : X \to X$ is a map and $G(x, y)$ means $\delta(fx, fy) < \delta(x, y)$ for $x, y \in X$ with $x \neq y$.

Let $\mathrm{Cl}(X)$ denote the family of all nonempty closed subsets of X (not necessarily bounded). For $A, B \in \mathrm{Cl}(X)$, set

$$H(A, B) = \max\{\sup\{\delta(a, B) : a \in A\},\ \sup\{\delta(b, A) : b \in B\}\},$$

where $\delta(a, B) = \inf\{\delta(a, b) : b \in B\}$. Then H is called a generalized Hausdorff metric since it may have infinite values.

In the following, multimaps have nonempty closed values:

Theorem B. *Let (X, δ) be a quasi-metric space. Then the following statements are equivalent:*

- (α1) *If $f : X \to X$ is a map, there exists a point $v \in X$ such that $\delta(fv, fw) \geq \delta(v, w)$ for any $w \in X \backslash \{v\}$.*
- (α2) *If $F : X \to \mathrm{Cl}(X)$ is a multimap, there exists a point $v \in X$ such that $H(Fv, Fw) \geq \delta(v, w)$ for any $w \in X \backslash \{v\}$.*
- (β1) *If $f : X \to X$ is a map such that, for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(fx, fy) < \delta(x, y)$, then f has a fixed point $v \in X$, that is, $v = fv$.*
- (β2) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X$ with $x \neq fx$, there exists a $y \in X \backslash \{x\}$ satisfying $\delta(fx, fy) < \delta(x, y)$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*
- (γ1) *If If $f : X \to X$ is a map satisfying $\delta(fx, f^2x) < \delta(x, fx)$ for all $x \in X$ with $x \neq fx$, then f has a fixed point $v \in X$, that is, $v = fv$.*
- (γ2) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $\delta(fx, f^2x) < \delta(x, fx)$ for all $x \in X$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v = fv$ for all $f \in \mathfrak{F}$.*
- (δ1) *If $F : X \to \mathrm{Cl}(X)$ is a multimap such that, for any $x \in X \backslash Fx$ there exists a $y \in X \backslash \{x\}$ satisfying $H(Fx, Fy) < \delta(x, y)$, then F has a fixed point $v \in X$, that is, $v \in F(v)$.*
- (δ2) *If $\mathfrak{F}$ is a family of multimaps $F : X \to \mathrm{Cl}(X)$ such that $H(Fx, Fy) < \delta(x, y)$ holds for any $x \in X$ and any $y \in Fx \backslash \{x\}$, then $\mathfrak{F}$ has a common fixed point $v \in X$, that is, $v \in Fv$ for all $F \in \mathfrak{F}$.*
- (ϵ1) *If $F : X \to \mathrm{Cl}(X)$ is a multimap such that, $H(Fx, Fy) < \delta(x, y)$ holds for any $x \in X$ and any $y \in Fx \backslash \{x\}$, then F has a stationary point $v \in X$, that is, $\{v\} = Tv$.*

(ϵ2) *If $\mathfrak{F}$ is a family of multimaps $F : X \to \mathrm{Cl}(X)$ such that $H(Fx, Fy) < \delta(x, y)$ holds for any $x \in X$ and any $y \in Fx\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary point $v \in X$, that is, $\{v\} = Fv$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in X\backslash Y$ there exist a $z \in X\backslash\{x\}$ and a map $f : X \to X$ satisfying $\delta(fx, fz) < \delta(x, z)$, then there exists a $v \in X \cap Y = Y$.*

Proof. Equivalency follows from Metatheorem by putting $\delta(fx, fy) < \delta(x, y)$ instead of $G(x, y)$. □

Remark 5.3. (1) In case all f is continuous and X is compact in Theorem B, all (α)-(η) are true. In fact, let a map $\varphi : X \to \mathbb{R}^+$ by putting

$$\varphi(x) = \delta(x, fx), \quad x \in X.$$

Then φ is continuous and bounded below, so it has a minimum value at a point $v \in X$. Hence (α1) holds. Moreover, (β1)–(η) also hold by Metatheorem.

(2) Theorem B is a new source of many consequences. Note that (δ1) and (ϵ1) would be the Covitz-Nadler type extensions of the well-known Edelstein fixed point theorem [8] in 1962.

Corollary B1. *Let f be an orbitally continuous selfmap of a compact quasi-metric space (X, δ). Then the equivalent statements (α1)-(η) in Theorem B hold.*

Proof. Since (γ1) is true by Theorem 5.1, we have Theorem B. Then the conclusion follows from Theorem B. □

Corollary B2. *Let (X, δ) be a compact quasi-metric space and $f : X \to X$ be an orbitally continuous weakly contractive selfmap. Then f has a fixed point.*

Note that Corollary B2 generalizes the Edelstein fixed point theorem [8].

6. Applications to weakly contractive maps and others

In this section, we recall historically well-known examples of the weakly contractive maps or generalizations of contractive maps. We show that they can be derived from Basic Principle or Theorem A. Moreover, our Metatheorem can be applied to some examples and hence we can obtain many new facts on them. It should be emphasized that all metric spaces in this section can be extended to the spaces in Basic Principle or Theorem A. This is rather surprising thing. Moreover, our Metatheorem can be applied to such examples and hence we can obtain many new facts on them.

Banach in 1922

From Basic Principle, we obtain the essence of the Banach contraction principle as follows:

Theorem 6.1. *Let (X, δ) be a quasi-metric space, $f : X \to X$ be a map, and $u \in X$ such that*

(i) *f is asymptotically regular at u (that is, $\lim_i \delta(f^i u, f^{i+1}u) = 0$);*
(ii) *$\{f^i u\}$ has a cluster point $\xi \in X$.*

Then $\xi = f\xi$ if and only if f is orbitally continuous at ξ.

Nemytskiĭ [19] in 1936

This paper gave a typical example of Theorem A with the following important Corollary.

Theorem 6.2 ([19])**.** *If the metric space (X, ρ) is compact, then every contractive map $f : X \to X$ has a unique fixed point in X. Moreover, for any $x_1 \in X$ the sequence defined by $x_{n+1} = f(x_n)$, $n \in \mathbb{N}$, converges to the fixed point of the map f.*

We borrowed this from Cobzaş [6]. This theorem follows from our Basic Principle or Theorem A.

In fact, for any $x_1 \in X$, consider the sequence $\{x_i\}$ defined by $x_{i+1} = fx_i$ for $i \in \mathbb{N}$. If $x_n = fx_n$ for some $n \in \mathbb{N}$, then we have done. If $x_n \neq fx_n$ for all $n \in \mathbb{N}$, then there exists a cluster point $\xi \in X$ of $O_f(x_1)$ since X is compact. Therefore (i) and (ii) of Theorem A are satisfied. Since f is continuous, by Theorem A, ξ is a fixed point of f. Uniqueness follows from $d(x, y) = d(fx, fy) < d(x, y)$ if x, y are different fixed points.

Edelstein [8] in 1962

Edelstein established a relative of the Banach contraction principle for contractive maps and state the following version of the Banach contraction principle:

Theorem 6.3 ([8])**.** *Let (X, ρ) be a metric space and $f : X \to X$ be a contractive map. If there exists $x \in X$ such that the sequence of iterates $f^n(x)$ has a cluster point $\zeta \in X$ (that is, $\zeta = \lim_{k\to\infty} f^{n_k}(x)$ for some subsequence of $\{f^n(x)\}$, then ζ is the unique fixed point of f.*

This follows from Theorem A. Note that f is continuous.

Theorem 6.4 ([8])**.** *Let (X, ρ) be a compact metric space and $f : X \to X$ be a contractive map. If there exists $x \in X$ such that the sequence of iterates $f^n(x)$ has a limit point $\zeta \in X$, then ζ is the unique fixed point of f.*

This is same to Theorem 5.2 of Nemytskiĭ [19] in 1936, and holds any topological space X with a function $\delta : X \times X \to \mathbb{R}^+ = [0, \infty)$ such that $\delta(x, y) = 0$ implies $x = y$ for $x, y \in X$.

Note that contractive maps satisfy all requirements of Basic Principle or Theorem A for compact metric spaces.

Kannan [13] in 1969

Kannan proved the following theorem:

Theorem 6.5 ([13])**.** *If a self-mapping T of a complete metric space (X, d) satisfies the condition:*

$$d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)],\ 0 \leq b < 1/2,\ \textit{for each } x, y \in X,$$

then T has a unique fixed point.

The Kannan contractive condition does not require the continuity of the mapping T for the existence of the fixed point. However, a map T satisfying Kannan contractive condition turns out to be orbitally continuous at the fixed point.

Kannan's paper was followed by a flood of papers involving contractive definitions, many of which did not require continuity of the mapping.

Note that T is a weakly contractive map and that Theorem 6.5 follows from Basic Principle.

Boyd and Wong [5] in 1969

One of the most interesting generalizations of the Banach-Picard-Caccioppoli contraction principle consists of replacing the Lipschitz constant k by some real valued function whose functional values are less than 1. In 1969, Boyd and Wong [5] initiated the work along these lines and proved the following theorem:

Theorem 6.6 ([5])**.** *Let T be a mapping of a complete metric space (X, d) into itself. Suppose there exists a function ϕ, upper semicontinuous from right from $\mathbb{R}^+$ into itself such that*

$$d(Tx, Ty) \leq \phi(d(x, y)) \quad \textit{for all } x, y \in X.$$

If $\phi(t) < t$ for each $t > 0$, then T has a unique fixed point.

For $x \neq y = Tx$, the Boyd-Wong map becomes a weakly contractive map satisfying $d(Tx, T^2x) < d(x, Tx)$ for $x \neq Tx$.

In [5], "for each $x \in X$, $\{T^n x\}$ is a Cauchy sequence. This will complete the proof, since the limit of this sequence is a fixed point of T which is clearly unique."

Note that (i) and (ii) of Theorem A hold for Theorem 6.6. Moreover,

$$d(Tx, Ty) \leq \phi(d(x, y)) < d(x, y) \text{ for all } x, y \in X \text{ with } x \neq y.$$

This implies T is continuous. Hence Theorem 6.6 follows from Theorem A.

Meir and Keeler [16] in 1969

Let (X, d) be a metric space. A map $T : X \to X$ is called a Meir-Keeler map on X (on short, MK-map), if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \implies d(Tx, Ty) < \varepsilon$$

for every $x, y \in X$.

Theorem 6.7 ([16]). *A MK-map has a fixed point if X is complete.*

A MK map is a contractive map, that is, $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$ with $x \neq y$, and a weakly contractive map, that is, $d(Tx, T^2x) < d(x, Tx)$ for all $x \in X$ with $x \neq Tx$. If X is complete, then we can show (i) of Theorem A holds. Moreover, any T is continuous. Consequently, Theorem 6.7 follows from Theorem A.

Sehgal [33] in 1972

Sehgal introduced the following contractive definition:

$$d(fx, fy) < \max\{d(x, y), d(x, fx), d(y, fy)\}$$

with $x \neq y$, the continuity of f and a convergent subsequence of $\{f^n x_0\}$ for an x_0.

Note that $d(fx, f^2x) < d(x, fx)$ for $x \neq fx$. Hence, f is a weakly contractive map and satisfies all requirements of Basic Principle or Theorem A.

Hardy and Rogers [9] in 1973

The following theorem is the principal result of this paper.

Theorem 6.8. *Let (M, d) be a metric space and T a self-mapping of M satisfying the condition*

(1) *for $x, y \in M$,*

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y),$$

where a, b, c, e, f are nonnegative and we set $\alpha = a + b + c + e + f$. Then

(a) *If M is complete and $\alpha < 1$, T has a unique fixed point.*

(b) *If (1) is modified to the condition*

(1′) *$x \neq y$ implies $d(Tx, Ty) < ad(x, Tx) + bd(y, Ty) + cd(x, Ty) + ed(y, Tx) + fd(x, y)$,*

and in this case we assume M is compact, T is continuous and α=1, then T has a unique fixed point.

Reich in [28] obtained a similar conclusion to that in (a) in the case that $\alpha = a + b + f$. Reich's result in turn generalizes the fixed point theorem of Kannan [13] in which $\alpha = a + b$. The conclusion in part (b) is a limiting version of the theorem contained in part (a), and Edelstein obtained this result in [8] for the case $\alpha = f = 1$.

Billy E. Rhodes [29] in 1977

In this historical article, he listed the basic 25 conditions all of them are RHR maps except

$$(6), (10), (13), (22), (24), \text{ and } (25).$$

Each of them implies as follows:

(6), (10), (22): $d(fx, f^2x) < d(x, fx)$
(13): $d(fx, f^2x) < d(x, f^2x)$
(24): Ćirić (1974) $d(fx, f^2x)) \leq h \max\{d(x, fx), d(x, f^2x)\},\ h \in [0, 1)$
(25): $d(fx, f^2x) < \max\{d(x, fx), d(x, f^2x)\},\ x \neq fx$

Therefore, the other 19 conditions are applicable our RHR Theorem in [25], [26] for f-orbitally complete quasi-metric spaces. For weakly contractive maps in (6), (10), (22), we can apply our methods in this paper.

Sehgal's condition [33] in 1972 is extended as follows:

$$d(fx, fy) < \max\{d(x, y), d(x, fx), d(y, fy), [d(x, fy) + d(y, fx)]/2\}$$

with the same requirements of Sehgal [33]. Note that $d(fx, f^2x) < d(x, fx)$ for $x \neq fx$. Hence, f is a weakly contractive map and satisfies all requirements of Basic Principle or Theorem A. Therefore, f has a fixed point.

Park and Rhoades [27] in 1981 also noted that the continuity and the existence of a convergent subsequence are also necessary, although the continuity of f need hold only in a neighborhood of the fixed point. This is extended to the orbital continuity at the fixed point as in Basic Principle or Theorem A.

Pant [20] in 1999

In 1982 Billy Rhoades raised an open question [30, p.242] on the existence of a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point and provide one such contractive definition. This was solved by Pant [20] as follows:

If f is a self-map of a metric space (X, d), we denote:

$$m(x, y) = \max\{d(x, fx), d(y, fy)\}.$$

Also, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ denote a function such that $\phi(t) < t$ for each $t > 0$.

Theorem 6.9. ([20]) *Let f be a self-mapping of a complete metric space (X, d) such that for any x, y in X*

(i) $d(fx, fy) \leq \phi(m(x, y))$,
(ii) *given $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\varepsilon < m(x, y) < \varepsilon + \delta \implies d(fx, fy) \leq \varepsilon.$$

Then f has a unique fixed point, say z. Moreover, f is continuous at z if and only if $\lim_{x\to z} m(x, z) = 0$.

For $x \neq y = fx$, (i) and (ii) implies

$$d(fx, f^2x) \leq \phi(m(x, fx)) < m(x, fx) = d(x, fx).$$

Hence f is a weakly contractive map which is not continuous at the fixed point z. However, it is orbitally continuous at z by the Basic Principle or Theorem A.

Berinde [2] in 2004

Berinde [2] introduced the notion of almost contractions: A map $T : U \to U$, where (U, ρ) is a metric space, is called almost contraction provided that it satisfies

$$\rho(Tu, Tv) \leq q\,\rho(u, v) + K\rho(v, Tu),$$

where $q \in (0, 1)$ and $K \geq 0$.

Berinde [2] generalized the Banach contraction principle by proving the existence of fixed points for almost contractions defined on complete metric spaces.

Similarly, we can define an almost contractive map $T : U \to U$ provided that it satisfies $\rho(Tu, Tv) < \rho(u, v) + K\rho(v, Tu)$, where $K \geq 0$.

Then an almost contractive map $T : U \to U$ is a weakly contractive map; that is, $\rho(Tx, T^2x) < \rho(x, Tx)$ for all $x \in X$ with $x \neq Tx$. Therefore, all requirements of Theorem A hold for such almost contractive maps.

Jachymski [11] in 2009

From [11] : Let T be a selfmap of a metric space (X, d). We say that $x_* \in X$ is a contractive fixed point (CFP) of T if $x_* = Tx_*$ and $\{T^n x\}_{n\in\mathbb{N}}$ converges to x_* for any $x \in X$. (T is called a Picard operator.)

We call $x_* \in X$ is an approximate fixed point (AFP) of T if $x_* = Tx_*$ and, for any sequence $\{x_n\}_{n\in\mathbb{N}}$,

$$d(x_n, Tx_n) \to 0 \text{ implies } x_n \to x_*.$$

Proposition 6.10. ([11]) *Let T be a continuous selfmap of a compact metric space (X, d). The following statements are equivalent:*

(i) *T has a unique fixed point;*
(ii) *T has an AFP.*

Corollary 6.11. ([11]) *Let (X, d) be a compact metric space and $T : X \to X$ be an Edelstein contractive map. Then T has a fixed point which is both approximate and contractive.*

Note that Proposition 6.10 is closely related to Basic Principle and Theorem A.

Suzuki [34] in 2009

Suzuki proved a generalization of Edelstein's fixed point theorem. Though there are thousands of fixed point theorems for metric spaces, he claimed his theorem is a new type of theorem.

The following is his main result:

Theorem 6.12. ([34]) *Let (X, d) be a compact metric space and let T be a selfmap on X. Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \quad \textit{implies} \quad d(Tx, Ty) < d(x, y)$$

for $x, y \in X$. Then T has a unique fixed point.

Here we should assume $x \neq y$. Suzuki showed that T is orbitally continuous.

In fact, in the proof of Suzuki's Theorem 6.12, he put $\beta = \inf\{d(x, Tx) : x \in X\}$ and choose a sequence $\{x_n\}$ in X satisfying $\lim_n d(x_n, Tx_n) = \beta$. Since X is compact, without loss of generality, we may assume that $\{x_n\}$ and $\{Tx_n\}$ converge to some elements $v, w \in X$, respectively. He showed $d(v, w) = \beta = 0$.

Then Theorem 6.12 follows from the Basic Principle.

Another proof is possible. For $y = Tx$, the map T in Theorem 6.12 satisfies

$$d(Tx, T^2x) < d(x, Tx) \text{ for } x \in X \text{ with } x \neq Tx.$$

Hence T is a weakly contractive map and Theorem A works.

Suzuki next proved that 1/2 in Theorem 6.12 is the best constant:

Theorem 6.13 ([34]) *For every $\eta \in (1/2, \infty)$, there exist a compact metric space (X, d) and a mapping T on X satisfying the following:*

- *T has no fixed points.*
- *$\eta\, d(x, Tx) < d(x, y)$ implies $d(Tx, Ty) < d(x, y)$ for all $x, y \in X$.*

For $x \neq y = Tx$, $\eta\, d(x, Tx) < d(x, y)$ holds for $\eta \in (0, 1)$. In this case, Theorem 6.13 violates the Edelstein theorem.

Karapinar [14] in 2011

In this manuscript, main theorem is on the form of Edelstein's fixed point theorem and generalizes the results of both [14] and Suzuki [34].

Theorem 6.14. ([14]) *Let T be a self mapping on a compact metric space (X, d). Assume that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \Longrightarrow d(Tx, Ty) < M(x, y) \text{ for all } x, y \in X,$$

where

$$M(x, y) = \max\{d(x, y), d(Tx, x), d(y, Ty), \frac{1}{2}d(Tx, y), \frac{1}{2}d(x, Ty)\}.$$

Then, T has a unique fixed point $z \in X$, that is, $Tz = z$.

In [15], Suzuki's method is applied to show the orbital continuity of T.

For $x \neq y = Tx$, we have $\frac{1}{2}d(x, Tx) < d(x, Tx)$ and

$$\begin{aligned} M(x, Tx) &= \max[d(x, Tx), d(Tx, T^2x), \frac{1}{2}d(x, T^2x)] \\ &\leq \max\{d(x, Tx), d(Tx, T^2x), \tfrac{1}{2}[d(x, Tx) + d(Tx, T^2x)]\}. \end{aligned}$$

Therefore

$$d(Tx, T^2x) < M(x, Tx) \implies d(Tx, T^2x) < d(x, Tx),$$

and hence T is a weakly contractive map. Then Theorem 6.14 follows from Theorem A.

Karapinar [15] in 2012

We introduce two theorems closely related to Basic Principle.

Theorem 6.15. ([15]) *Let T be a self mapping on a compact metric space (X, d). Assume that*

$$\tfrac{1}{2}d(x,Tx) < d(x,y) \Longrightarrow$$
$$d(Tx,Ty) < \frac{1}{2}[d(Tx,x) + d(Ty,y)] + L\min[d(y,Tx), d(x,Ty), d(x,y)]$$

for all $x, y \in X$ with $x \neq y$ and $L > 0$. Then, T has a fixed point $z \in X$, that is, $Tz = z$.

In [15], Suzuki's method is applied to show the orbital continuity of T.

For $x \neq y = Tx$, we have

$$\frac{1}{2}d(x,Tx) < d(x,Tx) \text{ and } \min[d(y,Tx), d(x,Ty), d(x,y)] = d(y,Tx) = 0.$$

Then

$$d(Tx,T^2x) < \frac{1}{2}[d(Tx,x) + d(Ty,y)] \implies d(Tx,T^2x) < d(x,Tx),$$

and hence T is a weakly contractive map. Then Theorem 6.15 follows from Theorem A.

In [15], several variants of Theorem 6.15 were given.

Bisht and Pant [3] in 2017

Abstract: In this paper, we investigate some contractive definitions which are strong enough to generate a fixed point but do not force the mapping to be continuous at the fixed point. We also obtain a fixed point theorem for generalized nonexpansive mappings in metric spaces by employing Meir-Keeler type conditions.

Bisht-Pant [3] stated: "In a survey paper of contractive definitions, Rhoades [30] compared 250 contractive definitions and showed that majority of the contractive definitions do not require the mapping to be continuous in the entire domain. However, in all the cases the mapping is continuous at the fixed point. He further demonstrated that the contractive definitions force the mapping to be continuous at the fixed point though continuity was neither assumed nor implied by the contractive definitions. The question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point was reiterated by Rhoades in [31] as an existing open problem."

The question of the existence of contractive mappings which are discontinuous at their fixed points was settled in the affirmative by Pant [20]. Recently, Bisht and Pant [3] also gave a contractive definition which does not force the map to be continuous at the fixed point. In this note we provide more solutions to the open question of the existence of contractive definitions which are strong enough to generate a fixed point but which do not force the mapping to be continuous at the fixed point.

In what follows we shall denote

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/2\};$$

$$N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), a[d(x,Ty)+d(y,Tx)]/2\},\ 0 \le a < 1.$$

Theorem 6.16. ([3]) *Let (X,d) be a complete metric space. Let T be a self-mapping on X such that for any $x, y \in X$;*

(i) *$d(Tx,Ty) \le \phi(N(x,y))$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $\phi(t) < t$ for each $t > 0$;*
(ii) *for a given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\epsilon < M(x,y) < \epsilon + \delta$ implies $d(Tx,Ty) \le \epsilon$.*

Suppose T is orbitally continuous. Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is continuous at z iff $\lim_{x\to z} M(x,z) = 0$.

Note that, for $y = Tx$, we have

$$M(x,Tx) = \max\{d(x,Tx), d(Tx,T^2x)\};\ N(x,Tx) = \max\{\{d(x,Tx), d(Tx,T^2x)\}.$$

Hence (i) and (ii) becomes

$$d(Tx,T^2x) < N(x,Tx) < d(x,Tx);\quad d(Tx,T^2x) < M(x,Tx) < d(x,Tx),$$

respectively. Hence T is a weakly contractive map.

Miñana and Valero [17] in 2019

It is stated in [17]: Taking into account the exposed facts about G-metric spaces and quasi-metric spaces, we are able to show that most fixed point results obtained in G-metric spaces can be deduced from a fixed point result stated in quasi-metric spaces obtained by Park in [21]. To this end, let us recall such a result:

Theorem 6.17. ([17]) *Let (X,τ) be a topological space, let $\delta : X \times X \to [0,\infty)$ be a continuous mapping, such that $\delta(x,y) = 0 \iff x = y$, and let $f : X \to X$ be a mapping. Suppose that there exist $x,\ x_0 \in X$, such that the following conditions hold:*

(1) $\lim_{n\to\infty} \delta(f^n(x_0), f^{n+1}(x_0)) = 0$;
(2) $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to x with respect to δ;*
(3) *f is orbitally continuous at x with respect to δ.*

Then $x \in Fix(f) = \{y \in X : f(y) = y\}$.

This is a variant of our Basic Principle.

7. Conclusion

In the previous section, we gave simple proofs of a number of well known theorems in the metric fixed point theory by applying our Basic Principle or Theorem A. Moreover, we can obtain some equivalent formulations of known theorems by applying our Metatheorem.

In the metric fixed point theory, many theorems based on the completeness or compactness of the metric spaces, and were used even to characterize the metric completeness. In this paper, we showed that such completeness or compactness can be replaced by some mild conditions as shown in Basic Principle or Theorem A.

Our methods in this paper will be helpful to destroy certain artificial results in the metric fixed point theory.

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