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SINGULAR PDE'S GEOMETRY AND BOUNDARY VALUE PROBLEMS

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Abstract. Local and global existence theorems for boundary value problems in singular PDE's are considered. In particular, surgery techniques and integral bordism groups are utilized, following previous works by A.Prástaro on PDE's, in order to build global solutions crossing also singular points and to study their stability properties.

1. Introduction

The geometric theory of PDE's generally works in some regularity conditions where general theorems can be built in order to obtain local and global solutions existence theorems. However, the great complexity of natural and mathematical phenomena, requires also to handle "singular PDE's". The aim of this paper is just to give a geometric characterization of such mathematical objects and to find general existence theorem of local and global solutions passing through singular points of PDE's. The characterization of singularities can be obtained by means of algebraic-geometric techniques and algebraic topological differential techniques. These induce effects on the integral structure of PDE's, described by means of their Cartan distributions, and formal prolongations properties. There the symbol of a PDE plays a fundamental role. Singular points in PDE's are sources of interesting phenomena, thus we can say that they instead to disturb the integral structure, they constitute a more richness and contribute to more versatile behaviours in PDE's solutions.

On the other hand singularities are also origins of unstabilities. In some sense we could say that singular points in PDE's are like doors that open new possibilities to solutions passing through them.

The main results of this paper are the following. Theorem 3.10 that relates singular integral bordism groups of PDE's to global solutions passing through singular points of PDE's. Theorem 4.2 that characterizes singular ODE's singular solutions and their stability. Theorem 4.4 that completes Theorem 4.2 on the same subject emphasizing the role played by singular points to conserve the smoothness or to create bifurcation in solutions of singular ODE's.

Some examples of singular PDE's and ODE's are considered in some details in order to show how our general theory works. In particular, for singular ODE's are considered also some examples just studied by R.P.Agarwal and D.O'Regan in the framework of the functional analysis [1, 3]. We show how by using only our

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geometric framework one can obtain local and global existence theorems also for singular boundary value problems.

2. Singular PDE's and local solutions existence theorems

In this section we resume some fundamental definitions and results on the geometry of PDE's in the category of commutative manifolds, emphasizing some our recent results on the algebraic geometry and topology of PDE's, that allowed us to characterize singular PDE's [2, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33].[1]

Let W be a smooth manifold of dimension $m+n$. For any n-dimensional submanifold $N \subset W$ we denote by $[N]^k_a$ the *k-jet of N at the point $a \in N$*, i.e., the set of n-dimensional submanifolds of W that have in a a contact of order k. Set $J^k_n(W) \equiv \bigcup_{a\in W} J^k_n(W)_a$, $J^k_n(W)_a \equiv \{[N]^k_a | a \in W\}$. We call $J^k_n(W)$ the *space of all k-jets of submanifolds* of dimension n of W. $J^k_n(W)$ has the following natural structures of differential fiber bundles: $\pi_{k,k-1} : J^k_n(W) \to J^{k-1}_n(W)$, with affine fiber $J^k_n(W)_{\bar{q}}$, where $\bar{q} \equiv [N]^{k-1}_a \in J^{k-1}_n(W)$, $a \equiv \pi_{k,0}(\bar{q})$, with associated vector space $S^k(T^*_aN) \otimes \nu_a$, $\nu_a \equiv T_aW/T_aN$. For any n-dimensional submanifold $N \subset W$ one has the canonical embedding $j^k : N \to J^k_n(W)$, given by $j^k : a \mapsto j^k(a) \equiv [N]^k_a$. We call $j^k(N) \equiv N^{(k)}$ the *k-prolongation* of N. In the following we shall also assume that there is a fiber bundle structure on W, $\pi : W \to M$, where $\dim M = n$. Then there exists a canonical submanifold $J^k(W)$ of $J^k_n(W)$ that is called the *k-jet space for sections* of π. $J^k(W)$ is diffeomorphic to the k-jet-derivative space of sections of π, $JD^k(W)$, and has the same dimension of $J^k_n(W)$ [12, 20]. Then, for any section $s : M \to W$ one has the following commutative diagram:

$$
\begin{tikzcd}
JD^k(W) \arrow[r, equal, "\sim"] & J^k(W) \arrow[r, hook] & J^k_n(W) \arrow[d, "\pi_k"] \\
M \arrow[u, "D^k s"] \arrow[r, equal] & M \arrow[u, "j^k s"'] \arrow[r, equal] & M \arrow[u]
\end{tikzcd}
\tag{2.1}
$$

where $D^k s$ is the k-derivative of s and $j^k(s)$ is the k-jet-derivative of s. If $s(M)^{(k)} \subset J^k_n(W)$ is the k-prolongation of $s(M) \subset W$, then one has $j^k(s)(M) \cong s(M)^{(k)} \cong s(M) \cong M$. Of course there are also n-dimensional submanifolds $N \subset W$ that are not representable as image of sections of π. As a consequence, in these cases, $N^{(k)} \cong N$ is not representable in the form $j^k(s)(M)$ for some section s of π. The condition that N is image of some (local) section s of π is equivalent to the following local condition: $s^*\eta \equiv s^*dx^1 \wedge \cdots \wedge dx^n \neq 0$, where $(x^\alpha, y^j)_{1\leq\alpha\leq n, 1\leq j\leq m}$, are fibered coordinates on W, with y^j vertical coordinates. In other words $N \subset W$ is locally representable by equations $y^j = y^j(x^1, \dots, x^n)$. This is equivalent to say that N is transversal to the fibers of π or that the tangent space TN identifies an horizontal distribution with respect to the vertical one $vTW|_N$ of the fiber bundle structure

[1]For general informations on the geometric theory of PDE's see, e.g.,[4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 38, 39, 40]. For basic informations on differential topology and algebraic topology see e.g., [4, 9, 11, 15, 16, 17, 18, 35, 36, 37, 38, 39, 40, 41, 42].

$\pi : W \to M$. Conversely, a completely integrable n-dimensional horizontal distribution on W determines a foliation of W by means of n-dimensional submanifolds that can be represented by images of sections of π.

Definition 2.1. A *partial differential relation* (PDR) of order k on the fibre bundle $\pi : W \to M$, defined in the category of manifolds, $\mathfrak{M}$, is a subset $E_k \subset JD^k(W)$ of the jet-derivative space $JD^k(W)$ over M.

A *partial differential equation* (PDE) of order k on the fibre bundle $\pi : W \to M$, defined in the category of manifolds, $\mathfrak{M}$, is a submanifold $E_k \subset JD^k(W)$ of the jet-derivative space $JD^k(W)$ over M.

A PDE E_k is *regular* if the r-prolongations $(E_k)_{+r} \equiv E_{k+r} \equiv JD^r(E_k) \cap JD^{k+r}(W)$ are subbundles of $\pi_{k+r,k+r-1} : JD^{k+r}(W) \to JD^{k+r-1}(W)$, $\forall r \geq 0$. Furthermore, we say that E_k is *formally integrable* if E_k is regular and if the mappings $E_{k+r+1} \to E_{k+r}$, $\forall r \geq 0$, and $\pi_{k,0} : E_k \to W$ are surjective.

Remark 2.2. In the following we shall consider PDEs on a fiber bundle $\pi : W \to M$, where M is a manifold of dimension n and W is a manifold of dimension $m + n$.

Definition 2.3. The *symbol* g_{k+r} of E_{k+r} is a family of $\mathbb{R}$-modules over E_k characterized by means of the following short exact sequence of $\mathbb{R}$-modules:

$$0 \to \pi^*_{k+r} g_{k+r,k} \to vTE_{k+r} \to \pi^*_{k+r,k+r-1} vTE_{k+r-1}.$$

Proposition 2.4. *If E_k is a formally integrable PDE, then we can complete above exact sequence with the following:*

$$0 \to \pi^*_{k+r,k} g_{k+r} \to vTE_{k+r} \to \pi^*_{k+r,k+r-1} vTE_{k+r-1} \to 0,$$

which means that E_{k+r} is an affine bundle over E_{k+r-1} with associated vector bundle $\pi^*_{k+r-1,k} g_{k+r} \to E_{k+r-1}$.

Proposition 2.5. *The symbol $(g_k)_q$, $q \in E_k$, of a PDE $E_k \subset JD^k(W)$ is a $\mathbb{R}$-submodule of $Hom_{\mathbb{R}}(S^k_0(T_pM); vT_{\bar{q}}W)$, $p = \pi_k(q) \in M$, $\bar{q} = \pi_{k,0}(q) \in W$.*

Theorem 2.6 (δ-complex)**.** *One has the following complex of $\mathbb{R}$-modules over E_k (δ-*complex*):*

$$\begin{aligned} &0 \to g_m \xrightarrow{\delta} Hom_{\mathbb{R}}(TM; g_{m-1}) \xrightarrow{\delta} Hom_{\mathbb{R}}(\Lambda^2_0 M; g_{m-2}) \xrightarrow{\delta} \\ &\dots \xrightarrow{\delta} Hom_{\mathbb{R}}(\Lambda^{m-k}_0 M; g_k) \xrightarrow{\delta} \delta(Hom_{\mathbb{R}}(\Lambda^{m-k}_0; g_k)) \to 0 \end{aligned} \tag{2.2}$$

Proof. See [7]. □

Definition 2.7. We call *Spencer cohomology* of $\hat{E}_k$ the homology of such complex. We denote by $\{H^{m-j,j}_q\}_{q \in E_k}$ the homology at $Hom(\Lambda^j_0 M; g_{m-j})_q$. We say that E_k is r-*acyclic* if $H^{m,j}_q = 0$, $m \geq k$, $0 \leq j \leq r$, $\forall q \in E_k$. We say that E_k is *involutive* if $H^{m,j}_q = 0$, $m \geq k$, $j \geq 0$. We say that E_k is δ-*regular* if there exists an integer $\kappa_0 \geq \kappa$, such that g_{κ_0} is involutive or 2-acyclic.

The following two theorems are fundamental in order to characterize the integrability of PDEs. (For their proofs see [7].)

Theorem 2.8 (δ-Poincaré lemma for PDEs)**.** *Let $E_k \subset JD^k(W)$ be a regular PDE.Then E_k is a δ-regular PDE.*

Theorem 2.9 (Criterion of formal integrability for PDEs)**.** *Let $E_k \subset JD^k(W)$ be a regular, (δ-regular), PDE. Then if g_{k+r+1} is a bundle of $\mathbb{R}$-modules over E_k, and $E_{k+r+1} \to E_{k+r}$ is surjective for $0 \leq r \leq h$, for suitable h, then E_k is a formally integrable PDE.*

Definition 2.10. An *initial condition* for PDE $E_k \subset JD^k(W)$ is a point $q \in E_k$. A *solution* of E_k passing for the initial condition q is a manifold, $N \subset E_k$ such that $q \in N$, $\dim N = n$, and such that N can be represented in a neighboroud of any of its points $q' \in N$, except for a nonwhere dense subset $\Sigma(N) \subset N$ of dimension $\leq n-1$, as image of the k-derivative $D^k s$ of some C^k-section s of $\pi : W \to M$. We call $\Sigma(N)$ the set of *singular points (of Thom-Boardman type)* of N. If $\Sigma(N) \neq \emptyset$ we say that N is a *regular solution* of $E_k \subset J\hat{D}^k(W)$. We shall also consider singular solutions of E_k, manifolds of dimension n, $N \subset E_k$, that can be obtained as projections of ones of the previous type, but contained in some s-prolongation E_{k+s}, $s > 0$.

Definition 2.11. The *Cartan distribution* of $JD^k(W)$, is the distribution $\mathbf{E}_k(W) \subset TJD^k(W)$, spanned by tangent spaces to graphs of k-derivatives of sections of the bundle $\pi : W \to M$. The *Cartan distribution* of $J^k_n(W)$ is the distribution $\mathbf{E}^k_n(W) \subset TJ^k_n(W)$, spanned by the tangent spaces to k-prolongation of manifolds, of dimension n, contained in W.

We call *integral plane* at the point $q \in J^k_n(W)$ the linear subspace in $T_q J^k_n(W)$ of the form, $T_q N^{(k)}$, for some C^k manifold $N \subset W$, of dimension n, passing for $a \in W$, with $a = \pi_{k,0}(q)$. Any point $q \in J^k_n(W)$ identifies a unique regular integral plane L_q at $q' \equiv \pi_{k,k-1}(q) \in J^{k-1}_n(W)$: $L_q \subset T_{q'} J^{k-1}_n(W)$. Set: $I_k(W) \equiv \bigcup_{u \in J^k_n(W)} I_k(W)_u$, $I_k(W)_u \equiv$ Grassmannian of integral planes at q. An *integral plane* at a point $u \in J^k_n(W)$ is defined to be a n-dimensional space, subspace of $(\mathbf{E}^k_n)_u$, tangent to some integral manifold of the Cartan distribution $\mathbf{E}^k_n \subset TJ^k_n(W))$. Let $I(E_{k+s})$ be the fiber bundle of Grassmannian n-dimensional integral planes of the Cartan distribution $\mathbf{E}^{k+s}_n$ on E_{k+s} being $E_k \subset J^k_n(W)$ a PDE. If $E_{k+s} = J^{k+s}_n(W)$ one has $I(J^{k+s}_n(W)) = I_{k+s}(W)$.

Remark 2.12. (A) An *integral manifold* on $J^k_n(W)$ is an integral manifold of its Cartan distribution $\mathbf{E}^k_n$ on $J^k_n(W)$. In particular: *(i)* Maximal integral manifolds of $J^\infty_n(W)$ are of dimension n and are regular ones (without singular points). *(ii)* Maximal integral manifolds, $V \subset J^k_n(W)$, are divided into types, (type$V$). (For details see, e.g., [12, 19].) If $\dim V > \dim V' \Rightarrow$ type$V <$ typeV'. *(iii)* The zero type integral manifolds coincide with the fibers of projections $\pi_{k,k-1}$. *(iv)* Maximal integral manifolds of type n are of dimension n. They have the representation $Z \setminus \Sigma(Z) = \bigcup_i V_i$, as said before, where the regular components V_i are the k^{th} prolongations of n-dimensional submanifolds of W. We call such integral manifolds *solutions* of $J^k_n(W)$ and the corresponding set is denoted by $\mathcal{S}_{ol}(J^k_n(W))$. If the set of singular points is empty, we call such solutions *regular solutions* and the corresponding set is denoted by $\underline{\mathcal{S}}_{ol}(J^k_n(W))$.

(B)(Exceptional cases). $(\alpha)(n = m = 1)$. Integral manifolds are one-dimensional and glued from pieces of type zero or type 1. A piece of type 0 is an open subset of the fibre $\pi^{-1}_{k,k-1}(\bar{q})$, $\bar{q} \in J^{k-1}_n(W)$; A piece of type 1 is an open subset of regular integral manifold (of dimension). $(\beta)(k = m = 1)$. In this case $J^1_n(W)$ becomes a classic contact manifold [20].

Definition 2.13. Let $E_k \subset JD^k(W)$ be a PDE. For a *p-dimensional integral manifold* V of E_k, $0 \leq p \leq n$, with boundary ∂V, (or eventually with $\partial V = \emptyset$), we mean an element $V \in C_p(E_{k+h})$, $h \geq 0$, such that $TV \subset \mathbf{E}_{k+h}$. Here $C_p(E_{k+h})$ denotes the vector space of p-chains in E_{k+h}. So, if $V = \sum_i a^i u_i$, $a_i \in \mathbb{R}$, one has $\partial V = \sum_i (-1)^i a^i \partial_i u$.

Definition 2.14. We call *PDR* (resp. *PDE*) for n-dimensional manifolds, contained into W, a subset, (resp. submanifold), $E_k \subset J^k_n(W)$.

We say that the PDE $E_k \subset J^k_n(W)$ is *completely integrable* if for any point $q \in E_k$ passes a manifold V of dimension n, that is the k-prolongation of a n-dimensional manifold $X \subset W$: $V = X^{(k)}$.

Definition 2.15. By using the natural embedding $JD^k(W) \subset J^k_n(W)$, we can consider PDR's, (resp. PDE's), $E_k \subset JD^k(W)$ like PDEs $E_k \subset J^k_n(W)$, hence we can consider solutions of E_k as n-dimensional manifolds $V \subset E_k$ such that V can be representable in the neighborhood of any of its points $q' \in V$, except for a nonwhere dense subset $\Sigma(V) \subset V$, of dimension $\leq n-1$, as $N^{(k)}$, where $N^{(k)}$ is the k-prolongation of a n-dimensional manifold $N \subset W$. In the case that $\Sigma(V) = \emptyset$, we say that V is a *regular solution* of $E_k \subset J^k_n(W)$. Of course, solutions V of $E_k \subset J^k_n(W)$, even if regular ones, are not, in general diffeomorphic to their projections $\pi_k(V) \subset M$, hence are not representable by means of sections of $\pi : W \to M$. We shall also consider solutions of $E_k \subset J^k_n(W)$, manifolds of dimension n, $V \subset E_k$, that can be obtained as projections of ones of the previous type, but contained in some s-prolongation $E_{k+s} \subset J^{k+s}_n(W)$, $s > 0$.

Theorem 2.16. *Let $E_k \subset JD^k(W)$ be a PDE such that the following conditions are satisfied:* (i) *E_k is regular;* (ii) *The symbol g_{k+r+1} of E_{k+r+1} is a bundle of $\mathbb{R}$-modules over E_k, $r \geq 0$;* (iii) *$E_{k+r+1} \to E_{k+r}$ is surjective with $0 \leq r \leq l$, for suitable l. Then, E_k, is formally integrable.*

Furthermore, if $E_k \subset JD^k(W)$ is a formally integrable PDE in the category of manifolds of class C^ω_w, then the PDE $E_k \subset J^k_n(W)$ is completely integrable.

Proof. See [7]. □

Theorem 2.17 (Cauchy problem for PDE's)**.** *Let $E_k \subset J^k_n(W)$ be a formally integrable and completely integrable PDE. Let $N \subset E_k$ be a regular integral manifold of dimension $n-1$. Let us assume that the symbols g_k and g_{k+1} are not trivial. Then there exists a solution $V \subset E_k$, such that $N \subset V$. In particular, if there exists a symmetry vector field ζ of E_k, transverse to N, that is a characteristic vector field for a sub-equation $\tilde{E}_k \subset E_k$, then there exists a solution $V \subset \tilde{E}_k$, passing through N, having ζ as characteristic vector field.*

Proof. Since N is a $(n-1)$-dimensional regular integral manifold of $E_k \subset J^k_n(W)$, then there exists a n-dimensional submanifold $Y \subset W$ such that $X \equiv \pi_{k,0}(N) \subset Y$ and $N = X^{(k)}$. In general $Y^{(k)} \subset J^k_n(W)$, but $Y^{(k)} \not\subset E_k$, even if $Y^{(k)} \cap E_k = N$. Let us consider the first prolongation $E_{k+1} \subset J^{k+1}_n(W)$ of E_k. Taking into account that

E_k is formally integrable and completely integrable, we can use the fact that E_{k+1} is a strong retract of $J^{k+1}_n(W)$, and that $g_{k+1}\neq 0$, to deform $Y^{(k+1)}\subset J^{k+1}_n(W)$ in E_{k+1}, obtaining a solution $\tilde{Y}$ of E_{k+1}. Then $\pi_{k+1,k}(\tilde{Y})\equiv V\subset E_k$ is a solution of E_k, passing for N, since $N^{(1)}=X^{(k+1)}\subset Y^{(k+1)}$, and $\pi_{k+1,k}(N^{(k)})=N$. (See also [25].)

Finally, if there exists a symmetry vector field ζ of E_k, that is transverse to N and characteristic for a sub-equation $\tilde{E}_k\subset E_k$, then $\hat{V}\equiv\bigcup_{\lambda\in]-\epsilon,\epsilon[}\phi_\lambda(N)$, with $\partial\phi=\zeta$, is a solution of $\tilde{E}_k$, hence a solution of E_k too, having just ζ as characteristic vector field. □

Definition 2.18 (Algebraic formulation of PDE's)**.** Let $\mathfrak{A}_k$ be the sheaf of germs of differentiable functions $JD^k(W)\to\mathbb{R}$. It is a sheaf of rings, but also a sheaf of $\mathbb{R}$-modules. A subsheaf of ideals $\mathfrak{B}_k$ of $\mathfrak{A}_k$ that is also a subsheaf of $\mathbb{R}$-modules is a *PDE of order* k on the fiber bundle $\pi:W\to M$. A *regular solution* of $\mathfrak{B}_k$ is a section $s:M\to W$ such that $f\circ D^ks=0,\forall f\in\mathfrak{B}_k$. The set of *integral points* of $\mathfrak{B}_k$ (i.e., the zeros of $\mathfrak{B}_k$ on $JD^k(W)$ is denoted by $J(\mathfrak{B}_k)$. The first *prolongation* $(\mathfrak{B}_k)_{+1}$ of $\mathfrak{B}_k$ is defined as the system of order $k+1$ on $W\to M$, defined by the $f\circ\pi_{k,k-1}$ and $f^{(1)}$, where $f^{(1)}$ on $D^{k+1}s(p)$ is defined by $f^{(1)}(D^{k+1}s(p))=(\partial x_\alpha.(f\circ D^ks(p)))$. In local coordinates $(x^\alpha,y^j,y^j_\alpha)$ the *formal derivative* $f^{(1)}$ is given by $f^{(1)}(x^\alpha,y^j,y^j_\alpha)=(\partial x_\alpha.f)+\sum_{[\beta]\leq k}y^j_{\beta\alpha}(\partial y^\beta_j.f)$. The system $\mathfrak{B}_k$ is said to be *involutive* at an integral point $q\in JD^k(W)$ if the following two conditions are satisfied: *(i)* $\mathfrak{B}_k$ is a *regular local equation* for the zeros of $\mathfrak{B}_k$ at q (i.e., there are local sections $F_1,...,F_t\in\Gamma(U,\mathfrak{B}_k)$ of $\mathfrak{B}_k$ on an open neighbourhood U of q, such that the integral points of $\mathfrak{B}_k$ in U are precisely the points q' for which $F_j(q')=0$ and $dF_1\wedge\cdots\wedge dF_t(q)\neq 0$, that is $F_1,\cdots,F_t$ are linearly independent at q; and *(ii)* there is a neighbourhood U of q such that $\pi^{-1}_{k+1,k}(U)\bigcap J((\mathfrak{B}_k)_{+1})$ is a fibered manifold over $U\bigcap J(\mathfrak{B}_k)$ (with projection $\pi_{k+1,k}$). For a system $\mathfrak{B}_k$ generated by linearly independent Pfaffian forms $\theta^1,\cdots,\theta^k$ (i.e., a Pfaffian system) this is equivalent to the involutiveness defined for distributions.

Theorem 2.19 ([13])**.** *Let $\mathfrak{B}_k$ be a system defined on $JD^k(W)$, and suppose that $\mathfrak{B}_k$ is involutive at $q\in J(\mathfrak{B}_k)$. Then, there is a neighbourhood U of q satisfying the following. If $\widetilde{q}\in J((\mathfrak{B}_k)_{+s})$ and $\pi_{k+s,k}(\widetilde{q})$ is in U, then there is a regular solution s of $\mathfrak{B}_k$ defined on a neighbourhood $p=\pi_{k+s,-1}(\widetilde{q})$ of M such that $D^{k+s}s(p)=\widetilde{q}$.*

Theorem 2.20 (Cartan-Kuraniski prolongation theorem [23,25])**.** *Suppose that there exists a sequence of integral points $q^{(s)}$ of $(\mathfrak{B}_k)_{+s},s=0,1,\cdots$, projecting onto each other, $\pi_{k+s,k+s-1}(q^{(s)})=q^{(s-1)}$, such that:* (a) *$(\mathfrak{B}_k)_{+s}$ is a regular local equation for $J((\mathfrak{B}_k)_{+s})$ at $q^{(s)}$; and* (b) *there is a neighborhood $U^{(s)}$ of $q^{(s)}$ in $J((\mathfrak{B}_k)_{+s})$ such that its projection under $\pi_{k+s,k+s-1}$ contains a neighborhood of $q^{(s-1)}$ in $J((\mathfrak{B}_k)_{+(s-1)})$ and such that $\pi_{k+s,k+s-1}:U^{(s)}\to\pi_{k+s,k+s-1}(U^{(s)})$ is a fibered manifold. Then, $(\mathfrak{B}_k)_{+s}$ is involutive at $q^{(s)}$ for s large enough.*

Let us consider, now, a relation between the algebraic localization of modules and the geometry of PDE's. This algebraic characterization of PDE's is principally useful to describe singularities in PDE's given by means of polynomial functions.

In this section we will start by considering commutative rings. (The proofs are almost omitted since the material is standard. For more details, see also, e.g., [24] and works quoted there.)

A *zero-divisor* in a ring A is an element $a \in A$ such that there exists $b \in A$, $b \neq 0$, such that $ab = 0$. An element $a \in A$ is *nilpotent* if $a^n = 0$, for some $n > 0$. (A nilpotent element is a zero-divisor (unless $A = 0$), but not conversely.)[2] The *identity element* $1 \in A$ is defined by $a1 = 1a = a$, for all $a \in A$. A *unit* in A is an element $a \in A$ that divides 1, i.e., $\exists b \in A$ such that $ab = 1$. Then b is called the *inverse* of a and denoted by a^{-1}. (a^{-1} is uniquely determined by a.) An *integral domain* is a ring with non-zero-divisors $\neq 0$. The set of units forms a multiplicative, abelian group in A, that we denote by $G(A)$. For any subset $S \subset A$, the smallest ideal $\mathfrak{a} \subset A$ containing S is called the *ideal generated by S* and denoted by $\langle S\rangle$. Any ideal $\mathfrak{p} = \langle a\rangle$, $a \in$A, is called a *principal ideal.* A ring is called *principal ideal ring* if every ideal is principal. In particular we denote $\mathfrak{a}+\mathfrak{b} = \langle\{a+b\}_{a\in\mathfrak{a},b\in\mathfrak{b}}\rangle$, and $\mathfrak{ab} = \langle\{ab\}_{a\in\mathfrak{a},b\in\mathfrak{b}}\rangle$. If $S \subset A$ is a subset and $\mathfrak{a} \subset A$ is an ideal, then the *quotient* $\frac{\mathfrak{a}}{S} \equiv \{a \in A | aS \subset \mathfrak{a}\}$ is an ideal of A. If $S \subset A$ is a multiplicatively closed subset of A, we call a *maximal ideal* with respect to S, a maximal member $\mathfrak{m} \subset A$ among the set of ideals do not meet S. In particular, if $S = \{1\}$, then $\mathfrak{m}$ is called a *maximal ideal* of A. An ideal $\mathfrak{m} \subset A$ is maximal iff $A/\mathfrak{m}$ is a field. An ideal $\mathfrak{p} \subset A$ is *prime* if $\mathfrak{p} \neq \langle 1\rangle$ and $xy \in \mathfrak{p}$ implies $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. $\mathfrak{p}$ is prime iff $A/\mathfrak{p}$ is any integral domain. A maximal ideal $\mathfrak{m}$ of A is prime. If $\mathfrak{a}$ is any ideal of A, the *radical* of $\mathfrak{a}$ is the following ideal

$$\sqrt{\mathfrak{a}} \equiv r(\mathfrak{a}) \equiv \{x \in A \mid x^n \in \mathfrak{a} \text{ for some } n > 0\} \equiv rad(\mathfrak{a}).$$

An ideal $\mathfrak{a}$ such that $\mathfrak{a} = \sqrt{\mathfrak{a}}$ is called *radical-ideal* (or *perfect*). One has the following properties: (i) $\sqrt{\{0\}}$ is the ideal consisting of all nilpotent elements of A and is denoted also by $nil(A)$: $\sqrt{\{0\}} = nil(A)$. If $\{0\}$ is a radical ideal, or equivalently, when $nil(A) = \{0\}$, A is said to be *reduced.* (ii) $r(\mathfrak{a}) \supseteq \mathfrak{a}$; (iii) $r(r(\mathfrak{a})) = r(\mathfrak{a})$; (iv) $r(\mathfrak{ab}) = r(\mathfrak{a}\cap\mathfrak{b}) = r(\mathfrak{a})\cap r(\mathfrak{b})$; (v) $r(\mathfrak{a}) = \langle 1\rangle \Leftrightarrow \mathfrak{a} = \langle 1\rangle$; (vi) $r(\mathfrak{a}+\mathfrak{b}) = r(r(\mathfrak{a})+r(\mathfrak{b}))$; (vii) If $\mathfrak{p}$ is prime, $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$;(viii) One has that $r(\mathfrak{a})$ is the intersection of all prime ideals containing $\mathfrak{a}$; (ix) If $\mathfrak{a}$ is a radical-ideal, $A/\mathfrak{a}$ is reduced, and in particular $A/nil(A)$ is reduced. (x) The radical $nil(A)$ of a commutative ring A is the intersection of all prime ideals of A. In particular $nil(A) \subset rad(A)$, where $rad(A)$ is the *radical* of A defined by:

$$rad(A) = \bigcap_{\mathfrak{m}\,=\,\text{maximal ideals of } A} \mathfrak{m}.$$

$rad(A)$ is an ideal of A. One has: $a \in rad(A)$ iff for all $x \in A$, one has that $1 - xa$ has a left inverse. An ideal $\mathfrak{q} \subset A$ is *primary* if $\mathfrak{q} \neq A$ and if $xy \in \mathfrak{q}$ implies $x \in \mathfrak{q}$ or $y^n \in \mathfrak{q}$ for some $n > 0$. One has the following properties: (i) $\mathfrak{q}$ is primary iff $A/\mathfrak{q} \neq 0$ and every zero-divisor in $A/\mathfrak{q}$ is nilpotent; (ii) Every prime ideal is primary; (iii) Let $\mathfrak{q}$ be a primary ideal in a ring A. Then $r(\mathfrak{q})$ is the smallest prime ideal containing $\mathfrak{q}$. If $\mathfrak{p} = r(\mathfrak{q})$ then $\mathfrak{q}$ is said to be $\mathfrak{p}$*−primary.*

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[2] A ring A, such that the set of its nilpotent elements is reduced to $\{0\}$, is called a *reduced ring.*

Definition 2.21. The *prime spectrum* of a ring A is the set $S_{pec}(A)$ of all prime ideals of A.

Proposition 2.22 ([24])**.** *Let A be a ring and let $\mathfrak{p} \in S_{pec}(A)$. Then one has the following properties:*

(i) *If $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$ are ideals of A and $\mathfrak{a}_1 \cdots \mathfrak{a}_r \subset \mathfrak{p}$, then $\mathfrak{a}_i \subset \mathfrak{p}$ for one $i \in \{1, \cdots, r\}$.*

(ii) *If $\mathfrak{a}_1, \cdots, \mathfrak{a}_r$ are deals of A and $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r \subseteq \mathfrak{p}$, then $\mathfrak{a}_i \subset \mathfrak{p}$ for one $i \in \{1, \cdots, r\}$. In particular if the equality holds, then $\mathfrak{a}_i = \mathfrak{p}$.*

Definition 2.23 (Localization)**.** Let $S \subset A$ be a multiplicatively closed subset of A with $1 \in S$. Let $S^{-1}A \equiv A \times S/\sim$, where $\sim$ is the following equivalence relation: $(a, s) \sim (a', s') \Leftrightarrow as' - a's = 0$. We denote by $\frac{a}{s}$ the equivalence class of (a, s) and call $S^{-1}A$ the set of *fractions* of A by S. In particular, if $\mathfrak{p} \subset A$ is a prime ideal we call $A_{\mathfrak{p}} \equiv S^{-1}A$, $S \equiv A - \mathfrak{p}$, the *localization* of A at $\mathfrak{p}$. If $a \in A$ and $S = \{a^n\}_{n \geq 0}$, then one writes $A_a \equiv S^{-1}A$.

Let M be a A-module. Set $S^{-1}M \equiv M \times S/\sim$, where $\sim$ is the following equivalence relation: $(p, s) \sim (p', s') \Leftrightarrow \exists t \in S : t(sp' - s'p) = 0$. We denote by $\frac{p}{s}$ the equivalence class of (p, s) and call $S^{-1}M$ the set of *fractions* of M by S. This is a $S^{-1}A$-module.

In particular, if $\mathfrak{p} \subset A$ is a prime ideal and $S \equiv A - \mathfrak{p}$, then $M_{\mathfrak{p}} \equiv S^{-1}M$ is called the *localization* of M at $\mathfrak{p}$.

Theorem 2.24 (Localization properties)**.** 1) *One has the following properties:*

(i) *S^{-1} is an exact functor;*

(ii) *$S^{-1}(M \oplus P) = (S^{-1}M) \oplus (S^{-1}P)$; In particular $(M \oplus P)_{\mathfrak{p}} = M_{\mathfrak{p}} \oplus P_{\mathfrak{p}}$, for any $\mathfrak{p} \in S_{pec}(A)$;*

(iii) *$S^{-1}(M + P) = (S^{-1}M) + (S^{-1}P)$; In particular $(M + P)_{\mathfrak{p}} = M_{\mathfrak{p}} + P_{\mathfrak{p}}$, for any $\mathfrak{p} \in S_{pec}(A)$;*

(iv) *$S^{-1}(M \cap P) = (S^{-1}M) \cap (S^{-1}P)$; In particular $(M \cap P)_{\mathfrak{p}} = M_{\mathfrak{p}} \cap P_{\mathfrak{p}}$, for any $\mathfrak{p} \in S_{pec}(A)$;*

(v) *$S^{-1}(M/N) \cong (S^{-1}M)/(S^{-1}N)$, (as $(S^{-1}A)$-module); In particular $(M/N)_{\mathfrak{p}} \cong M_{\mathfrak{p}}/N_{\mathfrak{p}}$, for any $\mathfrak{p} \in S_{pec}(A)$;*

(vi) *$S^{-1}A \otimes_A M \cong S^{-1}M$, for any $\mathfrak{p} \in S_{pec}(A)$; In particular $A_{\mathfrak{p}} \otimes_A M \cong M_{\mathfrak{p}}$, for any $\mathfrak{p} \in S_{pec}(A)$;*

(vii) *$S^{-1}A$ is a flat A-module; In particular $A_{\mathfrak{p}}$ is a flat A-module, for any $\mathfrak{p} \in S_{pec}(A)$;*

(viii) *$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N)$; In particular $M_{\mathfrak{p}} \otimes_{S^{-1}A} N_{\mathfrak{p}} \cong (M \otimes_A N)_{\mathfrak{p}}$, with $S \equiv A \setminus \mathfrak{p}$ and, for any $\mathfrak{p} \in S_{pec}(A)$.*

2) *If S is the set of non-zero-divisors of A, one denotes $S^{-1}A \equiv Q(A)$ and calls it the* full ring of fractions *of A. In that case, A can be considered as a subring of $Q(A)$. One has the following exact sequence: $0 \to t_S(M) \to M \to S^{-1}M$, where the map $M \to S^{-1}M$ is $x \mapsto \frac{x}{1}$ and $t_S(M) \equiv \{x \in M | sz = 0$, for some $s \in S\}$ is called the S-*torsion submodule *of M.*[3]

[3] A module is called *torsion-free* if it has no *torsion element*, i.e., $t_S(M) = 0$. For example the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$ is torsion free.

When A is an integral domain and $S = A \setminus \{0\}$ then $K \equiv Q(A)$ is a field called the field of fractions *of A. If $S = S_a$, then $S^{-1}A = A[\frac{1}{a}]$.*

One has the following local properties:

(i) *$M = 0$ iff $M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subset A$;*
(ii) *$M = 0$ iff $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset A$:*
(iii) *$\phi : M \to N$ injective iff $\pi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is so, for each prime ideal $\mathfrak{p} \subset A$;*
(iv) *$\phi : M \to N$ injective iff $\pi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is so, for each maximal ideal $\mathfrak{m} \subset A$;*
(v) *M flat iff $M_{\mathfrak{p}}$ is so, for each prime ideal $\mathfrak{p} \subset A$;*
(vi) *M flat iff $M_{\mathfrak{m}}$ is so, for each maximal ideal $\mathfrak{m} \subset A$;*
(vii) *A sequence $M \to N \to F$ of A-modules and homomorphisms is exact iff the corresponding* localized sequences*: $M_{\mathfrak{m}} \to N_{\mathfrak{m}} \to F_{\mathfrak{m}}$ is exact, for all maximal ideals $\mathfrak{m} \subset A$.*
(viii) *A short exact sequence $0 \to M \to N \to F \to 0$ of A-modules and homomorphisms, where M is* finitely presentable *splits iff the the corresponding* localized exact sequences*: $0 \to M_{\mathfrak{m}} \to N_{\mathfrak{m}} \to F_{\mathfrak{m}} \to 0$ splits for all maximal ideals $\mathfrak{m} \subset A$.*
(ix) *If M is a finitely presentable A-module and $N \subset M$ is a finitely generable submodule, then N is a direct summand of M iff $N_{\mathfrak{m}}$ is a direct summand of $M_{\mathfrak{m}}$, for any maximal ideals $\mathfrak{m} \subset A$.*
(x) *If A is an integral domain, and M is an A-module, then M is torsion-free iff $M_{\mathfrak{m}}$ is torsion-free for any maximal ideal $\mathfrak{m}$ of A.*
(xi) *If M is a finitely generated module over a Noetherian integral domain A, then M is* reflexive*, i.e., $M \cong Hom_A(Hom_A(M; A); A)$, iff $M_{\mathfrak{m}}$ is reflexive for any maximal ideal $\mathfrak{m}$ of A.*
(xii) *If M is a torsion-free module over an integral domain A and $S = A \setminus \{0\}$, then we have an exact sequence $0 \to M_{\mathfrak{m}} \to S^{-1}N$, for any maximal ideal $\mathfrak{m}$ of A and $N = \bigcap_{\mathfrak{m}} M_{\mathfrak{m}}$.*

3) *If $\phi : A \to A/\mathfrak{a}$ is the canonical homomorphism, it follows that $r(\mathfrak{a}) = \phi^{-1}(nil(A)_{A/\mathfrak{a}})$.*

A ring with a unique maximal ideal $\mathfrak{m}$ is called a local ring*. $A_{\mathfrak{p}}$ is a local ring, where $\mathfrak{p}$ is a prime ideal of A.*

Theorem 2.25 (Topological structure of $S_{pec}(A)$)**.** 1) *One has a natural structure of topological space (*Zariski topology*) on $S_{pec}(A)$. The open sets in this topology are the sets $V(E) \equiv \{\mathfrak{p} \supset E | E \subset A, \mathfrak{p}\, prime\ ideals\}$. With respect to such a topology, $S_{pec}(A)$ is a quasi-compact space, that is every open covering of $S_{pec}(A)$ has a finite subcovering. If the ring A is Noetherian, $S_{pec}(A)$ is a Noetherian space. (The converse of this is false.)*

2) $S_{pec}\left(\prod_{1\le i\le n} A_i\right) \cong \dot{\bigcup}_{1\le i\le n} S_{pec}(A_i)$.

3) *Let $\mathfrak{a}$ be an ideal of A. The ideals of $A/\mathfrak{a}$ are in correspondence one-to-one with the ideals of A that contain $\mathfrak{a}$, hence $S_{pec}(A/\mathfrak{a}) \cong V(\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of prime ideals of A that contain $\mathfrak{a}$. This is just a closed subspace of $S_{pec}(A)$.*

4) *S_{pec} is a contravariant functor from the category of rings and ring-homomorphisms to the category of topological spaces and continuous maps. In*

particular, if $\phi : A' \to A$ is surjective, then $S_{pec}(\phi) : S_{pec}(A) \to S_{pec}(A')$ is a closed embedding, i.e., a homeomorophism of $S_{pec}(A)$ onto a closed subset of $S_{pec}(A')$. Furthermore, if ϕ is injective, then $S_{pec}(\phi)$ is dominant*, i.e., $S_{pec}(\phi)(S_{pec}(A))$ is dense in $S_{pec}(A')$.*

5) *Let $a \in A$. Then $S_{pec}(A_a) = \{x \in S_{pec}(A) | a \notin x\}$. Let $\phi : A \to A_a$ be the canonical homomorphism associated to an element $a \in A$. Then the corresponding mapping $S_{pec}(\phi)$ is a homeomorphism of $S_{pec}(A_a)$ onto the open set: $D(a) \equiv S_{pec}(A) - V(a) \equiv \{x \in S_{pec}(A) | a(x) \neq 0\} \equiv$ support of a. Here $a(x)$ denotes the class of a mod x in A/x. Thus $a(x) = 0$ iff $a \in x$. ($D(a)$ is an open set.)*

In the following table we report some distinguished objects associated to the points of a prime spectrum of a ring. (For the definition of affine variety see [24] and Definition 2.32.)

TABLE 1. Objects associated to $x \in S_{pec}(A)$

local ring at x: $A_x \equiv S^{-1}A$, $S \equiv A \setminus x$
$A_x \equiv S^{-1}A$, $S \equiv A \setminus x$: residue field of A_x
$\kappa(x) \equiv A_x/\mathfrak{m}_x, \mathfrak{m}_x \subset A_x$ maximal ideal. $(\star)$
value of $a \in A$ at x, $a(x) \in A/x \subseteq \kappa(x)$, $A \to A/x$, $(a(x) = 0 \Leftrightarrow a \in x)$. $(\star\star)$
tangent space at x, $T_xS_{pec}(A) \equiv Hom_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2; \kappa(x))$. $(\star\star\star)$

$(\star)$ One has the exact sequence: $0 \to x \to A \to \kappa(x)$

$(\star\star)$ If $A \subset \kappa[X]$, $X \equiv$ affine variety, $x \in X$ determines a maximal ideal of A

$\kappa(x) = \kappa$, $a(x) \equiv$ value of the function a at x.

$(\star\star\star)$ $\mathfrak{m}_x/\mathfrak{m}_x^2$ is a $\kappa(x)$-vector space.

If A_x is a Noetherian ring (e.g. if A is Noetherian) then $\mathfrak{m}_x/\mathfrak{m}_x^2$ is finite dimensional.

Definition 2.26. The *support* of a A-module M is defined by the following:

$$supp_A(M) \equiv \{\mathfrak{p} \in S_{pec}(A) : M_{\mathfrak{p}} \neq 0\} \subseteq S_{pec}(A).$$

Theorem 2.27 (Properties of support of A-module)**.** 1) *$M \neq 0 \Leftrightarrow supp_A(M) \neq \emptyset$.*

2) *$V(\mathfrak{a}) = supp_A(A/\mathfrak{a})$, where $V(\mathfrak{a})$ is the set of prime ideals containing $\mathfrak{a}$.*

3) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence, then $supp_A(M) = supp_A(M') \cup supp_A(M'')$. If $M = \sum_i M_i \Rightarrow supp_A(M) = \bigcup_i supp_A(M_i)$.*

4) *If M is finitely generated A-module, it follows that*

$$supp_A(M) = V(ann_A(M))$$

and therefore is a closed subset of $S_{pec}(A)$. If M and N are finitely generated, then $supp_A(M \otimes_A N) = supp_A(M) \cap supp_A(N)$. If M is finitely generated and $\mathfrak{a}$ is an ideal of A, then $supp_A(M/\mathfrak{a}M) = V(\mathfrak{a} + ann_A(M))$. If $f : A \to B$ is a ring homomorphism and M is a finitely generated A-module, then $supp_A(B \otimes_A M) = f^{*-1}(supp_A(M))$, where $f^* : S_{pec}(B) \to S_{pec}(A)$ is the map induced by f.

5) *Let $A \equiv \bigoplus_{0 \leq n \leq \infty} A_n$ be a Noetherian graded ring. Then A_0 is a Noetherian ring and $A = A_0[x_1, \cdots, x_s]$, with $|x_j| > 0$,[4] i.e., A is a finitely generated A_0-algebra.*

[4]Here $|x_j| = n$ iff $x_j \in A_n$.

Theorem 2.28 (Hilbert-Serre)**.** 1) *Let $\mathcal{C}$ be a class of A-modules and $\lambda : \mathcal{C} \to \mathbb{Z}$. The function λ is* additive *if, for each short exact sequence* $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ *in $\mathcal{C}$, we have $\lambda(M) = \lambda(M') + \lambda(M'')$. Let $0 \to M_0 \to M_1 \to \cdots \to M_n \to 0$ be an exact sequence of A-modules in which all the modules M_i and the kernels of all the homomorphisms belong to $\mathcal{C}$. Then for any additive function λ on $\mathcal{C}$ we have $\sum_{0 \le i \le n} \lambda(M_i) = 0$. We call* Hilbert polynomial $P(M,t) = \sum_{0 \le n \le \infty} \lambda(M_n)t^n \in \mathbb{Z}[[t]]$*, where λ is an additive $\mathbb{Z}$-valued function. $P(M,t)$ is a rational function in t of the form $P(M,t) = \frac{f(t)}{\prod_{1 \le i \le s}(1-t^{|x_i|})}$, where $f(t) \in \mathbb{Z}[t]$, and s is the number of generators of A over A_0.*

2) *We denote by $d(M)$ the order of the pole of $P(M,t)$ at $t = 1$. It provides a measure of the "size" of M (relatively to λ). In particular $d(A)$ is defined. If each $|x_i| = 1$, then for all sufficiently large n, $\lambda(M_n)$ is a polynomial in n (with rational coefficients) of degree $d - 1$. (We adopt the convention that the degree of the zero polynomial is -1; also the binomial coefficient $\binom{n}{-1} = 0$ for $n \ge 0$, and $n = 1$ for $n = -1$.) This polynomial is called* Hilbert polynomial *of M (with respect to λ).*

Example 2.29. If $x \in A_\kappa$ is not a zero-divisor of M, then $d(M/xM) = d(M) - 1$.

If A_0 is an Artinian ring (in particular, a field), $A = A_0[x_1, \cdots, x_s]$, then A_n is a free A_0-module generated by the monomials $x_1^{m_1} \cdots x_s^{m_s}$, with $\sum_i m_i = n$; there are $\binom{s-n-1}{s-1}$ of these, hence $P(A,t) = (1-t)^{-s}$.

Definition 2.30. We define *dimension* of the $Supp(M)$ the degree of the Hilbert polynomial. We define *Krull* dimension of a ring A the sup of lengths of $(n+1)$-chains of prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ of length n:

$$\dim A = \sup_n \{\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n\}.$$

Theorem 2.31 (Dimension properties)**.** *One has the following properties.*

1) $\dim A = sup_{\mathfrak{p} \in S_{pec}(A)} \dim(A_\mathfrak{p})$.[5]

2) $\dim A \ge 0, +\infty$.

3) *A field has dimension* 0*; the ring $\mathbb{Z}$ has dimension* 1.

4) *A ring A is Artin iff A is Noetherian and* $\dim A = 0$.

5) *Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal. Then, exactly one of the following two statements is true: (i) $\mathfrak{m}^n \ne \mathfrak{m}^{n+1}$ for all n; (ii) $\mathfrak{m}^n = 0$ for some n, in which case A is an Artinian local ring.*

6) $\dim A = d(A) = \delta(A)$*, where $\delta(A)$ is the least number of generators of an $\mathfrak{m}$-primary ideal of A, Noetherian local ring.*

7) *Let M be an A-module. Then*

$$\dim_A(M) = \dim(A/ann_A(M)) = sup_{\mathfrak{p} \in supp_A(M)}\{\dim(A/\mathfrak{p})\}.$$

Definition 2.32. *1)* Let κ be a field of *characteristic zero*, that is a field containing the field $\mathbb{Q}$ of rational numbers as s subfield, and set $A = \kappa[\chi] \equiv \kappa[\chi_1, \cdots, \chi_n]$ the ring of polynomials in the indeterminates χ_i with coefficients in κ. For r given polynomials $P_1, \cdots, P_r \in \kappa[\chi]$, we define *algebraic set X* determined by the ideal $\mathfrak{a} \equiv \langle P_1, \cdots, P_r \rangle$ of $\kappa[\chi]$ the following one:

[5] For a Noetherian ring A one has $\dim(A_\mathfrak{p}) < +\infty$, but can be also $\dim(A) = +\infty$.

$$X \equiv \left\{ x = (x_1, \cdots, x_n) \middle| \left\{ \begin{array}{l} x_i,\ 1 \leq i \leq n, \text{ belong to an extension of } \kappa \\ \text{(for example its algebraic closure)} \\ \text{such that } P_j(x) = 0,\ 1 \leq j \leq r. \end{array} \right. \right\}$$

2) We define *irreducible algebraic set* (or *variety*) the set X of points in $\bar{\kappa}^n$, where $\bar{\kappa}$ is an algebraic closure of κ, which are zeros of solutions of a prime ideal $\mathfrak{p} \in S_{pec}(\kappa[\chi])$. We shall write $X = Z(\mathfrak{p})$ and we shall introduce the field $K \equiv \kappa(X) \equiv Q(\kappa[\chi]/\mathfrak{p})$ as the quotient field of the integral domain $A \equiv \kappa[X] \equiv \kappa[\chi]/\mathfrak{p}$ of polynomial functions on X. (We have the inclusions $\kappa \subseteq A \subseteq K$.)[6] We call $A \equiv \kappa[X]$ the *coordinates ring* of X. $K \equiv \kappa(X)$ is called the *field of rational functions* on X. A ring B is *integral* over a ring A if each element of B is *integral* over A, that is to say if each element of B is a root of a *unitary polynomial* with coefficients in A.[7] If $A \subset B$ are two rings, the *integral closure* of A in B is the subring of B consisting of all elements of B that are integral over A.

3) Let us consider a field extension K/κ.[8] Elements $a_1, \cdots, a_n \in K$ are said to be *algebraically independent* over κ in K if one cannot find a polynomial $P \in \kappa[\chi_1, \cdots, \chi_n]$ such that $P(a_1, \cdots, a_n) = 0$. Otherwise we say that $a_1, \cdots, a_n \in K$ are *algebraic* over κ. An extension K/κ is called an *algebraic (field) extension* if every element $a \in K$ is *algebraic* over κ, i.e., we can find a polynomial $P \in \kappa[\chi]$, such that $P(a) = 0$.

Theorem 2.33 (Existence of primitive element theorem)**.** *Every finitely generated algebraic extension is generated by a single element. More precisely, if $L = K(\eta^1, \cdots, \eta^m) \equiv Q(K[\eta^1, \cdots, \eta^m])$, then we can find $c_1, \cdots, c_m \in K$ such that $L = K(\zeta) \equiv Q(K[\zeta])$, with $\zeta = c_1\eta^1 + \cdots + c_m\eta^m$. An extension L/K is called* regular *if K is* algebraically closed *in L, that is no element of L is algebraic over K.*

Definition 2.34. Let us consider the following extensions of fields: $\kappa \subset K \subset L$. If L/κ is regular, then $K \otimes_\kappa L$ is an integral domain. A maximal subset of K which is algebraically independent over κ is called a *transcendence basis* of K/κ. The number of elements of a transcendence basis is uniquely defined and only depends on the extension K/κ. This numebr is called the *transcendence degree* of K/κ and denoted by $trd(K/\kappa)$. If A is an integral domain containing a field κ, we shall define $trd(A(\kappa)) = trd(K/\kappa)$ with $K = Q(A)$. Since $K = \kappa(A)$, transcendence basis exist that are subsets of A and any such transcendence basis will be called a transcendence basis of A/κ. If K/κ is a finitely generated extension, say $K = \kappa(a_1, \cdots, a_n)$ we may find a transcendence basis among the a_i, say $a_1, \cdots, a_r$ and $\kappa(a_1, \cdots, a_r)$, called a *purely* transcendental extension of κ-module K, is a finite algebraic extension of $\kappa(a_1, \cdots, a_r)$.

[6] We may consider A as a vector space over κ and K as a vector space over A or over κ. If we put $|A/\kappa| = \dim_\kappa A$, $|K/A| = \dim_A K$, $|K/\kappa| = \dim_\kappa K$, then we get $|K/\kappa| = |K/A|.|A/\kappa|$.

[7] A unitary polynomial is a polynomial where the coefficient of the term of highest degree is equal to 1.

[8] For simplicity we shall denote by K/κ an extension of fields $0 \to \kappa \to K \to K/\kappa \to 0$.

Theorem 2.35 (Noether normalization lemma)**.** *If $A = \kappa[a_1, \cdots, a_n]$ is a finitely integral domain with $trd(A/\kappa) = r$, then there exist r algebraically independent linear combinations $b_1, \cdots, b_r$ of the a_i such that A is integral over $B = \kappa[b_1, \cdots, b_r]$ and $K = Q(A)$ is algebraic over $\kappa(b_1, \cdots, b_r)$. If κ is an infinite field and $P \in \kappa[\chi_1, \cdots, \chi_n]$ is a non-zero polynomial, then we can find elements $\alpha_1, \cdots, \alpha_n \in \kappa$ such that $P(\alpha_1, \cdots, \alpha_n) \neq 0$ in an infinite number of fashions.*

Definition 2.36. If X is a variety defined over the ground field κ we define the *dimension* of X to be the integer $\dim_\kappa(X) = trd(\kappa(X)/\kappa)$. If X is an *affine variety*, that is to say $X = Z(\mathfrak{p})$ with $\mathfrak{p} \in S_{pec}(\kappa[\chi_1, \cdots, \chi_n])$, then we define the *codimension* of X by the integer $\mathrm{codim}(X) = n - \dim(X) \equiv n - d(X)$. Let X be an algebraic set defined as the zero of an ideal $\mathfrak{a}$, that is $X = Z(\mathfrak{a})$. Let us consider the decomposition of the ideal $I(X) = rad(\mathfrak{a})$ into prime ideals. Then, we define *dimension of algebraic set* (or of the ideal) to be the maximum of the dimensions of its irreducible components.

Example 2.37. An *irreducible plane curve* is a variety of dimension 1.

A *hypersurface* is an algebraic set which is defined by principal ideal in $\kappa[\chi_1, \cdots, \chi_n]$ and all its irreducible components are of codimension 1. Conversely, any algebraic set whose components all have codimension 1 is a hypersurface and its defining ideal is principal.

If a polynomial P does not vanish identically on a variety X, then the irreducible components of the intersection of X with the hypersurface defined by P all have dimension equal to $\dim(X) - 1$.

Theorem 2.38 (Hilbert theorem of zeros)**.** *A polynomial $P \in \kappa[\chi]$ vanishes on all the zeros of $P_1, \cdots, P_r \in \kappa[\chi]$ iff $P \in rad\langle P_1, \cdots, P_r\rangle$.*

The polynomials $P_1, \cdots, P_r \in \kappa[\chi]$ have no-common zero iff

$$\langle P_1, \cdots, P_r\rangle = A \equiv \kappa[\chi_1, \cdots, \chi_n].$$

Theorem 2.39. 1) *Let us denote by $X = Z(\mathfrak{a})$ the algebraic set deteremined by $\mathfrak{a}$ and by $I(X) \subset A$ the ideal of all polynomials vanishing on X. One has: $I(Z(\mathfrak{a})) = rad(\mathfrak{a})$. One has the following properties:*

- (i) $X \subset Y \Rightarrow I(X) \supset I(Y)$.
- (ii) $\mathfrak{a} \subset \mathfrak{b} \Rightarrow Z(\mathfrak{a}) \supset Z(\mathfrak{b})$.
- (iii) $I(X \cup Y) = I(X) \cap I(Y)$.
- (iv) $Z(\mathfrak{a} + \mathfrak{b}) = Z(\mathfrak{a}) \cap Z(\mathfrak{b})$.
- (v) $Z(\mathfrak{a}) = \emptyset \Rightarrow \mathfrak{a} = A$.
- (vi) *Let X and Y be algebraic sets defined over κ with $I(X) = \mathfrak{a} \subset \kappa[u_1, \cdots, u_r]$ and $I(Y) = \mathfrak{b} \subset \kappa[v_1, \cdots, v_s]$. Then one has $\kappa[X \times Y] \cong \kappa[X] \otimes_\kappa \kappa[Y]$, where $\kappa[X] \equiv \kappa[u]/\mathfrak{a}$, $\kappa[Y] \equiv \kappa[v]/\mathfrak{b}$ and $\kappa[X \times Y] = \kappa[u, v]/(\mathfrak{a}, \mathfrak{b})$.*
- (vii) *The ring $\kappa[X] = \kappa[\chi]/\mathfrak{a}$ is* reduced*, that is does not contain any nilpotent element. Such a ring is called the* ring of rational fractions *on X. $\mathfrak{a}$ is an intersection of prime ideals according to the decomposition theorem, say $\mathfrak{a} = \bigcap_i \mathfrak{p}_i$, then $X = \bigcup_i X_i$, with each $X_i = Z(\mathfrak{p}_i)$ a variety and $\kappa[X]$ is contained in a direct sum of integral domains $A_i \equiv \kappa[X]/\mathfrak{p}_i$: $\kappa[X] \subset \bigoplus_i A_i$. $\kappa(X)$ is contained in a direct sum of fields $K_i \equiv Q(\kappa[\chi]/\mathfrak{p}_i)$: $\kappa(X) \subset \bigoplus_i K_i$. Each field K_i is a finitely-generated extension of κ and so has a*

finite transcendence degree over κ, i.e., the maximum number of algebraically independent elements. Each variety X_i coincides with the set of maximal ideals of A_i. Then, the local dimension *of X_i at $p \in X_i$ is* $\dim(A_i)_{\mathfrak{m}_i}$, *where $\mathfrak{m}_i$ is the maximal ideal corresponding to p. The* local dimension *of X at $p \in \bigcap_j X_j$ is* $\dim_p X = sup_j\{\dim(A_j)_{\mathfrak{m}_j}\}$.

2) *If A is an integral algebra of finite type on a field κ, then for any maximal ideal $\mathfrak{m} \subset A$, one has* $\dim A_{\mathfrak{m}} = trd(Q(A)/\kappa)$.

3) *If A is a Noetherian local ring with* $\dim A = d$, *the minimal number of generators of the maximal ideal $\mathfrak{m}$ is always $\geq d$. We say that A is* regular *if the equality holds. A Noetherian ring A is called* regular *if $A_{\mathfrak{m}}$ is regular for any maximal ideal $\mathfrak{m}$ of A.*

Example 2.40. The ring of polynomials $A[\chi]$ and formal series $A[[\chi]]$ are regular.

Theorem 2.41 (Syzygies theorem of Hilbert-Serre)**.** *The A-modules M of finite type, with A regular, are characterized by the fact to admit a projective resolution: $0 \to L_d \to L_{d-1} \to \cdots \to L_1 \to M \to 0$. A regular local ring is integral and integrally closed.*

Theorem 2.42. 1) *Let X and Y be two varieties defined over the field κ. One has:* $\mathrm{codim}_\kappa(X \cap T) \leq \mathrm{codim}_\kappa(X) + \mathrm{codim}_\kappa(Y)$.

2) *Let A be a Noetherian ring then one has:* $\dim A[x_1, \cdots, x_n] = n + \dim A$.

Definition 2.43. Let Y be any variety. Y is *nonsingular at a point* $p \in Y$ if the local ring $A_p \equiv \kappa[Y]_p$ is a regular local ring. Y is *nonsingular* if it is nonsingular at every point. Y is *regular* if it is not nonsingular.

Proposition 2.44. *If A is a Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field κ, then* $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$. *If A is regular one has* $\dim_\kappa \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Let Y be a variety. Then the set, $Sing(Y)$, of singular points of Y, is a proper closed subset of Y.

(Topologies on a ring). *Let (A_n) be a filtration of a commutative ring A. Then (A_n) defines a topology on A compatible with the structure of ring, where (A_n) is a fundamental system of neighbourhoods of $0 \in A$. Set $A_\infty = \bigcap_n A_n$. Then A/A_∞ is a metrizable topological ring and its completion $\hat{A}$ is called the* (separated) completion *of A. The adherence in $\hat{A}$ of A_n/A_∞ forms a fundamental system of neighborhoods of 0. Then $\hat{A} = \varprojlim A/A_n$.*

Example 2.45 ($\mathfrak{a}$-adic topology)**.** Let $\mathfrak{a} \subset A$ be an ideal of A and let us consider the filtration $(A_n \equiv \mathfrak{a}^n)$ of A. Then the corresponding topology is called $\mathfrak{a}$-*adic*.

Furthermore, let us assume that $\bigcap_n \mathfrak{a}^n = \{0\}$ and that $\mathfrak{a}$ is of finite type. Then, one has $\overline{\mathfrak{a}^n} = (\overline{\mathfrak{a}})^n = \mathfrak{a}^n\hat{A}$, where $\overline{\mathfrak{a}}$ is the adherence of $\mathfrak{a}$ with respect to the $\mathfrak{a}$-adic topology.

In particular, if $A = R[\chi_1, \cdots, \chi_n]$ and $\mathfrak{a} \equiv \langle \chi_1, \cdots .\chi_n\rangle$, then one has $\hat{A} = R[[\chi_1, \cdots, \chi_n]]$, the ring of formal poweseries in n indeterminates on R, that is $\sum_\alpha c_\alpha \chi^\alpha$, $\chi^\alpha \equiv \chi_1^{\alpha_1} \cdots \chi_n^{\alpha_n}$, $c_\alpha \in R$, $\alpha \equiv (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, converging in the topology of $\hat{A}$. If R is integer, then so is $\hat{A} = R[[\chi_1, \cdots, \chi_n]]$.

Theorem 2.46. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, and let $\hat{A}$ be its completion with respect to the $\mathfrak{m}$-adic topology. Then one has the following properties.* (i) *$\hat{A}$ is a local Noetherian ring with maximal ideal $\widehat{\mathfrak{m}} = \mathfrak{m}\hat{A}$, and there is a natural injective homomorphism $A \to \hat{A}$. The residue field of $\hat{A}$ is canonically isomorphic to that of A. Furthermore, $\hat{A}$ is a flat A-module such that for any $\mathfrak{p} \in S_{pec}(A)$, one has $\mathfrak{p} = A \cap \mathfrak{p}'$, for some $\mathfrak{p}' \in S_{pec}(\hat{A})$, (that is the A-module $\hat{A}$ is* fidelly flat*).* (ii) *If M is a finitely generated A-module, its completion $\hat{M}$ with respect to its $\mathfrak{m}$-adic topology is isomorphic to $M \otimes_A \hat{A}$.* (iii) $\dim A = \dim \hat{A}$. (iv) *A is regular iff $\hat{A}$ is regular. (Furthermore, if A is regular any localized ring $A_{\mathfrak{p}}$ is also regular.)*

(Cohen structure theorem). *If A is a complete regular local ring of dimension n containing some field, then $A \cong \kappa[[x_1, \cdots, x_n]]$, the ring of formal power series over the residue field κ of A.*

(Elimination theorem). *We say two points $p \in X$ and $q \in Y$ are* analytically isomorphic *if there is an isomorphism $\hat{A}(X)_p \cong \hat{A}(Y)_q$ as κ-algebras. Here $A(X) \equiv \kappa[X]$ and $A(Y) \equiv \kappa[Y]$.*

Let $f_1, \cdots, f_r$ be homogeneous polynomials in $x_0, \cdots, x_n$ having indeterminates coefficients a_{ij}. Then there is a set $g_1, \cdots, g_t$ of polynomials in the a_{ij}, with integer coefficients, which are homogeneous in the coefficients of each f_i separately, with the following property: for any field κ, and for any set of spacial values of the $a_{ij} \in \kappa$, a necessary any sufficient condition for the f_i to have a common zero different from $(0, 0, \cdots, 0)$ is that the a_{ij} are a common zero of the polynomials g_j.

Example 2.47 (Examples of singular points)**.** In the following table are reported some examples of singular points of curves and surfaces.

TABLE 2. Examples of singular points

Name	Equation
Singular points of curves in κ^2 ($char \kappa \neq 2$).	
node	$x^4 + y^4 - x^2 = 0$
triple point	$x^6 + y^6 - xy = 0$
cusp	$x^4 + y^4 - x^3 + y^2 = 0$
tacnode	$x^4 + y^4 - x^2 y - xy^2 = 0$
Singular points of surfaces in κ^3 ($char \kappa \neq 2$)	
conical double point	$z^2 - xy^2 = 0$
double line	$z^2 - x^2 - y^2 = 0$
pinch point	$z^3 + y^3 + xy = 0$

Let $Y \subseteq \kappa^2$ be a curve defined by the equation $f(x, y) = 0$. Let $p = (a, b)$ be a point of κ^2. Make a linear change of coordinates so that p becomes the point $(0, 0)$. Then, write f as a sum $f = f_0 + \cdots + f_d$, where f_i is a homogeneous polynomial of degree i in x and y. Then, we define *multiplicity* of p on Y, denoted $\mu_p(Y)$, the least r such that $f_r \neq 0$. (Note that $p \in Y$ iff $\mu_p(Y) > 0$.) The linear factors of f_r are called the *tangent directions* at p. Then one has the following: $\mu_p(Y) = 1$ iff p is a nonsingular point of Y. If $Y, Z \subset \kappa^2$ are two distinct curves, given by equations $f = 0$ and $g = 0$, respectively, and if $p \in Y \cap Z$, we define the *intersection multiplicity*, $(Y.Z)_p$, of Y and Z at p to be the length of A_p-module $A_p/\langle f, g\rangle$. Then one has the following: (a) $(Y.Z)_p$ is finite and $(Y.Z)_p \geq \mu_p(Y).\mu_p(Z)$; (b) If $p \in Y$ for almost all lines L trough p (i.e., all but a finite number) $(L.Y)_p = \mu_p(Y)$.

The fields, the ring $\mathbb{Z}$, the complete Noetherian local rings, are called *excellent rings*.

Let A be an excellent ring. Then the *permanence properties* hold: (i) Any localized ring $S^{-1}A$ is an excellent ring. (ii) Any A-algebre of finite type is an excellent ring.

Furthermore, one has the following propositions: (iii) A is a reduced ring iff $\hat{A}$ is so. (iv) A is integral and integrally closed iff $\hat{A}$ is so. (v) If A is integral, its integral closure A' is a finite A-algebra.

Proposition 2.48 (Derivations and localization)**.** [24]. *Let A be a ring containing a field κ. Let $S \subset A$ be a subset of A. Then $\delta \in Der_\kappa(A, M)$ induces a unique derivation $\delta \in Der_\kappa(S^{-1}A; S^{-1}M)$. More precisely $\delta(\frac{a}{s}) = \frac{s\delta a - a\delta s}{s^2}$. In particular, δ induces a unique derivation $\delta \in Der_\kappa(A_\mathfrak{p}; M_\mathfrak{p})$, $\forall \mathfrak{p} \in S_{pec}(A)$. There are the following isomorphisms:*

$$S^{-1}A \bigotimes_A \Omega^1(A) \cong \Omega^1(S^{-1}A), \quad S^{-1}A \bigotimes_A Der_K(A, M) \cong Der_K(S^{-1}A; S^{-1}M).$$

In particular, one has the following isomorphisms:

$$A_\mathfrak{p} \bigotimes_A \Omega^1(A) \cong \Omega^1(A_\mathfrak{p}), \quad A_\mathfrak{p} \bigotimes_A Der_K(A, M) \cong Der_K(A_\mathfrak{p}; M_\mathfrak{p}),$$

for any $\mathfrak{p} \in S_{pec}(A)$.

Example 2.49. Let us consider a finitely generated integral domain $A = \kappa[\chi_1, \cdots, \chi_n]$ over a field κ, with quotient field $K \equiv Q(A)$ and extension of finitely elements $y_1, \cdots, y_r \in A$ such that A is *integral* over $\kappa[y] = \kappa[y_1, \cdots, y_r]$, that is each element $a \in A$ satisfies a polynomial equation of the form

$$P(a) \equiv a^d + c_1 a^{d-1} + \cdots + c_d = 0,$$

where $c_i \in \kappa[y]$. In that case K is an *algebraic extension* of $\kappa(y)$, that is each element of K satisfies a polynomial equation with coefficients in $\kappa(y) = Q(\kappa[y])$ and $(y_1, \cdots, y_r)$ is called a *trascendence basis* of the extension K/κ of degree r. Then we have that there exists $0 \neq c \in \kappa[y] \subseteq A$ such that $\Omega(A)$ is a free A_c-module with basis $\{dy_i\}$ and $Der_\kappa(A_c)$ is a free A_c-module with basis $\{\partial y_i\}$.

Proposition 2.50. [24] *Let $\kappa \subset K \subset L$ be a chain of extensions of fields. We have the following short exact sequence of vector spaces over L:*

$$0 \to L \bigotimes_K \Omega^1(K)_\kappa \to \Omega^1(L)_\kappa \to \Omega^1(L)_K \to 0.$$

Here we denote $\Omega^1(A)_\kappa \equiv \Omega^1(A)$, (resp. $\Omega^1(A)_K \equiv \Omega^1(A)$) to emphasize that we are talking about κ-derivations, (resp. K-derivations). If L/K is any algebraic extension, we have of course $\Omega(L)_K = 0$ because any derivation of K can be extended uniquely to L and we get the following isomorphism: $L \bigotimes_K \Omega(K)_\kappa \cong \Omega(L)_\kappa$.

Definition 2.51. A *differential ring* is a ring A with a finite number n of commutating derivations $d_1, \cdots, d_n$, $d_i d_j - d_j d_i = 0$, $\forall i, j = 1, \cdots, n$. A *differential ideal* is an ideal $\mathfrak{a} \subset A$ which is stable by each d_i, $i = 1, \cdots, n$.

A differential ring $(A, \{d_j\}_{1 \le j \le n})$ identifies a subring (*subring of constants*): $C \equiv cst(A) \equiv \{a \in A | d_j a = 0, \forall j = 1, \cdots, n\} \subset A$. We may extend each d_i to a derivation of $Q(A)$, still denoted by d_i and such that $d_i(a/r) = (r d_i a - a d_i r)/r^2$, for any $0 \neq r$, $a \in A$.

Example 2.52. If K is a differential field with derivations $\partial_1, \cdots, \partial_\mu$ and y^k, $k = 1, \cdots, m$, are indeterminates over K, we set $y^k_0 = y^k$. Then the polynomial ring $K[y]_d = K[y^k_\mu, k = 1, \cdots, m, \mu = \mu_1 \cdots \mu_s, |\mu| \ge 0]$, can be endowed with a structure of differential ring by defining the *formal derivations* $d_i \equiv \partial_i + y^k_{\mu+1i} \partial y^\mu_k$. Of course $K[y]_d$ is not a Noetherian ring. We write $K[y_q]_d = K[y^k_\mu | k = 1, \cdots, m; 0 \le |\mu| \le q]$ and one has $K(y_q)_d = Q(K[y_q]_d)$. We set also $K(y)_d = Q(K[y]_d)$.

Definition 2.53. A *differential subring* A of a differential ring B is a subring which is stable under the derivations of B. Similarly we can define a *differential extension* L/K of differential fields, and such an extension is said to be *finitely generated* if one can find elements $\eta^1, \cdots, \eta^m \in L$ such that $L = K(\eta^1, \cdots, \eta^m)$. Then the *evaluation epimorphism* is defined by $K[y]_d \to K[\eta]_d \subset L$, $y^k \mapsto \eta^k$. Its kernel is a prime differential ideal.

Proposition 2.54. [24] *Let $\langle S \rangle_d$ denote the differential ideal generated by the subset $S \subset A$, where A is a differential ring. If A is a differential ring and $a, b \in A$, then one has the following:*

(i) $a^{|\mu|+1} d_\mu b \in \langle d_\nu(ab) || \nu| \le |\mu| \rangle$.
(ii) $(d_i a)^{2r-1} \in \langle a^r \rangle_d$.
(iii) *If $\mathfrak{a}$ is a radical differential ideal of the differential ring A and S is any subset of A, then $\mathfrak{a} : S \equiv \{a \in A | \mathfrak{a} S \subset \mathfrak{a}\}$ is again a radical differential ideal of A.*
(iv) *If $\mathfrak{a}$ is a differential ideal of a differential ring A, then $rad(\mathfrak{a})$ is a differential ideal too.*
(v) *One has the following inclusion: $a\, rad\langle S \rangle_d \subset rad\langle aS \rangle_d$, $\forall a \in A$, and for all subset $S \subset A$.*
(vi) *If S and T are two subsets of a differential ring A, then*

$$rad\langle S \rangle_d . rad\langle T \rangle_d \subset rad\langle ST \rangle_d = rad\langle S \rangle_d \cap rad\langle T \rangle_d.$$

(vii) *If S is any subset of a differential ring A, then we have:*

$$rad\langle S, a_1, \cdots, a_r \rangle_d = rad\langle S, a_1 \rangle_d \cap \cdots \cap rad\langle S, a_r \rangle_d.$$

Definition 2.55. A *differential vector space* is a vector space V over a differential field $(K, \partial_i)_{1 \le i \le n}$ such that are defined n homomorphisms d_i, $i = 1, \cdots, n$, of the additive group V such that: $d_i(av) = (\partial_i a)v + a(d_i v)$, $\forall a \in K$, $\forall v \in V$. Then we say that K is a *differential field of definition*.

Proposition 2.56. [24] *Let V be a differential vector space over a differential field K, with derivations d_i, $i = 1, \cdots, n$, and let $\{e_j\}_{j \in I}$ be a basis of V. Then the field* *of definition κ of a differential subspace $W \subset V$ is a differential subfield of K if it contains the field of definition of each $d_1 e_i, \cdots, d_n e_i$ with respect to $\{e_i\}$.*

Definition 2.57. A family $\eta = (\eta^1, \cdots, \eta^m)$ of elements in a differential extension of the differential field K is said to be *differentially algebraically independent* (or a *family of differential indeterminates*) over K, if the kernel of the evaluation epimorphism $K[y]_d \to K[\eta]_d$ is zero. Otherwise the family is said to be *differentially algebraically dependent* (or *differentially algebraic*) over K.

Proposition 2.58. *If K/κ and L/κ are two given differential extensions with respective derivations d_K and d_L, there always exists a differential free composite field of K and L over κ.*

Proof. The ring $K \bigotimes_\kappa L$ has a natural differential structure given by $d(a \otimes b) = (d_K a) \otimes b + a \otimes (d_L b)$, as $d_K|_\kappa = d_L|_\kappa = \partial$. On the other hand there is a finite number of prime ideals $\mathfrak{p}_i \subset K \bigotimes_\kappa L$ such that $\bigcap_i \mathfrak{p}_i = 0$ and $\mathfrak{p}_i + \mathfrak{p}_j = \langle 1 \rangle$, $\forall i \neq j$. Now we have the following lemma.

Lemma 2.59. *If $\mathfrak{a}_1, \cdots, \mathfrak{p}_r$ are ideals of a differential ring A such that $\mathfrak{a}_i + \mathfrak{a}_j = A$, $\forall i \neq j$, and $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_r$ is a differential ideal of A, then each $\mathfrak{a}_i$ is a differential ideal too.*

Therefore we can conclude that each $\mathfrak{p}_i$ is a differential ideal, hence the proposition is proved. □

Lemma 2.60. *A family η is differentially algebraic over K iff a differential polynomial $P \in \mathfrak{p}$ exists such that $(\partial y_P.P) \notin \mathfrak{p}$, where y_P is the highest power of y_p appearing in P. $S_P \equiv (\partial y_P.P)$ is called the* separout *of P. (The* initial *of P is the coefficient of the highest power of y_P appearing in P and it is denoted by I_P. More precisely one has $P = I_P(y_P)^r +$ terms of lower degree.)*

Proposition 2.61 ([24])**.** *If S is any subset of a differential ring A and $r \geq 0$ is any integer, we call* r-prolongation *of S, the ideal*

$$(S)_{+r} = \langle d_\nu a | a \in S,\ 0 \leq |\nu| \leq r \rangle \subset A.$$

One has the following properties: (i) $(S)_{+(r+s)} = ((S)_{+s})_{+r}$. (ii) $(S)_{+\infty} = \langle S \rangle_d$.

(iii) *Let $\mathfrak{a}$ be a differential ideal of the differential ring $K[y]_d$. We set $\mathfrak{a}_q = \mathfrak{a} \cap K[y_q]_d$, $\mathfrak{a}_0 = \mathfrak{a} \cap K[y]_d$, $\mathfrak{a}_\infty = \mathfrak{a}$. We call the* r-prolongation *of $\mathfrak{a}_q$, the following ideal:*

$$(\mathfrak{a}_q)_{+r} = \langle d_\nu P | P \in \mathfrak{a}_q,\ 0 \leq |\nu| \leq r \rangle \subset K[y_{q+r}]_d.$$

One has:

$$(\mathfrak{a}_q)_{+r} \subseteq \mathfrak{a}_{q+r}, \quad (\mathfrak{a}_q)_{+\infty} \subseteq \mathfrak{a}, \quad (\mathfrak{a}_q)_{+r} \cap K[y_q]_d = \mathfrak{a}_q,\ \forall q, r \geq 0.$$

With algebraic sets it is better to consider radical ideals. Hence if $\mathfrak{r} \subset K[y]_d$ is a radical differential ideal, then $\mathfrak{r}_q$ is a radical ideal of $K[y_q]_d$, for all $q \geq 0$. Then if $E_q = Z(\mathfrak{r}_q)$ is the algebraic set defined over K by $\mathfrak{r}_q = I(E_q)$, we call *r-prolongation* of E_q the following algebraic set: $(E_q)_{+r} = Z((\mathfrak{r}_q)_{+r})$. In general one has $(\mathfrak{r}_q)_{+r} \subseteq \mathfrak{r}_{q+r}$, hence $rad((\mathfrak{r}_q)_{+r}) \subseteq \mathfrak{r}_{q+r}$. Therefore, in general one has: $E_{q+r} \subseteq (E_q)_{+r}$.

Proposition 2.62 ([24]). *Let $\mathfrak{p} \subset K[y]_d$ be a prime differential ideal. Then we can identify each field $L_q = Q(K[y_q]_d/\mathfrak{p}_q)$ with a non-differential subfield of $L = Q(K[y]_d/\mathfrak{p})$ and we have: $K \subseteq L_0 \subseteq \cdots \subseteq L_\infty = L$. Then there are vector spaces R_q over L_q or L defined by the following linear system:*

$$(\partial y_k^\mu.P_\tau)(\eta)v_\mu^k = 0, \ \{1 \leq \tau \leq t, \quad 1 \leq k \leq m, \quad |\mu| = q\},$$

*where η is a generic solution of $\mathfrak{p}$ and $P_1, \cdots, P_t$ are generating $\mathfrak{p}_q$. Such result does not depend on the generating polynomials. We can also define the vector space g_q (*symbol*) over L_q or L, by means of the linear system:*

$$(\partial y_k^\mu.P_\tau)(\eta)v_\mu^k = 0, \ \{1 \leq \tau \leq t, \quad 1 \leq k \leq m, \quad 0 \leq |\mu| \leq q\}.$$

For the prolongations $(g_q)_{+r}$ one has, in general, $g_{q+r} \subseteq (g_q)_{+r}$, $\forall q, r \geq 0$.

Definition 2.63. We say that R_q or g_q is *generic* over E_q, if one can find a certain number of maximum rank determinants D_α that cannot be all zero at a generic solution of $\mathfrak{p}$.

Proposition 2.64. *R_q or g_q is generic if we may find polynomials $A_\alpha, B_\tau \in K[y_q]_d$ such that:*

$$\sum_\alpha A_\alpha D_\alpha + \sum_\tau B_\tau P_\tau = 1.$$

Furthermore, R_q or g_q are projective modules over the ring $K[y_q]_d/\mathfrak{p}_q \subset K[y]_d/\mathfrak{p}$.

Proof. It follows directly from the Hilbert theorem of zeros. □

Theorem 2.65 (Primality criterion [24]). *Let $\mathfrak{p}_q \subset K[y_q]_d$ and $\mathfrak{p}_{q+1} \subset K[y_{q+1}]_d$ be prime ideals such that $\mathfrak{p}_{q+1} = (\mathfrak{p}_q)_{+1}$ and $\mathfrak{p}_{q+1} \cap K[y_q]_d = \mathfrak{p}_q$. If the symbol g_q of the variety R_q defined by $\mathfrak{p}_q$ is 2-acyclic and its first prolongation g_{q+1} is generic over E_q, then $\mathfrak{p} = (\mathfrak{p}_q)_{+\infty}$ is a prime differential ideal with $\mathfrak{p} \cap K[y_{q+r}]_d = (\mathfrak{p}_q)_{+r}$, for all $r \geq 0$.*

Let $\mathfrak{r}_q \subset K[y_q]_d$ and $\mathfrak{r}_{q+1} \subset K[y_{q+1}]_d$ be radical ideals such that $\mathfrak{r}_{q+1} = (\mathfrak{r})_{+1}$ and $\mathfrak{r}_{q+1} \cap K[y_q]_d = \mathfrak{r}_q$. If the symbol g_q of the algebraic set E_q defined by $\mathfrak{r}_q$ is 2-acyclic and its first prolongation g_{q+1} is generic over E_q, then $\mathfrak{r} = (\mathfrak{r}_q)_{+\infty}$ is a radical differential ideal with $\mathfrak{r} \cap K[y_{q+r}]_d = (\mathfrak{r}_q)_{+r}$, for all $r \geq 0$.

Theorem 2.66 (Differential basis). *If $\mathfrak{r}$ is a differential ideal of $K[y]_d$, then $\mathfrak{r} = rad((\mathfrak{r}_q)_{+\infty})$ for q sufficiently large.*

Proof. In fact one has the following lemma.

Lemma 2.67. *If $\mathfrak{p}$ is a prime ideal of $K[y]_d$, then for q sufficiently large, there is a polynomial $P \in K[y_q]_d$ such that $P \notin \mathfrak{p}_q$ and $P\mathfrak{p}_{q+r} \subset rad((\mathfrak{p}_q)_{+r}) \subset \mathfrak{p}_{q+r}$, for all $r \geq 0$.*

After above lemma the proof follows directly. □

Every radical differential ideal of $K[y]$ can be expressed in a unique way as the non-redundant intersection of a finite number of prime differential ideals. The smallest field of definition κ of a prime differential ideal $\mathfrak{p} \subset K[y]$ is a finitely generated differential extension of $\mathbb{Q}$.

Example 2.68. With $n = 2$, $m = 2$, $q = 1$. Let us consider the differential polynomial $P = y^1_1 y^2_2 - y^2_1 y^1_2 - 1$. We obtain for the symbol g_1: $y^1_1 v^2_2 + y^2_2 v^1_1 - y^1_2 v^2_1 - y^2_1 v^1_2 = 0$. Setting $v^k_i = y^k_l w^l_i$ we obtain $(y^1_1 y^2_2 - y^2_1 y^1_2)(w^2_2 + w^1_1) = 0$ and thus $w^2_2 + w^1_1 = 0$ on E_1. Hence g_1 is generic. One can also set $P_1 = y^1_2$, $P_2 = y^2_1$ and we get the relation: $y^2_2 P_1 - y^1_2 P_2 - P \equiv 1$. A similar result should hold for E_1. g_1 is involutive and the differential ideal generted by P in $\mathbb{Q}\langle y^1, y^2\rangle$ is therefore a prime ideal.

Definition 2.69. A *differentially algebraic extension* L over of a differential field K is a differential extension over K where every element of L is differentially algebraic over K.

The *differential transcendence degree* of a differential extension L/K is the number of elements of a maximal subset S of elements of L that are differentially transcendental over K and such that L becomes differentially algebraic over $K(S)$. We shall denote such number by $trd_d(L/K)$.

Theorem 2.70 ([24])**.** *One has the following formula:*

$$\dim(\mathfrak{p}_{q+r}) = \dim(\mathfrak{p}_{q-1}) + \sum_{1\le i\le n} \frac{(r+i)!}{r!i!}\alpha^i_q, \ \ \forall r \geq 0,$$

where α^i_q is the character of the corresponding system of PDE's. The character α^i_q of a q-order PDE $E_q \subset JD^q(W)$, $\pi : W \to M$, $\dim M = n$, with symbol g_q, is the integer $\alpha^i_q \equiv \dim(g^{(i-1)}_q)_p - \dim(g^{(i)}_q)_p$, $p \in E_q$, where $(g^{(i)}_q)_p \equiv \{\zeta \in (g_q)_p | \zeta(v_1) = \cdots = \zeta(v_i) = 0\}$, where $(v_1, \cdots, v_n)$ is the natural basis in $T_{\pi_k(p)}M$.

The character α^n_q and the smallest non-zero character only depend on the differential extension L/K and not on the generators. In particular, one has: $trd_d(L/K) = \alpha^n_q$.

If ζ is differentially algebraic over $K(\eta)_d$ and η is differentially algebraic over K, then ζ is differentially algebraic over K.

If L/K is a differential extension and $\xi, \eta \in L$ are both differentially algebraic over K, then $\xi + \eta$, $\xi\eta$, ξ/η, $(\eta \neq 0)$, and $d_i\xi$ are differentially algebraic over K.

Definition 2.71. If L/K is a differential extension, the set of elements of L that are algebraic over K is an intermediate field K_0, called the *algebraic closure* of K in L. Furthermore, the set of elements of L that are differentially algebraic over K is an intermediate differential field K', called the *differential algebraic closure* of K in L. (K' is differentially algebraic closed in L.)

If K is a differential field with derivations $\partial_1, \cdots, \partial_n$, we say that the derivative operators $\{\partial_\mu\}_{|\mu|\geq 0}$ are *algebraically independent* over K if there does not exist a differential indeterminate z over K and a nontrivial differential polynomial in $K[z]_d$ vanishing on any element of K.

Proposition 2.72 ([24])**.** *The derivative operators $\{\partial_\mu\}_{|\mu|\geq 0}$ are algebraically independent over K iff one of the following equivalent propositions are verified:*

- (i) *The derivative operators $\{\partial_\mu\}_{|\mu|\geq 0}$ are linearly independent over K.*
- (ii) *The derivatives $\partial_1, \cdots, \partial_n$ are linearly independent over K. (In such a case we simply say that $\partial_1, \cdots, \partial_n$ are* independent *over K.)*

Theorem 2.73 (Differentially primitive element [24])**.** *If K is a differential field with independent derivations, then every finitely generated differentially algebraic extension of K can be generated by a simple element.*

If L/K is a finitely generated differentially algebraic extension with derivations $d_1, \cdots, d_n$, then L/K can be considered as a finitely generated extension with derivations $d'_1, \cdots, d'_{n-1}$ such that $d'_j = c^i_j d_i$ for certain $c^i_j \in C = cst(K)$.

If L/K is a finitely generated differential extension, then one has the following propositions: (i) *Any intermediate differential field between K and L is also finitely generated over K.* (ii) *If $C = cst(K)$, $D = cst(L)$, then D/C is a finitely generated extension.* (iii) *If K_0 is the algebraic closure of K in L, then K_0 is a finitely generated extension of K and $|K_0/K| < \infty$.* (iv) *If K' is the differential algebraic closure of K in L, then K'/K is a finitely generated differential extension.*

If ζ is differentially algebraic over $K(\eta)_d$ but η is not differentially algebraic over $K(\zeta)_d$, then ζ is differentially algebraic over K.

If $K \subset L \subset M$ are differential fields and S is a differential transcendence basis of L/K while T is a differential transcendence basis of $M(L)$, then $S \cap T = \emptyset$ and $S \cup T$ is a differential transcendence basis of M/K. Furthermore

$$trd_d(M/K) = trd_d(M/L) + trd_d(L/K).$$

Let $(A, \{d_j\}_{1 \le j \le n}$ be a differential ring such that A is an integral domain with field quotients $K = Q(A)$, then $cst(A)$ is integrally closed in A while $cst(K)$ is algebraically closed in K. In general one has $cst(A) \subseteq cst(K)$.

Let M and N be filtered modules over the filtered ring A. This means that there are increasing filtrations, $\cdots \subset M_{q-1} \subset M_q \subset \cdots \subset M$ and $\cdots \subset N_{q-1} \subset N_q \subset \cdots \subset M$, such that $A_r M_q \subseteq M_{q+r}$ and $A_r N_q \subseteq N_{q+r}$ respectively, for all $q, r \ge 0$. To such filtered modules we can associate graded modules $gr(M) \equiv G = \bigoplus_{0 \le q \le \infty} G_q$ and $gr(N) \equiv H = \bigoplus_{0 \le q \le \infty} H_q$ respectively, such that $G_q \equiv M_q/M_{q-1}$ and $H_q \equiv N_q/N_{q-1}$.[9] *Then any morphism $f : M \to N$,* compatible *with above filtrations, that is $f(M_q) \subset N_q$, induces by restriction morphisms $f_q : M_q \to N_q$ that pass to quotient inducing morphisms $gr_q(f) : G_q \to H_q$, hence a morphism $gr(f) : G \to H$, such that the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & M_{q-1} \arrow[r] \arrow[d, "f_{q-1}"'] & M_q \arrow[r] \arrow[d, "f_q"'] & G_q \arrow[r] \arrow[d, "gr_q(f)"'] & 0 \\ 0 \arrow[r] & N_{q-1} \arrow[r] & N_q \arrow[r] & H_q \arrow[r] & 0 \end{tikzcd}$$

is commutative with exact horizontal lines. In general $f(M) \cap N_q \ne f_q(M_q)$, for all $q \ge 0$.

Definition 2.74. A *strict* morphism $f : M \to N$ between filtered modules is a compatible morphism such that the following equivalent propositions are verified: (i) $f(M) \cap N_q = f_q(M_q)$, for all $q \ge 0$. (ii) The following short exact sequences are exact $0 \to \mathrm{coker}(f_q) \to \mathrm{coker}(if)$, for all $q \ge 0$.

[9] Note that G and H are modules over the graded ring $gr(A)$.

Proposition 2.75. [24] 1) *For filtered modules and strict compatible morphisms hold the following propositions:*

(i) *A short sequence*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact iff the associated sequence

$$0 \to gr(M') \xrightarrow{gr(f)} gr(M) \xrightarrow{gr(g)} gr(M'') \to 0$$

is exact.

(ii) *Let $M' \xrightarrow{f} M \xrightarrow{g} M''$ be an exact sequence then the associated sequence $gr(M') \xrightarrow{gr(f)} gr(M) \xrightarrow{gr(g)} gr(M'')$ is exact too.*

(iii) *If the sequence $gr(M') \xrightarrow{gr(f)} gr(M) \xrightarrow{gr(g)} gr(M'')$ is exact (without assuming that f and g are strict), and $M = \bigcup_{q \in \mathbf{Z}} M_q$ with $M_q = 0$ for $q \ll 0$, then f and g are strict and the filtered sequence $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact.*

If the sequence of filtered modules $M' \xrightarrow{f} M \xrightarrow{g} M''$ is exact does not imply that sequences $M'_q \xrightarrow{f_q} M_q \xrightarrow{g_q} M''_q$ are exact for all $q \geq 0$.

Let

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

be an exact sequence of Noetherian filtered modules and compatible morphisms. If the associated sequence

$$G'_q \xrightarrow{gr_q(f)} G_q \xrightarrow{gr_q(g)} M''_q$$

are exact for sufficiently large q, then the sequence

$$M'_q \xrightarrow{f_q} M_q \xrightarrow{g_q} M''_q$$

are exact for sufficiently large q.

2) *Let M be a filtered module over a filtered ring A. One has the following propositions:* (i) *M is finitely generated iff $gr(M)$ is finitely generated over $gr(A)$. This is equivalent to say that there exist homogeneous elements $\bar{x}_1, \cdots, \bar{x}_r \in G$, with $\bar{x}_i \in G_{q_i}$ being the canonical projection of $x_i \in M_{q_i}$ such that any element of G_q may be written as finite sum $\sum_{1 \leq i \leq r} \bar{a}_i \bar{x}_i$ where $\bar{a}_i \in gr_{q-q_i}(A)$.*

M is Noetherian iff $gr(M)$ is Noetherian.

3) *If A is a Noetherian filtered ring, the following three conditions are equivalent:*

(i) *M is a finitely generated A-module.*

(ii) *G is a finitely generated $gr(A)$-module.*

(iii) *There exist q_0 such that $A_1 M_q = M_{q+1}$ for any $q \geq q_0$.*

Theorem 2.76. *Let $(A, \{\partial_j\}_{1 \leq j \leq n})$ be a differential ring. The set $D(A)$ of differential operators over $(A, \{\partial_j\}_{1 \leq j \leq n})$ is a non-commutative filtered ring and a filtered bimodule over A.*

Proof. If y is a differential indeterminate over A, we may introduce the *formal derivatives* $d_1, \cdots, d_n$ which are such that $d_i d_j - d_j d_i = 0$, $\forall i, j = 1, \cdots, n$, and are defined by: $d_i(ay) = (\partial_i a)y + a(d_i y)$. We shall write $d_i y = y_i$, $d_i y_\mu = y_{\mu+1i}$, where μ is the multi-index $\mu = (\mu_1, \cdots, \mu_n)$ with length $|\mu| = \mu_1 + \cdots + \mu_n$. If

$y = (y^1, \cdots, y^m)$, we set $d_\mu = (d_1)^{\mu_1} \cdots (d_n)^{\mu_n}$ and $d_\mu y^k = y^k_\mu$. Any differential operator of order q over A can be written in the form $P = \sum_{0\le\mu\le q} a^\mu d_\mu$, $a^\mu \in A$. Set $ord(P) = q$. Then, we can write $D(A) \cong A[d_1, \cdots, d_n] \equiv A[d]$ the *ring of partial differential operators* over A with derivatives $d_1, \cdots, d_n$. The addition rule is clear. The multiplication rule comes from the Leibniz formula:

$$\left\{ \begin{array}{l} \partial_\nu(ab) = \sum_{\lambda+\mu=\nu} \frac{\nu!}{\lambda!\mu!}(\partial_\lambda a)(\partial_\mu b) \\ d_\nu(ay) = \sum_{\lambda+\mu=\nu} \frac{\nu!}{\lambda!\mu!}(\partial_\lambda a) d_\mu y \end{array} \right\} \Rightarrow d_\nu a = \sum_{\lambda+\mu=\nu} \frac{\nu!}{\lambda!\mu!}(\partial_\lambda a) d_\mu.$$

Here we have put $\mu! = \mu_1! \cdots \mu_n!$. With these rules $D(A)$ becomes a non-commutative ring and a bimodule over A. In fact, the previous formula defines the right action of A on $D(A)$. The left action of A on $D(A)$ is simply the multiplication on the left by A, that is $aP = a(\sum_{0\le\mu\le q} a^\mu d_\mu) = \sum_{0\le\mu\le q} aa^\mu d_\mu$. Now, the filtration of $D(A)$ is naturally induced by filtration of spaces of differential operators. More precisely $D_q(A) = \{P \in D(A) | ord(P) \le q\}$, where $ord(P) = sup\{|\mu| | a^\mu \ne 0\}$. We set $D_{-1}(A) = 0$ and $D_0(A) = A$. Then, $D_q(A) \subset D_{q+1}(A)$, $D(A) = \bigcup_{q\ge0} D_q(A)$ and $D_q(A)D_p(A) \subseteq D_{p+q}(A)$. □

Definition 2.77. If $\chi_1, \cdots, \chi_n$, are indeterminates over A, we define the *prinipal symbol* of $P \in D(A)$, with respect to χ, by setting: $\sigma_\chi(P) = \sum_{|\mu|=ord(P)} a^\mu \chi_\mu$.

If $f, g \in A[\chi]$, we define the *Poisson bracket*:

$$\{f, g\} = \sum_{1\le i\le n} (\partial\chi_i.f)(\partial_i.g) - (\partial\chi_i.g)(\partial_i.f) \in A[\chi],$$

where $\partial_i.f$ is the polynomial obtained by applying ∂_i to the coefficients of the polynomial f.

Proposition 2.78. *For any $P, Q \in D(A)$ with $ord(P) = p$, $ord(Q) = q$, we have: $\sigma_\chi(PQ) = \sigma_\chi(P)\sigma_\chi(Q)$; $ord([P, Q]) \le p + q - 1$, with $[P, Q] = PQ - QP$; $\{\sigma_\chi(P), \sigma_\chi(Q)\} \ne 0 \Rightarrow ord([P, Q]) = p + q - 1$ and $\sigma_\chi([P, Q]) = \{\sigma_\chi(P), \sigma_\chi(Q)\}$.*

$D_q(A)$ identifies a representable functor D_q on the category of modules. More precisely, for any A-module E one has the following A-module

$$D_q(E) = Hom_A(\mathcal{I}^q(E); A),$$

where $\mathcal{I}^q(E) \equiv \mathcal{I}^q(A) \bigotimes_A E$ and $\mathcal{I}^q(A) \equiv Hom_A(D_q(A); A)$. One has the isomorphism: $D_q(E) \cong Hom_A(\mathcal{I}^q(A); E^*)$, where $E^* \equiv Hom_A(E; A)$.

If E is finitely generated and projective or free one has the isomorphism $D_q(E) \cong D_q(A) \bigotimes_A E^*$.

$D(A)$ identifies a functor from the category of modules to the category of filtered modules. More precisely, for any A-module E one has: $D(E) = \bigcup_{q\ge0} D_q(E)$.

If E is finitely generated projective module over A, one has the isomorphism: $D(E) \cong D(A) \bigotimes_A E^*$.

$D(A)$ identifies a representable functor of two variables, controvariant in the first and covariant in the second. More precisely one has; $D_q(E; F) = Hom_A(\mathcal{I}^q(E); F)$.

Proof. Let $\mathcal{I}^q(A) \equiv Hom_A(D_q(A); A) \equiv D_q(A)^*$. One has the isomorphism $D_q(A) \cong Hom_A(\mathcal{I}^q(A); A) \equiv \mathcal{I}^q(A)^*$. Then, if E is a module over A, setting $\mathcal{I}^q(E) = mathcalI^q(A) \bigotimes_A E$, we get on $\mathcal{I}^q(E)$ a left-module structure over A. If

$\{e_k\}$ is a set of generators for E, typical terms of $\mathcal{I}^q(E)$ are $\xi^k_\mu \otimes e_k$, $0 \leq |\mu| \leq q$, $\xi^k_\mu \in A$. One has also the *q-jet operator*: $j_q : E \to \mathcal{I}^q(E)$, $j_q(\xi^k e_k) = (\partial_\mu \xi^k) \otimes e_k$, $0 \leq |\mu| \leq q$. One has the isomorphism:

$$D_q(E) \equiv Hom_A(\mathcal{I}^q(E); A) \cong Hom_A(\mathcal{I}^q(A); E^*).$$

If E is finitely genertaed and projective or free one has the following isomorphisms:

$$Hom_A(\mathcal{I}^q(A); E^*) \cong Hom_A(\mathcal{I}^q(A); A) \bigotimes_A E^* \cong \mathcal{I}^q(A)^* \bigotimes_A E^* \cong D_q(A) \bigotimes_A E^*.$$

It is enough to consider that to the inclusions $D_q(A) \subset D_{q+1}(A)$ there correspond the projection $\mathcal{I}^{q+1}(A) \to \mathcal{I}^q(A)$. Then by taking the inductive limit on the filtration of $D(A)$ we get $D(A) = \lim\limits_{\leftarrow} \mathcal{I}^q(A)^* \equiv \mathcal{I}^\infty(A)^*$ and the corresponding projective limit gives $\mathcal{I}^\infty(A) = \lim\limits_{\rightarrow} \mathcal{I}^q(A)$. So $D(E) = \bigcup_{q \geq 0} D_q(E)$.

This follows directly from the above points.

Furthermore,

$$\begin{aligned} D(E,F) &= F \bigotimes_A D(A) \bigotimes_A E^* \cong F \bigotimes_A D(E) \\ &\cong F \bigotimes_A Hom_A(\mathcal{I}^\infty(E); A) \cong Hom_A(\mathcal{I}^\infty(E); F). \end{aligned} \tag{2.3}$$

The ring A can be considered a left-module over $D(A)$: $D(A) \times A \to A$, $(P,a) \mapsto P(a)$. Then one has the isomorphisms: $Hom_{D(A)}(D(A); A) \cong A$; $A \cong D(A)/I(A)$, $D(A) \cong A \bigoplus I(A)$, where $I(A) \equiv \{P \in D(A) | P(1) = 0\}$. $Hom_{D(A)}(A; A) = C = cst(A)$. $T(A) \equiv D_1(A)/A \cong \{P \in D_1(A) | P(1) = 0\} \subset Der_C(A)$; $D_1(A) \cong A \bigoplus T(A)$.

The left action of $D(A)$ on A coincides with the evaluation of the differential operators, i.e., $(P,a) \mapsto P(a)$. This induces the isomorphism (i). $I(A)$ is a left ideal of $D(A)$ generated by $d_1, \cdots, d_n$. This induces the isomorphism (ii). By means of the inclusions $A = D_0(A) \subset D(A)$, we can identify A with a subring of $D(A)$, hence any $D(A)$-module can be considered also as a A-module, just by forgetting about the differential structure. Then, one has $Hom_{D(A)}(A; A) = cst(A) \equiv C$. Furthermore, for the A-module $D_1(A)/D_0(A) = D_1(A)/A$ one has the natural short exact sequence: $0 \to D_1(A)/A \to Der_C(A)$, as any element $\delta \in D_1(A)/A$ is of the form $\delta = a^1 d_1 + \cdots + a^n d_n$, hence can be identified with a vector $\delta = a^1 \partial_1 + \cdots + a^n \partial_n \in Der_C(A)$. Therefore, $T(A) \equiv D_1(A)/A$ is a free module over A. Furthermore, as $T(A)$ can be considered also an element of $D_1(A)$, we get isomorphism (iv).

If A is an integral domain then $D(A)$ is also an integral domain.

Identifying $T(A) \subset D(A)$ as a submodule, the commutator in $D(A)$ restricts to $T(A)$ to produce the standard bracket for vector fields.

We have the following short exact sequences:

$$0 \to S_q T^*(A) \to \mathcal{I}^q(A) \to \mathcal{I}^{q-1}(A) \to 0 \quad 0 \to D_{q-1}(A) \to D_q(A) \to S_q T(A) \to 0$$

where $T^*(A) \equiv Hom_A(T(A); A)$.

For the graded module $gr(D(A))$ associated with the filtered module $D(A)$, i.e., $gr_q(D(A)) = D_q(A)/D_{q-1}(A)$, we have: $gr_q(D(A)) \cong S_qT(A)$. Furthermore, introducing indeterminates $\chi_1, \cdots, \chi_n$ over A, we get: $gr(D(A)) \cong A[\chi_1, \cdots, \chi_n]$.

If A is a Noetherian ring, then $D(A)$ is also a Noetherian ring.

The module $D(E)$ can be considered also a left-module over $D(A)$: $D(A) \times D(E) \to D(E)$, $(P,Q)(e) = P(Q(e))$. In particular, if E is a finitely generated projective module over A, one has the isomorphism $Hom_{D(A)}(D(E);A) \cong E$. One has a canonical C-linear map (*adjoint involution*):

$$ad \in Hom_C(D(A);D(A)), ad(P) = ad(a^\mu d_m) = (-1)^{|\mu|} d_\mu a^\mu.$$

One has the following properties: (i) $ad^2 = id_{D(A)}$; (ii) $ad|_{D0=A} = id_A$; (iii) $ad(d_i) = -d_i$; (iv) $ad(PQ) = ad(Q).ad(P)$; If $P \in D(A)$ is a differential operator of order q and $Q \in D(E)$ is a differential operator of order r, $(PQ) \in D(E)$ is a differential operator of order $q+r$, such that for any $e \in E$ one has $P(Q(e))$.

In fact, under our hypotheses one has:

$$Hom_{D(A)}(D(E);A) \cong Hom_{D(A)}(D(A) \bigotimes_A E^*;A) \cong Hom_A(E^*;A) \cong E^{**} \cong E. \ \square$$

Theorem 2.79 (Localization in noncommutative ring)**.** *Let A be a noncommutative ring and let S be a* multiplicative set *in A, that is $1 \in S$ and S is closed under multiplication. Let us assume that the following two conditions are satisfied:* (i) $Sa \cap As \neq 0$, $\forall a \in A$, $s \in S$. (Left Ore set condition). (ii) *If $s \in S$ and $a \in A$ are such that $as = 0$, then there is $t \in S$ such that $ta = 0$. Then there exists a ring $S^{-1}A$ (*left ring fractions *or* left localization *of A with respect to S) and a homomorphism $\theta = \theta_S : A \to S^{-1}A$, with the following properties:* (iii) *$\theta(s)$ is invertible in $S^{-1}A$;* (iv) *Each element of $S^{-1}A$ or* fraction *has the form $\theta(s)\theta(a)$ for some $s \in S$ and $a \in A$;* (v) $\ker(\theta) = \{a \in A | \exists s \in S, sa = 0\}$. *More precisely $S^{-1}A = S \times A/\sim$, where $\sim$ is the following equivalence relation: $(s,a) \sim (t,b) \Leftrightarrow \exists u,v \in A$ such that $us = vt \in S$, $ua = vb$. We will denote the equivalence class of (s,a) by $s^{-1}a \in S^{-1}A$.*[10] *For symmetry we have a similar theorem for* right ring fractions *or* right localization *of A with respect to S. If A is Noetherian and $S \subset A$ is simply a left Ore set in A, then the existence of $S^{-1}A$ is assured.*[11]

Proof. We shall use the following lemmas.

Lemma 2.80. *If there exists a left localization of A with respect to S, that is a homomorphism $\theta_S = \theta : A \to S^{-1}A$ such that the above conditions* (iii), (iv), (v) *are satisfied, then also conditions* (i) *and* (ii) *must be satisfied.*

Lemma 2.81. *If S is a left Ore set in a ring A, then one has the following properties:* (a) *If $s,t \in S$ then $A_s \cap A_t \cap S \neq 0$;* (b) *Two fractions $\theta(s)^{-1}\theta(a)$ and $\theta(t)^{-1}\theta(b)$, $a,b \in A$, can be reduced to the same common denominator $\theta(us) = \theta(vt)$, where $u,v \in A$ such that $us = vt \in S$.*

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[10] Let us emphasize that for a non-commutative ring A we cannot write $\frac{a}{s}$ as it does not distinguish between $s^{-1}a$ and as^{-1}.

[11] Compare with a recent result on the noncommutative localization introduced by A.Ranicki [34]

Now, in order to conclude the proof it is enough to prove that the above equivalence relation is well defined in $S \times A$. This can be down just by using above lemmas. The other proof can be conduced for similarity.

furthermore, to end it is enough to prove that from the fact that A is Noetherian, the assumption that S satisfies the left Ore condition (i) implies also condition (ii). Now let $s \in$ and $a \in A$ such that $as = 0$. Let us denote yet by ann the left annihilator in A. Then one has $ann(s^n) \subseteq ann(s^{n+1})$, $\forall n \in \mathbf{N}$. As A is Noetherian, we have $ann(s^n) = ann(s^{n+1})$ for $n \gg 0$. From the left Ore condition we may find $t \in S$, $b \in A$ such that $ta = bs^n$ and we get $bs^{n+1} = tas = 0$. Hence, $b \in ann(s^{n+1}) = ann(s^n)$. Therefore $ta = 0$. □

Theorem 2.82 (Localization of module over noncommutative ring)**.** *Let M be a left module over the noncommutative ring A and let $S \subset A$ be a set in A that satisfies conditions* (i) *and* (ii) *in Theorem 3.124(1). Then there exists a left module $S^{-1}M$ over $S^{-1}A$ (*left module of fractions *or* left localization *of M with respect to S) and a homomorphism $\theta_S = \theta : M \to S^{-1}M$ with the following properties:*

(iii) *Each element of $S^{-1}M$ has the form $s^{-1}\theta(x)$ for $s \in S$, $x \in M$;*

(iv) $\ker(\theta_S) = t_S(M) \equiv \{x \in M | \exists s,\ sx = 0\} \equiv S$-torsion submodule *of M. More precisely $S^{-1}M = S \times M/\sim$, where $\sim$ is the following equivalence relation: $(s, x) \sim (t, y) \Leftrightarrow \exists u, v \in A$ such that $us = vt \in S$, $ux = vy$. For symmetry we have a similar theorem for* right module fractions *or* right localization *of M with respect to S. One has the following isomorphism of modules over $S^{-1}A$:*

$$(S^{-1}A) \bigotimes_A M \cong S^{-1}M.$$

Proof. The proof can be conduced for similarity with one of the previous Theorem 3.124(1). Let us simply emphasize, here, that $t_S(M)$ is just a sub-module of M. For symmetry with respect to the point. This isomorphism is induced by the *multiplication map* $S^{-1}A \times M \to S^{-1}M$. □

Proposition 2.83 (Internal operations)**.** *Let A be a differential ring with n derivations $\{\partial_i\}_{1 \le i \le n}$. A vector $\zeta \in T(A)$ can be written $\zeta = \zeta^i \partial_i$, $\zeta^i \in A$, and a r-form $\alpha \in \Lambda^r T^*(A)$, $T^*(A) \equiv Hom_A(T(A); A)$ can be written in the form $\alpha = \sum_{1 \le i_1 < \cdots < i_r \le n} \alpha_{i_1, \cdots, i_r} \delta^{i_1} \wedge \cdots \wedge \delta^{i_r} \equiv \alpha_I \delta^I$, where $\alpha_{i_1, \cdots, i_r}, \alpha_I \in A$ and $I \equiv (i_1 < \cdots < i_r)$ is a multi-index and $\delta^I \equiv \delta^{i_1} \wedge \cdots \wedge \delta^{i_r}$.*[12] *One has the following distinguished properties:* (1) (Exterior differential). $d : \Lambda^r T^*(A) \to \Lambda^{r+1} T^*(A)$, *$d\alpha = (\partial_i a_I)\delta^i \wedge \delta^I$, $a_I.(\partial_i a_I) \in A$. One has the following property:*

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^r \alpha \wedge d\beta,\ \alpha \in \Lambda^r T^*(A),\ \beta \in \Lambda^s T^*(A).$$

(2) (Interior multiplication by a vector $\xi \in T(A)$). $\xi \rfloor : \Lambda^{r+1} T^*(A) \to \Lambda^r T^*(A)$, $\xi \rfloor \alpha = (\xi^i \alpha_{i i_1 \cdots i_r}) \delta^{i_1} \wedge \cdots \wedge \delta^{i_r}$. *One has the following properties:* (i) $(\xi \rfloor d + d\xi \rfloor) a = \xi \rfloor da = \xi(a) = \xi^i(\partial_i a) \in A$, $a \in A$. (ii) $\xi \rfloor d + d\xi \rfloor d = d(\xi \rfloor) d + d\xi \rfloor$, *(as $d^2 = 0$).* (iii) $\xi \rfloor (\alpha \wedge \beta) = (\xi \rfloor \alpha) \wedge \beta + (-1)^r \alpha \wedge (\xi \rfloor \beta)$, $\alpha \in \Lambda^r T^*(A)$, $\beta \in \Lambda^s T^*(A)$.

(3) (Lie derivative with respect to a vector $\xi \in T(A)$). $\mathcal{L}_\xi : \Lambda^r T^*(A) \to \Lambda^r T^*(A)$, $\mathcal{L}_\xi \alpha = \xi^i \rfloor (\alpha) + d(\xi \rfloor \alpha)$, $\alpha \in \Lambda^r T^*(A)$. *One has the following property:* $[\mathcal{L}_\xi, \mathcal{L}_\zeta] = \mathcal{L}_{[\xi, \zeta]}$, $\forall \xi, \zeta \in T(A)$.

[12]For example if $A = \mathbb{Q}[\chi^1, \cdots, \chi^n]$, one has $\partial_i = \partial x_i$, $\delta^i = dx^i$, $i = 1, \cdots, n$.

Let A be Noetherian integral domain. Then $D(A)$ is also a Noetherian integral domain and if $S \subset D(A)$ is a left Ore domain in $D(A)$, then one has the following isomorphisms: $S^{-1}D(A) \cong D(A)S^{-1}$.

(Adjoint operation). *The adjoint operation identifies a functor from the category of left $D(A)$-modules to the category of right $D(A)$-modules. More precisely $m.P = ad(P)m$, for any $m \in M$, where M is a A-module, and $P \in D(A)$. One has $m.PQ = ad(PQ)m = ad(Q)ad(P)m = (m.P).Q$. $ad(ad(P)) = P$.*

$\Lambda^n T^*(A)$ has a right module structure over $D(A)$, if A is a differential ring with derivatives $\{\partial_i\}_{1\le i\le n}$.

Proof. If $\alpha = a\delta^1 \wedge \cdots \wedge \delta^n$, $a \in A$, one has $\alpha.P = ad(P)(a)\delta^1 \wedge \cdots \wedge \delta^n$. An alternative proof can be given also by taking into account that $D(A)$ is generated by $D_1(A) = A \bigoplus T(A)$. Then, any P can be considered as linear combinations of iterated operators of first order. Such operators can be written as linear combinations of term of the form $a+\xi$, with $a \in A$, $\xi \in T(A)$. Then, it is enough define the right action $\alpha.(a+\xi) = \alpha.a + \alpha.\xi$. The action $\alpha.a$ is defined by $\alpha(\zeta)a$, $\forall\zeta \in T(A)$. Furthermore, one has: $\alpha.\xi = -\mathcal{L}_\xi\alpha$. □

Proposition 2.84 ([24]). *If M and N are left $D(A)$-modules, then $Hom_A(M;N)$ and $M \bigotimes_A N$ become left $D(A)$-modules. If M is a left $D(A)$-module and N is a right $D(A)$-module, then $M \bigotimes_A N$ and $Hom_A(M;N)$ become right $D(A)$-modules. If M and N are right $D(A)$-modules, $Hom_A(M;N)$ becomes a left $D(A)$-modules. If M is a left $D(A)$-module and N is a right $D(A)$-module, then one has the following properties:* (i) *$M_r \equiv \Lambda^n T^*(A) \bigotimes_A M$ is a right $D(A)$-module, called the* converted right $D(A)$-module *of M.* (ii) *$N_l \equiv Hom_A(\Lambda^n T^*(A); N) \cong \Lambda^n T(A) \bigotimes_A N$ is a left $D(A)$-module, called the* converted left $D(A)$-module *of N. Above conversions identify corresponding functors that are inverse to each other.*

Definition 2.85. A *linear differential system of order q, on the A-module E*, is a submodule over A, $E_q \subset \mathcal{I}^q(E)$ of the A-module $\mathcal{I}^q(E)$. We say that $E_q \subset \mathcal{I}^q(E)$ is a *normal differential system* if $F \equiv \mathcal{I}^q(E)/E_q$ and E_q are free or projective modules over A. We call *r-prolongation* of E_q, $E_{q+r} \equiv \mathcal{I}^r(E_q) \cap \mathcal{I}^{q+r}(E)$. We say that E_q is *regular* if E_{q+r}, $r \ge 0$, is a normal differential system of order $q+r$. We say that E_q is *formally integrable* if it is regular and $E_{q+r+1} \to E_{q+r} \to 0$ are short exact sequences for all $r \ge 0$.

Theorem 2.86 ([24]). *Let $E_q \subset \mathcal{I}^q(E)$ be a differential system of order q. The following properties hold:*

(i) *The following sequences are exact:*

(a) $0 \to E_q \to \mathcal{I}^q(E) \to F \to 0$
(b) $0 \to F^* \to D_q(A) \bigotimes_A E^* \to E_q^*$
(c) $0 \to E_{q+r} \to \mathcal{I}^{q+r}(E) \to \mathcal{I}^r(F) \to Q_r \to 0$
(d) $0 \to Q_r^* \to D_r(F) \to D_{q+r}(E)$

Furthermore, the exact sequence $D(F) \to D(E) \to M \to 0$, defines M equipped with the quotient filtration $M_q \subset \cdots \subset M_{q+r} \subset \cdots \subset M$. In general the induced sequences $D_r(F) \to D_{q+r}(E) \to M_{q+r} \to 0$ are not necessarily exact.

(ii) *If E_q is normal then the exact sequence* (a) *splits and the exact sequence* (b) *can be completed obtaining the following one:*

$$\text{(bb)}\ 0 \to F^* \to D_q(A) \bigotimes_A E^* \to E_q^* \to 0.$$

(iii) *If E_{q+r} is a regular system, then the sequence* (c) *splits and* (d) *can be completed giving the following exact sequence:*

$$\text{(dd)}\ 0 \to Q_r^* \to D_r(F) \to D_{q+r}(E) \to E_{q+r}^* \to 0.$$

(iv) *If the morphism $D(F) \to D(E)$ is strict the sequences*

$$D_r(F) \to D_{q+r}(E) \to M_{q+r} \to 0$$

are exact. In this case we have the exact sequences: $0 \to E_{q+r}^* \to M$, $0 \to E_{q+r}^* \to E_{q+r+1}^*$, $E_{q+r+1} \to E_{q+r} \to 0$, for all $r \geq 0$, where E_q is regular. Therefore, in such a case E_q is formally integrable.

(v) *If $E_q \subset \mathcal{I}^q(E)$ is a formally integrable linear differential system, one has*

$$E_{q+r} \subseteq (E_q)_{+r} \ \Leftrightarrow\ D_r M_q \subseteq M_{q+r}, \forall q, r \geq 0.$$

More precisely one has: $E_{q+r} = (E_q)_{+r}$ for $r \geq 0$ and $E_q \subseteq (E_p)_{+(q-p)}$, for all $p \leq q$. One has the filtration of $M = E_\infty^* \Leftrightarrow E_\infty = M^*$ by means of $M_q = E_q^* \Leftrightarrow E_q = M_q^*$. The corresponding gradiation is $gr(M) \equiv G$, with $g_q^* = G_q \Leftrightarrow g_q = G_q^*$. g_q is the symbol *of E_q. One has $g_q \subseteq (g_p)_{+(q-p)}$, for all $p \leq q$, and $g_{q+r} = (g_q)_{+r}$, for all $r \geq 0$.*

Definition 2.87. A *solution* of a q-order *linear differential system* $E_q \subset \mathcal{I}^q(E)$, is an element $e \in E$ such that $j^q(e) \in E_q$. The set of solutions is a submodule $\underline{Sol}(E_q) \equiv \Theta \subset E$, over A.

Theorem 2.88. *Given a formally integrable q-order linear differential system $E_q \subset \mathcal{I}^q(E)$, one has the following isomorphism:*

$$\Theta \cong Ext^0_{D(A)}(M; A) \cong Hom_{D(A)}(M; A).$$

Proof. As E and F are projective modules of finite rank over A, we obtain the exact sequence of filtered left $D(A)$-modules:

$$D(F) \to D(E) \to 0. \tag{2.4}$$

If N is another left $D(A)$-module, any sequence $D(F) \to D(E) \to N$ induces a morphism $M \to N$ of left $D(A)$-modules. We can define a *solution* of M in N such a morphism $M \to N$. We denote the set of such solutions by $\underline{Sol}_N(M)$. Let us apply the controvariant functor $Hom_{D(A)}(-; N)$ to the sequence (2.4), and taking into account the following isomorphisms:

$$\left\{ \begin{array}{ll} Hom_{D(A)}(D(E); N) & \cong Hom_{D(A)}(D(A) \bigotimes_A E^*; N) \cong Hom_A(E^*; N) \\ & \cong E^{**} \bigotimes_A N \cong E \bigotimes_A N \end{array} \right. \tag{2.5}$$

we get the following exact sequence:

$$0 \to Hom_{D(A)}(M; N) \to E \bigotimes_A N \to F \bigotimes_A N.$$

So we get thet $\underline{Sol}_N(M) \cong Hom_{D(A)}(M;N) \cong Ext^0_{D(A)}(M;N)$. In the case that $N = A$ we get that $\Theta \cong Ext^0_{D(A)}(M;A)$. $\square$

Theorem 2.89 (Algebraic criterion for formal integrability [24])**.** *Let $Z_q = Z(\mathfrak{p}_q)$ be the variety defined by means of ideal $\mathfrak{p}_q \subset K[y_q]_d$ such that the following conditions are verified:*

(i) $(\mathfrak{p}_q)_{+1} = \mathfrak{p}_{q+1} \subset K[y_{q+1}]_d$ *is also a prime ideal.*
(ii) $\mathfrak{p}_{q+1} \cap K[y_q]_d = \mathfrak{p}_q$.
(iii) g_{q+1} *is generic over* E_q.
(iv) g_q *is* 2*-acyclic.*

Then $(\mathfrak{p}_q)_{+\infty} = \mathfrak{p} \subset K[y]_d$ is a prime differential ideal, where $\mathfrak{p}$ is the differential ideal generated by a finite number of differential polynomials $P_1, \cdots, P_t$, defining E_q, and E_q is formally integrable. If one of these conditions is not satisfied we get that $\mathfrak{p}$ is not a prime ideal, hence we have a factorization of $\mathfrak{p}$. In other words the PDE is not formally integrable.

Proof. It follows from above propositions. $\square$

Example 2.90 (D'Alembert equation)**.** Let us consider the d'Alembert equation $(d'A) \subset JD^2(W)$, defined by means the following differential polynomial $F \equiv u_{xy}u - u_xu_y$, over $JD^2(W)$, where $\pi : W \equiv \mathbf{R}^3 \to M \equiv \mathbf{R}^2$, $(x,y,u) \mapsto (x,y)$. The ideal $\langle F \rangle$ is not prime in $\mathbb{R}[u,u_x,u_y,u_{xy}]$. One irreducible component $\mathfrak{p}_1$ is generated by (u_x,u_y), while the other is described by the system in solved form: $(d'A)'\, u_{xy} = \frac{u_xu_y}{u}$, with the localization $u \neq 0$, that is formally integrable. The intersection of these two components is described by the system

$$(A)\bigcap(B) = (C) \subset JD^2(W) : \{u_x = 0, u_y = 0, u_{xy} = 0\}. \tag{2.6}$$

Let us see under which condition a solution of the formally integrable component $(B) \subset JD^2(W) : \{u_{xy} = \frac{u_xu_y}{u}\}$, can pass for such intersection, say $(A)\bigcap(B) = (C) \subset JD^2(W)$. For example, let us consider the following solution $u(x,y) = (\frac{\beta}{2}y^2 + \alpha y + 1)h(x)$, where $\alpha, \beta \in \mathbb{R}$ and $h(x)$ is an arbitrary function on one real variable [24]. Then let us require that such a solution pass for $(C) \subset JD^2(W)$. We get that this condition requires $h = \gamma \in \mathbb{R}$, $\alpha = \beta = 0$. Therefore, we get that $u = u(x,y)$ is also a solution of (C) iff $u = \gamma \in \mathbb{R}$, i.e., a constant function on $\mathbb{R}^2$. Since $\pi_2(C) = \mathbb{R}^2 = M$, it follows that we can also find nontrivial solutions of (B) that pass for some point $(x_0,y_0) \in \mathbb{R}^2 = M$ for (C). For example

$$u(x,y) = (be^x - bx)(\frac{\beta}{2}y^2 + 1), \quad b, \beta \neq 0, \tag{2.7}$$

is of the type above considered. This gives, for $(x_0,y_0) = (0,0)$, $u(0,0) = b \neq 0$. Thus the two solutions $u(x,y) = b$ and the one given in (2.7) are both solutions of (B), having a contact of second order in $p = (0,0) \in M$, but the first, i.e., $u(x,y) = b$ is a solution of (A) also, and meets the solution (2.7) just in the intersection (C).

Example 2.91 (Some singular PDE's)**.** In Table 3 we report some singular PDE's having singularities of the type just reported in Table 2.

Table 3. Examples of singular PDE's

Name	Singular PDE
PDE with node and triple point $E_1\subset JD(E)$	$(u^1_x)^4+(u^2_y)^4-(u^1_x)^2=0$ $(u^2_x)^6+(u^1_y)^6-u^2_xu^1_y=0$
PDE with cusp and tacnode $\bar{E}_1\subset JD(E)$	$(u^1_x)^4+(u^2_y)^4-(u^1_x)^3+(u^2_y)^2=0$ $(u^2_x)^4+(u^1_y)^4-(u^2_x)^2(u^1_y)-(u^2_x)(u^1_y)^2=0$
PDE with conical double point, double line and pinch point $\tilde{E}_1\subset JD(F)$	$(u^1)^2-(u^1_x)(u^2_y)^2=0$ $(u^2)^2-(u^2_x)^2-(u^1_y)^2=0$ $(u^3)^3+(u^3_y)^3+(u^2_x)(u^3_y)=0$

$\pi{:}E\equiv\mathbb{R}^4\to\mathbb{R}^2$, $(x,y,u^1,u^2)\mapsto(x,y)$. $\bar{\pi}{:}F\equiv\mathbb{R}^5\to\mathbb{R}^2$, $(x,y,u^1,u^2,u^3)\mapsto(x,y)$.

3. Surgery, integral bordism groups and global solutions existence theorems

In order to characterize the global properties of solutions of PDE's we shall characterize the integral bordism groups of PDE's. Here we shall report on some results given by A.Prástaro in [14, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32].

Let $V\subset J^k_n(W)$ be an integral manifold, $\dim V=p$, $p\le n-1$. We denote by $\Sigma(V)\subset V$ the set of all singular points of the mapping $\pi_{k,0}:V\to W$, and $\Sigma_l(V)\equiv\big\{q\in V|\dim(\ker(\pi_{k,0}|_V)_{*,q})=l\big\}\subset\Sigma(V)$. Set $L_{0,q}\equiv ker(\pi_{k,0}|_V)_{*,q}\subset T_qV$. Then, one has $L_{0,q}\subset S^k(\Xi_q)\bigotimes\nu_a$ for some subspace $\Xi_q\subset L^*_q$. We will consider only the minimal subspace Ξ_q with the above property. An integral manifold $V\subset J^k_n(W)$ will be called *admissible* if the following properties hold:

(i) $\Sigma(V)\subset V$ has no open subsets and has no *frozen singularities*, i.e., for any $q\in\Sigma(V)$, $L_{0,q}$ is a degenerate subspace in $S^k(\Xi_q)\bigotimes\nu_a$, with respect to the exterior 2-form $\Omega(q)(\lambda)\in C^\infty(\Lambda^2(\mathbf{E}^k_n(W)^*_q))$. Recall that

$$ker\Omega(q)(\lambda)=\big\{\theta\in S^k(T^*_aN)\otimes\nu_a|\delta\theta\in T^*_aN\otimes g_\lambda\big\}\tilde{=}g^{(1)}_\lambda$$

where g_λ is a symbol space of tensor $\lambda\in S^{k-1}(T_aN)\otimes\nu^*_a$: $g_\lambda\equiv\big\{\gamma\in S^{k-1}(T^*_aN)\otimes\nu_a\mid\langle\lambda,\gamma\rangle=0\big\}$. (ii) There is a vector bundle $e:H\to V$ such that the fibers $H_q\supset\Xi_q$ for any point $q\in V$, satisfy the following conditions: $\Xi_q\subset H_q\subset L^*_q$, $\dim H_q\le l$. (iii) The family $L_0:q\mapsto L_{0,q}$ over $\Sigma(V)$ can be prolonged to some subbundle $L_0\subset H\bigotimes\nu$. (iv) If V is a compact closed integral manifold of dimension p, $0\le p\le n-1$, for V passes at least one integral manifold of dimension $(p+1)$ that satisfies above conditions (i), (ii), (iii). More precisely, we shall assume that an integral manifold V of dimension $p\le n-1$, is *admissible* if its set of singular points $\Sigma(V)$ can be solved by means of integral deformations. Furthermore, we say that V is *integral admissible* with respect to a PDE E_k, if V is admissible in the above sense, is contained into E_k, and the $(p+1)-$dimensional integral manifold mentioned in the above point (iv) is also contained into E_k. Let $E_k\subset J^k_n(W),k\ge0$, be a k-order PDE, $\dim W=n+m$. If $f_i:P_i\to E_k$, $i=1,2$, are C^∞ mappings that represent p-dimensional integral admissible manifolds $N_i\subset E_k$ respectively, then $N_1\sim_{E_k}N_2\Leftrightarrow\exists$ a C^∞ mapping $f:R\to E_k$ such that R is a $(p+1)$-dimensional smooth manifold with $\partial R=P_1\dot{\bigcup}P_2$, $f|_{P_i}=f_i$, $i=1,2$, and the following condition is verified: (i) $f(R) \equiv V \subset E_k$ is a $(p+1)$-dimensional piecewise admissible integral manifold of E_k. Then we say that N_1 and N_2 are *E_k-bordant*. We call $\Omega_p^{E_k}$, the *integral p-bordism group* of E_k. Note that if $N = \partial V$ is nonorientable, V cannot be simply connected. In fact, a compact nonorientable p-manifold without boundary cannot be embedded in a simply connected $(p+1)$-manifold [26]. The empty set $\emptyset$ will be regarded as a p-dimensional compact closed admissible integral manifold for all $p \geq 0$. $\sim_{E_k}$ is an equivalence relation. We write $\Omega_p^{E_k}$ for the set of all E_k-bordism classes $[N]_{E_k}$ of compact closed p-dimensional admissible integral manifolds of $E_k, 0 \leq p \leq n-1$. The operation of taking disjoint union $\dot{\bigcup}$ defines a sum $+$ on $\Omega_p^{E_k}$ such that $\Omega_p^{E_k}$ becomes an abelian group. The class $[\emptyset]_{E_k}$ defines the zero element. For $k=0$ we set $\Omega_p^{E_0} = \underline{\Omega}_p(E_0)$, where $E_0 \subset J_n^0(W) \equiv W$, and $\underline{\Omega}_p(E_0)$ is the p-bordism group of E_0. We shall denote by Ω_p, $p \in \mathbf{N}$, the p-bordism groups of unoriented smooth compact p-dimensional manifolds.

For any couple $(J_n^k(W), E_k)$, where $E_k \subset J_n^k(W)$ is a PDE, we will denote by $\Omega_p(E_k)$, $p \in \{0, \ldots, n-1\}$, the corresponding relative integral p-bordism groups, and we call them the *quantum p-bordism groups* of E_k.

The existence of admissible p-dimensional manifolds is obtained solving Cauchy problems of order $p \in \{0, \cdots, n-1\}$, i.e., finding n-dimensional admissible integral manifolds (solutions) of a PDE $E_k \subset J_n^k(W)$, that contains some fixed integral manifolds of dimension $p < n$. We call *low dimension Cauchy problems*, Cauchy problems of dimension $0 \leq p \leq n-2$. We simply say *Cauchy problems*, Cauchy problems of dimension $p = n-1$.

In a satisfactory theory of PDE's it is necessary to consider in a systematic way also *weak solutions*, i.e., solutions V, where the set $\Sigma(V)$ of singular points of V, contains also discontinuity points, $q, q' \in V$, with $\pi_{k,0}(q) = \pi_{k,0}(q') = a \in W$, or $\pi_k(q) = \pi_k(q') = p \in M$. We denote such a set by $\Sigma(V)_S \subset \Sigma(V)$, and, in such cases we shall talk more precisely of *singular boundary* of V, like $(\partial V)_S = \partial V \setminus \Sigma(V)_S$. However for abuse of notation we shall denote $(\partial V)_S$, (resp. $\Sigma(V)_S$), simply by (∂V), (resp. $\Sigma(V)$), also if no confusion can arise. Solutions with such singular points are of great importance and must be included in a geometric theory of PDE's too.

Let $\Omega_{n-1}^{E_k}$, (resp. $\Omega_{n-1,s}^{E_k}$, resp. $\Omega_{n-1,w}^{E_k}$), be the integral bordism group for $(n-1)$-dimensional smooth admissible regular integral manifolds contained in E_k, borded by smooth regular integral manifold-solutions,[13] (resp. piecewise-smooth or singular solutions, resp. singular-weak solutions), of E_k.

Theorem 3.1 ([25]). *Let us assume that E_k is formally integrable and completely integrable, and such that $\dim E_k \geq 2n+1$. Then, one has the following canonical isomorphisms: $\Omega_{n-1,w}^{E_k} \cong \oplus_{r+s=n-1} H_r(W;\mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s \cong \Omega_{n-1}^{E_k}/K_{n-1,w}^{E_k} \cong \Omega_{n-1,s}^{E_k}/K_{n-1,s,w}^{E_k}$. Furthermore, if $E_k \subset J_n^k(W)$, has non zero symbols: $g_{k+s} \neq 0$, $s \geq 0$, (this excludes that can be $k = \infty$), then $K_{n-1,s,w}^{E_k} = 0$, hence $\Omega_{n-1,s}^{E_k} \cong \Omega_{n-1,w}^{E_k}$.*

Here we want relate the concept of conservation laws with the existence of smooth solutions bording Cauchy manifolds in PDE's, and with the integral bordism groups.

[13]This means that $N_1 \in [N_2] \in \Omega_{n-1}^{E_k}$, iff $N_1^{(\infty)} \in [N_2^{(\infty)}] \in \Omega_{n-1}^{E_\infty}$. (See refs.[26] for notations.)

Definition 3.2 ([25])**.** We define *(differential) conservation law* of a PDE $E_k \subset J^k_n(W)$, any differential $(n-1)$-form β belonging to the following quotient space:

$$\mathfrak{Cons}(E_k) \equiv \Omega^{n-1}(E_\infty) \cap \frac{d^{-1}C\Omega^n(E_\infty)}{C\Omega^{n-1}(E_\infty)\bigoplus d\Omega^{n-2}(E_\infty)},$$

where $\Omega^q(E_\infty)$, $q=0,1,2,\dots$, is the space of differential q-forms on E_∞, $C\Omega^q(E_\infty)$ is the space of all Cartan q-forms on E_∞, $q=1,2,\dots$, and $C\Omega^o(E_\infty)\equiv 0$, $C\Omega^q(E_\infty)\equiv \Omega^q(E_\infty)$, for $q>n$, $\Omega^{-1}(E_\infty)=0$.[14] Thus a conservation law is a $(n-1)$-form on E_∞ non trivially closed on the (singular) solutions of E_k.

Theorem 3.3 ([25])**.** *There exists a canonical representation of the space of differential conservation laws of E_k in $\mathbf{H}(E_\infty)\equiv \mathbb{R}^{\Omega^{E_\infty}_{n-1}}$.*

Definition 3.4 ([25])**.** We call *full p-Hopf algebra*, (or *space of the full p-conservation laws*), of $E_k \subset J^k_n(W)$ the following Hopf algebra: $\mathbf{H}_p(E_k)\equiv \mathbb{R}^{\Omega^{E_k}_p}$.[15] In particular for $p=n-1$ we write $\mathbf{H}(E_k)\equiv \mathbf{H}_{n-1}(E_k)$ and we call *full Hopf algebra* of E_k, $\mathbf{H}(E_\infty)\equiv \mathbf{H}_{n-1}(E_\infty)$. If $\langle E^{0,n-1}\rangle \cong \mathbf{H}(E_\infty)\equiv \mathbb{R}^{\Omega^{E_\infty}_{n-1}}$, we say that E_k is *wholly Hopf-bording.*

Theorem 3.5 ([25])**.** *If $\Omega^{E_\infty}_{n-1}$ is trivial then E_k is wholly Hopf-bording. Furthermore, in such a case $E^{0,n-1}=0$.*

In order to distinguish between integral manifolds V representing singular solutions, where $\Sigma(V)$ has no discontinuities, and integral manifolds where $\Sigma(V)$ contains discontinuities, we shall consider "conservation laws" valued on integral manifolds N representing the integral bordism classes $[N]_{E_k}\in \Omega^{E_k}_p$.

Definition 3.6 ([23, 24, 25])**.** Set: $\mathfrak{I}(E_k)^p \equiv \frac{\Omega^p(E_k)\cap d^{-1}(C\Omega^{p+1}(E_k))}{d\Omega^{p-1}(E_k)\oplus\{C\Omega^p(E_k)\cap d^{-1}(C\Omega^{p+1}(E_k))\}}$. Then we define *integral characteristic numbers* of N, with $[N]_{E_k}\in\Omega^{E_k}_p$, the numbers

$$i[N]\equiv \langle [N]_{E_k},[\alpha]\rangle,\ \forall[\alpha]\in\mathfrak{I}(E_k)^p.$$

Theorem 3.7. [22, 23] *Let $E_k\subseteq J^k_n(W)$ be a PDE. Let us consider admissible p-dimensional, $p\in\{0,1,\cdots,n-1\}$, integral manifolds that are orientable. Let $N_1\in [N_2]_{E_k}\in\Omega^{E_k}_p$, then there exists a $(p+1)$-dimensional integral manifold $V\subset E_k$, such that $\partial V = N_1\dot{\bigcup} N_2$, where V is without discontinuities iff the integral numbers of N_1 and N_2 coincide.*

Remark 3.8. Integral bordism groups allow us to identify a large class of PDE's, *(extended crystal PDE's)*, where their integral bordism groups are extensions of crystallographic subgroups. There we can identify obstructions to existence of global smooth solutions with some suitable charactersic classes. (For more details see [29, 30, 31, 32, 33].)

[14] $C\Omega^q(E_\infty)\equiv\{\beta\in\Omega^q(E_\infty)|\beta(\zeta_1,\cdots,\zeta_q)(p)=0,\zeta_i(p)\in(\mathbf{E}^\infty_n)_p,\forall p\in E_\infty\}$.

[15] $\mathbf{H}(E_\infty)$ has a natural structure of Hopf algebra if $\Omega^{E_\infty}_{n-1}$ is a finite group, otherwise it is an extension of an Hopf subalgebra. (See [22, 23].)

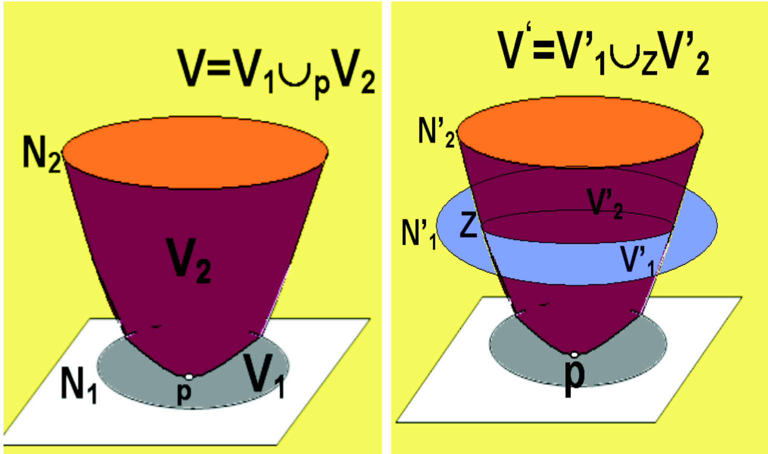

FIGURE 1. d'Alembert singular solutions (a) e (b).

Definition 3.9. Let $E_k \subset J^k_n(W)$ be a singular PDE, that splits in irreducible components A_i, i.e., $E_k = \bigcup_i A_i$. Then, we say that E_k admits an *algebraic singular solution* $V \subset E_k$, if $V \bigcap A_r \equiv V_r$ is a solution (in the usual sense) in A_r for at least two different components A_r, say A_i, A_j, $i \neq j$, such that $A_i \bigcap A_j \neq \emptyset$.

Let $N_1, N_2 \subset E_k \subset J^k_n(W)$ be two $(n-1)$-dimensional admissible integral manifolds. We say that N_1 *algebraic integral bords* with N_2, if N_1 and N_2 belong to two different irreducible components, say $N_1 \subset A_i$, $N_2 \subset A_j$, $i \neq j$, such that there exists an algebraic singular solution $V \subset E_k$ such that $\partial V = N_1 \dot{\bigcup} N_2$.

In the singular integral bordism group $\Omega^{E_k}_{n-1,s}$ of a singular PDE $E_k \subset J^k_n(W)$, we call *algebraic class* a class $[N] \in \Omega^{E_k}_{n-1,s}$, with $N \subset A_j$, such that there exists a $(n-1)$-dimensional admissible integral manifolds $X \subset A_i \subset E_k$, algebraic integral bording with N, i.e., there exists an algebraic singular solution $V \subset E_k$, with $\partial V = N \dot{\bigcup} X$.

Theorem 3.10. (Singular integral bordism group of singular PDE.) *Let $E_k \subset J^k_n(W)$ be a singular PDE, that splits in irreducible components A_i, i.e., $E_k = \bigcup_i A_i$. Then a smooth solution $V \subset E_k$ is the one that belongs to some of its irreducible components A_i. If $A_i \bigcap A_j \neq \emptyset$, then V is an algebraic singular solution belonging to both A_i and A_j iff $V_{ij} \equiv (V_i \equiv A_i \bigcap V) \bigcap (V_j \equiv A_j \bigcap V) \neq \emptyset$. In such a case we say that V* bifurcates *along $V_{ij} \subset V$.*[16]

Proof. It is a direct consequence of above results and definitions. □

Example 3.11 (d'Alembert equation singular PDE and global algebraic singular solution)**.** Let us consider again the singular d'Alembert equation (Example 2.90). If $N_1 \subset (A)$ and $N_2 \subset (B)$, it follows that $N_1 \dot{\bigcup} N_2 = \partial V$, where $V = V_1 \bigcup_Z V_2$, where Z is reduced to the point $p = (0,0)$, in $(A) \bigcap (B) = (C)$ and V_1 is the disk

[16]Note that the bifurcation does not necessarily imply that the tangent planes in the points of $V_{ij} \subset V$ to the components V_i and V_j, should be different. See, e.g. Fig.1(a).

D^2, identified with the solution $u(x,y)=b\in(C)$, with boundary N_1, and V_2 is identified with the solution in (B) passing for p and bording N_2. So V is an algebraic singular solution of $(d'A)$, obtained by surgering V_1 with V_2 at p. (See Fig.1(a).)

In this way the algebraic singular integral bordism allows to generalize integral bordism to all the components of $(d'A)$ and not only to the solved form (B). With this respect we can state that $\Omega_{1,s}^{(d'A)}=0$, by considering $(d'A)$ as a singular PDE. Note that $\Omega_{1,s}^{(A)}\cong\Omega_{1,s}^{(B)}=0$. Another algebraic singular solution of $(d'A)$ can be obtained by cutting, with the solution $u=b'\neq b$, the above solution V_2 of (B). In this way one obtains circular sector V_1' on the plane $u=b'$, with boundary $N_1'\dot{\bigcup}Z'$. Then the new algebraic singular solution is $V'=V_1'\bigcup_{Z'}V_2'$, where V_2' has boundary $Z'\bigcup N_2$. (See Fig.1(b).)

Finally note that $u(x,y)=b\in\mathbb{R}$ is a smooth solution of the singular PDE $(d'A)$, belonging to both components (A) and (B) since it belongs also to $(A)\bigcap(B)$. In the above examples of algebraic singular solutions one has that the bifurcation set is reduced to a point in Fig.1(a) and to a circle in Fig.1(b).

4. Surgery and global solutions in singular ordinary differential equations

Nonsingular ordinary differential equations (ODE's) have some particularities, with respect to PDE's, one of most important is that for ODE's the Cartan distribution is necessarily 1-dimensional and the symbol is trivial. Let us however note that some important 1-dimensional dimensional characteristic distributions of PDE's can exist also for PDE's. (See the following example. There, the characteristic distribution is related to solutions of Cauchy problems.) In this section we shall characterize singular ODE's and to see how global solutions passing through singular points, can be obtained by surgering techniques. Furthermore, we shall show that singular points in ODE's are sources of unstabilities in the global solutions passing through them.

Example 4.1. *1) (Cauchy problems and characteristic curves).* Let us consider the following PDE:

$$E_1\subset JD(W),\quad \pi:W\equiv\mathbb{R}^n\times\mathbb{R}\to\mathbb{R}^n;\quad \left\{P^\alpha(x^\beta,u)u_\alpha+Q(x^\beta,u)=0\right\}.$$

Its Cartan distribution $\mathbf{E}_1$ is generated by the following differential 1-forms:

$$\left\{\begin{array}{l}\omega_0\equiv[(\partial x_\gamma.P^\alpha)u_\alpha+(\partial x_\gamma.Q)]dx^\gamma+[(\partial u.P^\alpha)u_\alpha+(\partial u.Q)]du+P^\alpha du_\alpha\\ \omega_1\equiv du-u_\alpha dx^\alpha\end{array}\right\}.$$

Therefore the Cartan distribution $\mathbf{E}_1$ is generated by the following vector fields:

$$\zeta=X^\alpha(\partial x_\alpha+u_\alpha\partial u)+Z_\alpha\partial u^\alpha,\quad X^\alpha,Z_\alpha\in C^\infty(JD(W),\mathbb{R})$$

such that

$$Z_\alpha P^\alpha=-X^\gamma[(\partial x_\gamma.P^\alpha)u_\alpha+(\partial x_\gamma.Q)+(\partial u.P^\alpha)u_\alpha u_\gamma+(\partial u.Q)u_\gamma].$$

Therefore, $\dim \mathbf{E}_1 = 2n-1$. Its completely integrable sub-distribution *(characteristic distribution)*, is obtained by means of the following conditions:

$$\begin{array}{l}\left\{\begin{array}{l}\langle\zeta, d\omega_0\rangle = A\omega_0 + B\omega_1\\ \langle\zeta, d\omega_1\rangle = \bar{A}\omega_0 + \bar{B}\omega_1\end{array}\right\}, \quad A, B, \bar{A}, \bar{B} \in C^\infty(JD(W), \mathbb{R})\\ \Downarrow\\ \left\{\begin{array}{l}X^\alpha = \bar{A}P^\alpha\\ -Z_\alpha = \bar{A}[P^\beta_\alpha u_\beta + Q_\alpha + P^\beta_u u_\beta u_\alpha + Q_u u_\alpha]\end{array}\right\}.\end{array}$$

Therefore the characteristic distribution $\mathbf{C}har(E_1) \subset \mathbf{E}_1$ is 1-dimensionale and generated by the following vector:

$$\zeta = P^\gamma[\partial x_\gamma + u_\gamma \partial u] - [P^\beta_x u_\beta + Q_\alpha + P^\beta_u u_\beta u_\alpha + Q_u u_\alpha]\partial u^\alpha.$$

Therefore, if $N \subset E_1$ is a $(n-1)$-dimensional integral manifold, such that $\zeta(p) \notin T_pN, \forall p \in N$, the flow generated by ζ produces an admissible integrale manifold $V \subset E_1$, of dimension n: $V = \bigcup_t \phi_t(N)$, $\zeta = \partial\phi$. The corrisponding characteristic distribution on W is generated by $\zeta = P^\gamma[\partial x_\gamma + u_\gamma \partial u]$ and generates the solution on W.

For example let us consider the following first order PDE $E_1 \subset JD(W)$, $\{u_x x + u_y y = 1\}$. The corresponding characteristic vector field is given by $\zeta = x\partial x + y\partial y + \partial u - u_x \partial u^x - u_y \partial u^y$. The characteristic curves satisfy the following ordinary differential system:

$$\left\{\begin{array}{l}\dot{x} = x\\ \dot{y} = y\\ \dot{u} = 1\\ \dot{u}_x = -u_x\\ \dot{u}_y = -u_y\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\frac{dx}{x} = dt \Rightarrow x(t) = c_1e^t\\ \\ \frac{dy}{y} = dt \Rightarrow y(t) = c_2e^t\\ \\ du = dt \Rightarrow u(t) = t + u_0\\ u_x(t) = c_3e^{-t}\\ u_y(t) = c_4e^{-t}\end{array}\right\}.$$

Therefore, by considering an admissible 1-dimensionale integral manifold $y = y_0$, defined by the following equations: $F^I(x, y_0, u, u_x, u_y) = 0$, $1 \leq I \leq 3$, we get that the generated 2-dimensional integral manifold is identified by the equations: $F^I(xe^t, y_0e^t, u + t, u_xe^{-t}, u_ye^{-t}) = 0$, $1 \leq I \leq 3$.

In singular ODE's the Cartan distribution does not necessitate to be 1-dimensional in the singular points. In fact one has the following theorem.

Theorem 4.2 (Integral characterization singular points in ODE's)**.** *Let $E_k \subset JD^k(W)$ be an ODE on the fiber bundle $\pi : W \to M$, where* $\dim W = m+1$ *and* $\dim M = 1$*. One has the following propositions.*

(i) *If E_k is a determined non-singular ODE, then its Cartan distribution $\mathbf{E}_k$ is* 1*-dimensional and its symbol $(g_k)_q = 0$, $\forall q \in E_k$. Furthermore, any smooth solution $V \subset E_k$ necessitates to be stable at finite times.*[17]

(ii) *If E_k is a singular ODE, then for its Cartan distribution $(\mathbf{E}_k)_q$, in a singular point $q \in E_k$, can be verified the following situations:*

[17]But does not necessitate to be asymptotically stable. Note that the average stability for solutions of ODE's coincides with the asymptotic stability.

(a) (Full degeneration). *The jacobian matrix* $(\partial\xi_L.F^i)(q)=0$, *where* $F^i=0$ *are the local equations encoding* E_k *and* $(\xi^L)\equiv(t,\overset{(\cdot\alpha)}{u}{}^j)$, $0\le\alpha\le k$, *are the coordinates on* $JD^k(W)$.[18] *Then,* $(\mathbf{E}_k)_q=\mathbf{E}_k(W)_q$, *i.e., the Cartan distribution at* q, *coincides with the Cartan distribution of* $JD^k(W)$ *at the same point* q. *Furthermore, the symbol* $(g_k)_q\cong vT_{\bar q}W$, *i.e., the symbol of* E_k *at* q, *coincides with the symbol of the trivial ODE* $JD^k(W)\subseteq JD^k(W)$. *In such a case we say that the* integral singular dimension *of* q *is* m.

(b) *The rank of* $(\partial\xi_L.F^i)(q)$ *is* $m-s$, $m>1$, $s<m$. *Then* $(\mathbf{E}_k)_q$ *may be undetermined, whether some conditions are not satisfied,* (integral singular consistence conditions). *When these conditions are satisfied, the dimension of* $(\mathbf{E}_k)_q$ *is* $1+s$. *We call* s *the* integral singular dimension *of such a singular point* q. *There the symbol* $(g_k)_q$ *has dimension* s: $\dim(g_k)_q=s$. *If* $m=1$, $(\partial\overset{(\cdot k)}{u}.F)(q)=0$, *then* $\dim(g_k)_q=1$ *and* $\dim(\mathbf{E}_k)_q\in\{1,2\}$. *In such a case we say that* q *has integral singular dimension* $s=1$.

Furthermore, in both cases (a) *and* (b), *any smooth solution* $V\subset E_k$ *passing through a singular point does not necessitate to be functional stable, i.e. there can occurr finite times unstabilities.*[19]

Proof. (i) In this case we can assume that the local equations $F^i(t,u^j,\dot u^j,\cdots,\overset{(\cdot k)}{u}{}^j)=0$, $i=1,\dots,m$, can be solved with respect $\overset{(\cdot k)}{u}{}^j$. This implies that $A^i_j\equiv(\partial\overset{(\cdot k)}{u}{}_j.F^i)$ is a $m\times m$ matrix that in any point of E_k has determinant different from zero. Then in such a case one has that E_k is a smooth submanifold of $JD^k(W)$ of dimension $\dim E_k=\dim JD^k(W)-m=1+m(1+k)-m=1+mk$. Furthermore a Cartan vector field of E_k is a vector field ζ on $JD^k(W)$ of the type $\zeta=T(\partial t+\sum_{0\le\alpha\le k-1}\overset{(\cdot 1+\alpha)}{u}{}^j\partial\overset{(\cdot\alpha)}{u}{}_j)+Z^j\partial\overset{(\cdot k)}{u}{}_j$, where T and Z^j, $1\le j\le m$, are numerical functions on $JD^k(W)$, such that $\zeta.F^i=0$. So these functions must solve the system:

$$\left\{\begin{aligned}&Z^j(\partial\overset{(\cdot\alpha)}{u}{}_j.F^i)&&=-T[(\partial t.F^i)+\textstyle\sum_{0\le\alpha\le k-1}\overset{(\cdot 1+\alpha)}{u}{}^j(\partial\overset{(\cdot\alpha)}{u}{}_j.F^i)]\\&F^i(t,u^j,\cdots,\overset{(\cdot k)}{u}{}^j)=0\end{aligned}\right.\tag{4.1}$$

Under the condition that $\det(\partial\overset{(\cdot k)}{u}{}_j.F^i)=\det(A^i_j)\neq0$, one has the following unique solution of the system (4.1):

$$\left\{\begin{aligned}&Z^j(q)=-Tb^i(q)(A^{-1})^j_i(q)\\&b^i(q)\ =[(\partial t.F^i)+\textstyle\sum_{0\le\alpha\le k-1}\overset{(\cdot 1+\alpha)}{u}{}^j(\partial\overset{(\cdot\alpha)}{u}{}_j.F^i)]\\&q\qquad\in E_k.\end{aligned}\right.\tag{4.2}$$

[18] We adopt the notation that vertical coordinates in $JD^k(W)$ are written in the form $\overset{(\cdot\alpha)}{u}{}^j$, with $0\le\alpha\le k$, and we set $\overset{(\cdot 0)}{u}{}^j=u^j$.

[19] In the case where $(\mathbf{E}_k)_q$ is undertermined, there is no solution passing through the singular point $q\in E_k$. However, if in the neighbourhood of q there exist solutions tending to such a point q, we can, for continuity surgery such solutions.

Thus the Cartan distribution of E_k is the 1-dimensional distribution generated by the following vector field

$$\zeta = \partial t + \sum_{0 \leq \alpha \leq k-1} \overset{(\cdot 1+\alpha)}{u}{}^{j} \partial \overset{(\cdot\alpha)}{u}{}_{j} - b^{i}(A^{-1})^{j}_{i} \partial \overset{(\cdot k)}{u}{}_{j}. \tag{4.3}$$

The equation of the symbol can be written $X^i A^j_i = 0$. Since $\det(A^j_i) \neq 0$, it follows that the unique solution is $X^i = 0$, hence $(g_k)_q = 0$, for all $q \in E_k$. This has as a consequence that any smooth solution of E_k is functionally stable, i.e., there cannot occurr finite times unstabilities.

(ii) If, instead, there points $q \in E_k$, where $\det(A^j_i)(q) = 0$, and the rank of $(A^j_i)(q)$ is $m - s$, then the linear equation $(X^i A^j_i)(q) = 0$ has there s linearly independent solutions, say $(\overset{(1)}{X}, \cdots, \overset{(s)}{X})$. The same happens for the adjoint equation $A^*(\alpha) = 0$. Say $(\overset{(1)}{\alpha}, \cdots, \overset{(s)}{\alpha})$. Then equation (4.1) has, in such points, solutions iff the following consistence conditions are satisfied.

$$\langle \overset{(r)}{\alpha}, b \rangle = \sum_{1 \leq j \leq m} \overset{(r)}{\alpha}{}_{j} b^{j} = 0, \quad 1 \leq r \leq s. \tag{4.4}$$

In such a case equation (4.1) has an infinite number of solutions:

$$Z^j(q) = \sum_{1 \leq r \leq s} C_r \overset{(r)}{X}{}^{j}(q) - T(q) W^j(q) \tag{4.5}$$

where $C_r \in \mathbb{R}$ are arbitrary constants, and $W(q) \in (A^{-1})(q)(b(q))$, $q \in E_k$. This means that the Cartan distribution of E_k, in the points $q \in E_k$, where $\det(A^j_i)(q) = 0$, has dimension $1 + s$. Since, in such points, one has that the linearly indipendent solutions of the symbol equation $(X^i A^j_i)(q) = 0$ are just s, we get $\dim(g_k)_q = s$, when q is a singular point of E_k. Then, if V is a smooth solution passing through such a point, it can exhibit a finite time unstability. Finally, in the case where conditions (4.4) are not satisfied in the singular point $q \in E_k$, the system (4.1) has not solutions, hence the Cartan distribution is not determined there. In order to complete the proof of the theorem, let us assume that $(\partial \xi_L . F^i)(q) = 0$, i.e., case (a). Then equation (4.1) is satisfied for any vector field ζ of $\mathbf{E}_k(W)_q$, hence $(\mathbf{E}_k)_q = \mathbf{E}_k(W)_q$ and $\dim(\mathbf{E}_k)_q = \dim \mathbf{E}_k(W)_q = m$. Therefore, also in this case hold the same considerations on the stability of solutions. The particular case $m = 1$ can be similarly proved. (For a geometric theory of stability of PDE's and solutions of PDE's see some recent works on this subject by A.Prástaro [28, 30, 31, 32, 33].) □

Definition 4.3. A singular ODE $E_k \subset JD^k(W)$ is one that splits in irreducible components $E_k = \bigcup A_i$, where each A_i is an ODE.

An *algebraic singular solution* of a singular ODE, is one that can be obtained by surgering solutions of some components $A_i, A_j \subset E_k$, such that $A_i \bigcap A_j \neq \emptyset$.

Theorem 4.4 (Smooth and bifurcating solutions in singular points of ODE's)**.** *An algebraic singular solution V of a singular ODE is a smooth solution iff the Cartan distribution of E_k in the points where $V \subset A_i \bigcap A_j \neq \emptyset$ admits a smooth 1-dimensional sub-distribution.*

If V is not smooth, then one says that the singularity points are bifurcation points.[20]

An algebraic singular solution $V = V_i \bigcup V_j$, with $V_i \subset A_i$ and $V_j \subset A_j$, of a singular ODE E_k does not necessitate to conserve the stability behaviour when one pass through a singular point.

Proof. In fact, if in the points $V \subset A_i \bigcap A_j \neq \emptyset$ there exists a 1-dimensional smooth subdistribution of the Cartan distribution, the corresponding integral curve is smooth.

The existence of singular points in ODE's implies that the symbol of the equation there is not zero, in general. So do not necessitate to be functionally stable a solution passing there. Furthermore, the different branches in a global solution passing through a singular point, can have different asymptotic stability behaviours. Then, if there exists a bifurcation point on V, then the asymptotic stability of V_1 does not necessitate coincide with the one of V_2. □

Example 4.5 (Some singular ODE's)**.** In Table 4 we report some singular ODE's having singularities of the type just reported in Table 2.

TABLE 4. Examples of singular ODE's

Name	Singular ODE
ODE with node: $E_2 \subset JD^2(E)$	$\ddot{u}^4 + \dot{u}^4 - \ddot{u}^2 = 0$
ODE with triple point: $E_2 \subset JD^2(E)$	$\ddot{u}^4 + \dot{u}^4 - \ddot{u}\dot{u} = 0$
ODE with cusp: $\tilde{E}_2 \subset JD(E)^2$	$\ddot{u}^4 + \dot{u}^4 - \ddot{u}^3 + \dot{u}^2 = 0$
ODE with tacnode: $\hat{E}_2 \subset JD(E)^2$	$\ddot{u}^4 + \dot{u}^4 - \ddot{u}^2\dot{u} - \ddot{u}\dot{u}^2 = 0$
ODE with conical double point: $\widetilde{E_2} \subset JD^2(E)$	$\ddot{u}^2 - \dot{u}u^2 = 0$
ODE with double line: $\widehat{E_2} \subset JD^2(E)$	$\ddot{u}^2 - \dot{u}^2 - u^2 = 0$
ODE with pinch point: $\overline{E_2} \subset JD^2(E)$	$\ddot{u}^3 + u^3 + \dot{u}u = 0$

$\pi : E \equiv \mathbb{R}^2 \to \mathbb{R}, \quad (t, u) \mapsto t.$

For example, let us consider the first equation in Table 4. Let us consider the fiber bundle $\pi : E \equiv \mathbb{R}^2 \to M \equiv \mathbb{R}$, $(t, u) \mapsto t$. The corresponding fibered coordinates on $JD^2(E) \cong \mathbb{R}^4$ are $(t, u, \dot{u}, \ddot{u})$. In the plane $\mathbb{R}^2$, $(\dot{u}, \ddot{u})$, we can identify an algebraic curve Γ, with equation given just by the equation encoding E_2, i.e., $F \equiv \ddot{u}^4 + \dot{u}^4 -$

[20] An ordinary smooth differential equation, (without singular points), cannot be web-chaotic [2]. In fact, its Cartan distribution is 1-dimensional. Let us remark that even if a nonsingular ordinary differential equation cannot be web-chaotic, we can, in general, recognize for such equations web-structures on the configuration sapce W, associated to their solutions. However, these web-structures must not be confused with the web-chaotic concept. For example a first order ordinary differential equation of the type $E_1 \subset JD(W)$, $F(t, u, \dot{u}) = 0$, over the fiber bundle $\pi : W \equiv \mathbb{R}^2 \to \mathbb{R}$, $(t, u) \mapsto t$, that admits the following solved forms $\dot{u} = f_i(t, u)$, $i \in \{1, \cdots, s\}$, identifies on $W \equiv \mathbb{R}^2$, an s-web (of codimension 1), where each foliation is identied by means of solutions of each equation $\dot{u} = f_i(t, u)$. In other words the solutions of $F(t, u, \dot{u}) = 0$ identify on W an s-web. A more concrete example is the well-known equation $\dot{u}^3 + u\dot{u} - t = 0$, *(symmetric wave front)*, identifies a 3-web on $\mathbb{R}^2$. However, this equation is not web-chaotic.

$\ddot{u}^2 = 0$. Then we can represent E_2 as the following trivial fiber bundle

$$E_2 = \bigcup_{a\in\Gamma} \mathbb{R}^2 \cong \Gamma \times \mathbb{R}^2. \tag{4.6}$$

In this trivial fiber bundle the fiber is the vector space $\mathbb{R}^2$ and the basis is the algebraic curve Γ. Thus we can say that $\dim E_2 = 3$ and that the canonical surjective mapping $\pi_{2,0} : JD^2(E) \to E$, remains surjective, when restricted to E_2. In fact, one has the natural surjection $\pi_{2,0} : E_2 \to E \cong \mathbb{R}^2$, $(a, t, u) \mapsto (t, u)$. Note that the point of singularity in E_2 is just the fiber $(E_2)_{a=(0,0)} \cong \mathbb{R}^2$. In fact, one has

$$((\partial t.F)\ (\partial u.F)\ (\partial \dot{u}.F)\ (\partial \ddot{u}.F)) = (0\ 0\ 4\dot{u}^3\ 2\ddot{u}^2(2\ddot{u}^2 - 1)). \tag{4.7}$$

Therefore, with respect to Theorem 4.2 the singular points of this equation are of the type full degeneration. In fact, $(\mathbf{E}_2)_q \cong \mathbf{E}_2(E)_q \cong \mathbb{R}^2$ that is the vector space generated by the following two vectors $\{\partial t, \partial \ddot{u}\}$, i.e., the Cartan vector space of $JD^2(E)$ at the points $q = (t, u, 0, 0) \in E_2 \subset JD^2(E)$.

We can split E_2 in the following way

$$E_2 = A_1 \bigcup (E_2)_{a=(0,0)} \bigcup A_2 \tag{4.8}$$

where A_i, $i = 1, 2$ are the regular parts of E_2. So the singular part $(E_2)_{a=(0,0)}$ separates the regular parts. The Cartan distribution in the regular parts is the 1-dimensional distribution generated by the following vector field

$$\zeta = \partial t + \dot{u}\partial u + \ddot{u}\partial \dot{u} + \frac{2\dot{u}^3}{1 - 2\ddot{u}^2}\partial \ddot{u}. \tag{4.9}$$

Then one can see that $\lim_{(\dot{u},\ddot{u})\to(0,0)} \zeta = \partial t$. Therefore we can prolonge for continuity a solution approaching the singular points with a time-like curve tangent to ∂t, that belongs to the Cartan distribution at the singular points. Such an algebraic singular solution is not functionally stable since $\dim(g_2)_q = 1$, where $q \in (E_2)_{a=(0,0)}$. This is a bifurcation point.

Example 4.6 (Draining flow boundary value problem)**.** Let us consider the following ODE on the trivial vector fiber bundle $\pi : E \equiv \mathbb{R}^2 \to \mathbb{R}$, $(t, u) \mapsto t$,

$$E_3 \subset JD^3(E) \cong \mathbb{R}^5 : \{F \equiv u^2 \dddot{u} - 1 = 0\}. \tag{4.10}$$

Let us note that E_3 is not properly a singular equation as usually one considers. (See e.g.,[1, 3].) In fact the jacobian of the function F is given by the following 1×5 matrix

$$(j(F)) = ((\partial t.F)\,(\partial u.F)\,(\partial \dot{u}.F)\,(\partial \ddot{u}.F)\,(\partial \dddot{u}.F)) = (0\ \ 2u\dddot{u}\ \ 0\ \ 0\ \ u^2) \tag{4.11}$$

therefore it degenerates only in the points of $JD^3(E)$ with $u = 0$. But such points do not properly belong to E_3. In fact, E_3 can be represented as the following trivial unconnected fiber bundle

$$E_3 = {}_{(a)}E_3 \dot{\bigcup} {}_{(b)}E_3 \equiv \Gamma_{(a)} \times \mathbb{R}^3 \dot{\bigcup} \Gamma_{(b)} \times \mathbb{R}^3 \tag{4.12}$$

where $\Gamma_{(a)}$ and $\Gamma_{(b)}$ are algebraic curves in the plane $\mathbb{R}^2$ of $(u, \dddot{u})$, identified by the equation $F = 0$, and $\mathbb{R}^3$ is the space of $(t, \dot{u}, \ddot{u})$. So E_3 is a smooth submanifold of $JD^3(E)$. Furthermore, its Cartan distribution $\mathbf{E}_3 \subset TE_3$ is the 1-dimensional distribution generated by the following smooth vector field on E_3:

$$\zeta = \partial t + \dot{u}\partial u + \ddot{u}\partial \dot{u} + \frac{1}{u^2}\partial \ddot{u} - \frac{2}{u^3}\partial \dddot{u}\,. \tag{4.13}$$

Therefore, for any initial condition, i.e., any $q \in E_3$, there exists an unique smooth solution passing for there, i.e., an unique integral curve on E_3 tangent to ζ in q. Boundary value problems with two end points conditions, admit smoooth solutions if these points belong to the same connected component of E_3 and to the same flow line of ζ. In such a case one has an unique regular smooth solution, for such boundary conditions. Whether we require, instead, that the solution can be a weak solution, then the uniqueness is not more guaranteed and the two end points conditions do not necessitate to belong to the same connected component of E_3. Another, typical boundary value problem with such an equation is the following. (See [1, 3] and references quoted in the first above paper.)

$$\begin{cases} u\,\dddot{u} - 1 = 0 \\ u(0) = 1 \\ \dot{u}(0) = 0 \\ \lim_{t\to-\infty} \ddot{u}(t) = 0. \end{cases} \tag{4.14}$$

Since t, $\dot{u}$ and $\ddot{u}$ do not explicitly appear in equation $F = 0$, one has that points like $(0, 1, 0, \ddot{u}, \dddot{u}) \in E_3$ are on E_3 iff $\dddot{u} = 1$. Thus the initial condition in *(4.14)* implies that the initial points on E_3 are the following ones $q_0 \equiv (0, 1, 0, \ddot{u}, 1) \in E_3$. In such points the Cartan vectors are the following $\zeta_0 = \partial t + \ddot{u}\partial\dot{u} + \partial\ddot{u} - 2\partial\dddot{u}$. Therefore, since $\ddot{u}$ is arbitrary we get that there exists an infinity of solutions satisfying the infinity of initial conditions contained in the boundary value problem *(4.14)*. Furthermore, the asymptotic condition $\lim_{t\to-\infty} \ddot{u}(t) = 0$, in the boundary value problem *(4.14)* has sense, since the points $\{(t, u, \dot{u}, 0, \dddot{u}) \in JD^3(E) | u\,\dddot{u} - 1 = 0\}$ are on E_3. On the other hand the Cartan vector field in the points that satisfy such asymptotic condition is the following smooth vector field $\zeta_{-\infty} = \partial t + \dot{u}\partial u + \frac{1}{u^2}\partial\ddot{u} - \frac{2}{u^3}\partial\dddot{u}$. So we can try to find solutions of the boundary value problem *(4.14)* such that $\ddot{u} = ae^{bt}$, with $a, b \in \mathbb{R}$, $b > 0$. In fact for such a function one has that the asymptotic condition $\lim_{t\to-\infty} \ddot{u}(t) = 0$, in the boundary value problem *(4.14)* is satisfied. By direct integration of this function we get $u = \frac{a}{b^2}e^{bt} + ct + d$, $c, d \in \mathbb{R}$. Let us determine the constants by imposing to satisfy the initial conditions in the boundary value problem *(4.14)*. Then we get

$$u = \frac{1}{b^3}e^{bt} - \frac{1}{b^2}t + (1 - \frac{1}{b^3}), \quad b > 0. \tag{4.15}$$

So the solutions of the boundary value problem *(4.14)* is an open 1-dimensional submanifold of $\underline{S}_{ol}(E_3)$ identified with $\mathbb{R}^+$.

Example 4.7 (Gas dynamic through a semi-infinite porous medium boundary value problem)**.** Let us consider the following ODE on the trivial vector bundle $\pi : E \equiv \mathbb{R}^2 \to \mathbb{R}$, $(t, u) \mapsto t$,

$$E_2 \subset JD^2(E) \cong \mathbb{R}^4 : \{F \equiv \ddot{u}u^{1/2} + 2t\dot{u} = 0\}. \tag{4.16}$$

Note that the jacobian of F, on E_2 is given by the following 1×4 matrix

$$j(F)|_{E_2} = (2\dot{u} \quad -t\frac{\dot{u}}{u} \quad 2t \quad u^2). \tag{4.17}$$

Therefore, the unique singular point on E_2 is $q_0 \equiv (0\,0\,0\,0) \in E_2$. This is completely degenerate. The other points $q \in E_2$ are regular points and there the Cartan distribution is 1-dimensional, generated by the following smooth vector field on $E_2 \setminus \{q_0\}$:

$$\zeta = \partial t + \dot{u}\partial u - \frac{\dot{u}}{u^{1/2}}2t\partial\dot{u} - \frac{\dot{u}}{u^{1/2}}(2 - \frac{\dot{u}}{u}t - 4t^2\frac{1}{u^{1/2}})\partial\ddot{u}. \tag{4.18}$$

Thus for any regular point $q \in E_2$ one has a unique solution of E_2, i.e., an unique smooth integral curve passing for q. The Cartan distribution at q_0 coincides with the Cartan vector space $\mathbf{E}_2(E)_{q_0}$ of $JD^2(E)$ at q_0, i.e.,

$$\zeta(q_0) = [T(\partial t + \dot{u}\partial u + \ddot{u}\partial\dot{u}) + Z\partial\ddot{u}]_{q=q_0} = T\partial t + Z\partial\ddot{u}, \quad T, Z \in \mathbb{R}. \tag{4.19}$$

Therefore $\mathbf{E}_2(E)_{q_0}$ is a 2-dimensional vector space. A regular smooth solution belonging to $E_2 \setminus \{q_0\}$ can be eventually prolonged until to pass through q_0, since $(\mathbf{E}_2)_{q_0} = \mathbf{E}_2(E)_{q_0}$. However, such a prolonged solutions have in general finite times unstability, since the symbol $(g_2)_{q_0}$ of E_2 at q_0 is just the symbol of $JD^2(E)$ there, i.e., $(g_2)_{q_0} \cong vT_{\bar{q}_0}E \cong \mathbb{R}$, where $\bar{q}_0 = \pi_{2,0}(q_0) \in E$.

For such an equation a typically asymptotic boundary value problem is the following. (See, e.g. [1] and works quoted there.)

$$\begin{cases} \ddot{u}u^{1/2} + 2t\dot{u} = 0 \\ \lim_{t\to 0} u(t) = 1 - \alpha, \quad 0 < \alpha \leq 1 \\ \lim_{t\to\infty} u(t) = 1 \end{cases} \tag{4.20}$$

Let us assume $0 < \alpha < 1$ only. In such a case the point $(0, u(0) = 1-\alpha, \dot{u}, \ddot{u})$ is on E_2 iff $\ddot{u} = -2t\frac{\dot{u}}{u^{1/2}}|_{t=0,u=1-\alpha} = 0$. Therefore, the first condition in *(4.20)* on E_2 requires that the integral curve should start from a point $(0, 1-\alpha, \dot{u}, 0)$. Therefore, the corresponding Cartan vectors are the following ones $\zeta_{\to 0} = \partial t + \dot{u}\partial u - 2\frac{\dot{u}}{(1-\alpha)^{1/2}}\partial\ddot{u}$. Furthermore, the condition $\lim_{t\to\infty} u(t) = 1$ can be satisfied by looking that from equation $F = 0$ we get $\lim_{t\to\infty} u(t)^{1/2} = \lim_{t\to\infty} -2t\frac{\dot{u}}{\ddot{u}}$. Therefore, the above asymptotic condition is satisfied if $\lim_{t\to\infty} -2\frac{\dot{u}}{\ddot{u}} = \frac{1}{t}$. So let us consider the equation $-2\frac{\dot{u}}{\ddot{u}} = \frac{1}{t}$. Its integral gives $u = c_1\int e^{-t^2}dt + c_2$, $c_1, c_2 \in \mathbb{R}$. This means that the overdetermined system

$$\widetilde{E_2} \subset JD^2(E) : \begin{cases} \ddot{u}u^{1/2} + 2t\dot{u} = 0 \\ \ddot{u} + 2t\dot{u} = 0 \end{cases} \tag{4.21}$$

has the unique solution $u = 1$. Since $\widetilde{E_2} \subset E_2$, we can also say that $u = 1$ is a solution of the original equation E_2, and we call it the ∞-asymptotic solution of E_2 and denote it with the symbol $u_{(\infty)} \in \underline{\mathcal{S}}_{ol}(E_2)$. Thus we can obtain a solution of the boundary value problem *(4.20)* by gluing any solution of the initial condition, in such a way that $\dot{u} > 0$, with $u_{(\infty)}$. This is possible since $\dot{u}$ is arbitrary. Finally from the equation $F = 0$ we get $\ddot{u} = -2t\frac{\dot{u}}{u^{1/2}} < 0$. This means that such solutions have down-concavity. The solutions so built are every-where smooth except in the point where they solders with $u_{(\infty)}$.[21] By conclusion, the set of solutions of the boundary value problem *(4.20)* is a smooth curve in $\underline{\mathcal{S}}_{ol}(E_2)$, identified with $\mathbb{R}^+$.

Let us consider, now, the boundary value problem with $\alpha = 1$, i.e., the initial condition $u(0) = 0$. The difference with the previous one is that now the corresponding initial point on E_2, can be any point $\widetilde{q}_0 = (0, 0, \dot{u}, \ddot{u}) \in E_2$. Note, that the singular point q_0 of E_2 belongs to such a set of initial points $\widetilde{q}_0$. However, for the moment, let us exclude it. Then, solutions $u = u(t)$, starting from $(0, 0) \in \mathbb{R}^2 = E$, can have any nonzero initial inclination and curvature. In order to satisfy the asymptotic condition $\lim_{t\to\infty} u(t) = 1$ we can adopt the above surgery technique. Since we can use, now, any initial values of $\dot{u}$ and $\ddot{u}$, we get that the set of solutions of such a boundary value problems is a 2-dimensional submanifold of $\underline{\mathcal{S}}_{ol}(E_2)$, identified with $\mathbb{R}^+ \times (\mathbb{R} \setminus \{0\})$.

Finally, let assume that the initial condition should be just the singular point $q_0 \in E_2$. Then, we shall consider that the Cartan vector field there belongs the the 2-dimensional vector space given in *(4.19)*, i.e, generated by $\{\partial t, \partial \ddot{u}\}$. So we can consider the integral curve of $\xi = \partial t + a\partial \ddot{u}$, starting from q_0. This curve identifies the function $u = \frac{a}{6} t^3$, that can be a solution of E_2 iff $a = 0$. In fact, from *(4.18)* we get $\lim_{q\to q_0} \zeta = \partial t$. So a solution starting from the singular point q_0 should be necessarily $u(t) = 0$. We denote such solution $u_{(0)} \in \underline{\mathcal{S}}_{ol}(E_2)$. Then, we can choise an (intermediate) initial condition $q_1 \equiv (t_0, u(t_0) = 0, \dot{u}(t_0) = 0, \ddot{u}(t_0) > 0) \in E_2$, and consider the unique solution of E_2 starting from q_1. This solution will meet $u_{(\infty)}$ at some time t_2. In this way we get a regular solution of E_2, that satisfies the boundary value problem, starts from the singular point q_0 and it is everywhere smooth, except in two pints where it is continuous only. So it is a singular solution. The set of such solutions is a 2-dimensional submanifold of $\underline{\mathcal{S}}_{ol}(E_2)$ identified with $\mathbb{R}^+ \times \mathbb{R}^+ \equiv (\mathbb{R}^+)^2$. This proves that singular boundary value problems can admit solutions of class C^0 that are almost everywhere smooth curves in E.

It is important to underline that such solutions passing for the singular point $q_0 \in E_2$ are not functionally stable, i.e., admits unstability at finite times, and neither are asymptotic stable. In fact, let us consider the linear equation $E_2[u]$ associated to E_2 at one solution $u = u(t)$ of above type considered.

$$[2u]\ddot{\nu} + [4tu^{1/2}]\dot{\nu} + [\ddot{u}]\nu = 0 \tag{4.22}$$

Then one can see that at the singular point q_0 above equation is satisfied for any function $\nu = \nu(t)$. In particular we can consider the function $\nu(t) = 1/(t_a - t)$, that has just a singularity for $t = t_a > 0$. Furthermore, we can consider also $\nu(t) = e^{bt}$, $b > 0$, that gives asymptotic unstability.

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