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STRONGLY CONVERGENT ITERATIVE SCHEMES FOR A SEQUENCE OF NONLINEAR MAPPINGS

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Abstract. We deal with a sequence of nonlinear mappings satisfying certain conditions and generate a strongly convergent iterative sequence to a common fixed point of these mappings. To prove our main theorem, we use a technique of set convergence. This result can be applied for various types of mappings and therefore it includes many known results.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, $\mathbb{N}$ the set of all positive integers, and $\{T_n\}$ a sequence of mappings of H into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, where $F(T_n)$ is the set of all fixed points of T_n for $n \in \mathbb{N}$.

Approximating common fixed points of mappings $\{T_n\}$ has been studied by many researchers under various settings. One of the most important methods is so called the hybrid method introduced by Haugazeau [5] as follows: Let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_1 = x \in H, \\ y_n = T_n x_n, \\ C_n = \{z \in H : \langle x_n - y_n, y_n - z \rangle \geq 0\}, \\ Q_n = \{z \in H : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x \end{cases}$$

for each $n \in \mathbb{N}$, where $P_{C_n \cap Q_n}$ is the metric projection onto $C_n \cap Q_n$. Haugazeau proved a strong convergence theorem when $T_n = P_{C_{(n \bmod m)+1}}$ for every $n \in \mathbb{N}$, where P_{C_i} is the metric projection onto a nonempty closed convex subset C_i of H for each $i = 1, 2, \dots, m$ and $\bigcap_{i=1}^{m} C_i \neq \emptyset$. Bauschke and Combettes [1] extended the result of [5] (see also [10]) and Nakajo, Shimoji and Takahashi [12] generalized the result of [1, 10] to a uniformly convex and smooth Banach space.

Recently, Nakajo and Takahashi [11] proved the unified convergence theorems for an improved hybrid method in a real Hilbert space. On the other hand, Kimura and Takahashi [7] proved convergence theorems for an improved hybrid method, which is a generalization of the result proved by Takahashi, Takeuchi, and Kubota [19].

Motivated by these results, we extend the results of [11] to a strictly convex, smooth, and reflexive Banach space with the Kadec-Klee property using the method of [7] and get the result for convex feasibility problem under weaker conditions than [12].

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2. Preliminaries and lemmas

Throughout this paper, E is a real Banach space with norm $\|\cdot\|$. We write $x_n \to x$ to indicate that a sequence $\{x_n\}$ converges strongly to x.

The normalized duality mapping of E is denoted by J, that is,

$$Jx = \{x^* \in E^* : \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\}$$

for $x \in E$. If E is strictly convex, smooth, and reflexive Banach space, then J is a single-valued one-to-one mapping onto E^*.

A set-valued operator A of E into E^* is said to be monotone if $\langle x - y, x^* - y^* \rangle \geq 0$ for every $x, y \in E$ and $x^*, y^* \in E^*$ such that $x^* \in Ax$ and $y^* \in Ay$. A monotone operator A is said to be maximal monotone if the graph of A is not properly contained in the graph of any other monotone operator. It is known that a monotone operator A is maximal if and only if for $(u, u^*) \in E \times E^*$, $\langle x - u, x^* - u^* \rangle \geq 0$ for every $(x, x^*) \in E \times E^*$ with $x^* \in Ax$ implies $u^* \in Au$. We know the following result [3]: Let E be a strictly convex, smooth and reflexive Banach space and let A be a monotone operator of E into E^*. Then, A is maximal if and only if $R(J + rA) = E^*$ for all $r > 0$, where $R(J + rA)$ is the range of $J + rA$. By this result, it is also known that if E is a strictly convex, smooth and reflexive Banach space and A is a maximal monotone operator of E into E^*, then, for any $x \in E$ and $r > 0$, there exists a unique element $x_r \in D(A)$ such that

$$J(x_r - x) + rAx_r \ni 0,$$

where $D(A)$ is the domain of A. We define a mapping J_r of E into itself by $J_r x = x_r$ for every $x \in E$ and $r > 0$ and such J_r is called the resolvent of A; see [18] for more details. We have the following result for the resolvents of maximal monotone operators, which was essentially proved in [15].

Lemma 2.1. *Let E be a strictly convex, smooth, and reflexive Banach space and let A be a maximal monotone operator of E into E^* such that $A^{-1}0 \neq \emptyset$. Let $\{r_n\}$ be a positive real sequence with* $\inf_n r_n > 0$ *and let $\{J_{r_n}\}$ be a sequence of resolvents of A. Let $\{z_n\}$ be a sequence in E and $z \in E$ such that $z_n \to z$ and $J_{r_n} z_n \to z$. Then, $z \in A^{-1}0$.*

For the sake of completeness, we give the proof.

Proof. Let $(u, u^*) \in E \times E^*$ with $u^* \in Au$. Since $(1/r_n)J(z_n - J_{r_n} z_n) \in AJ_{r_n} z_n$ and A is monotone, we have

$$\left\langle J_{r_n} z_n - u, \frac{1}{r_n} J(z_n - J_{r_n} z_n) - u^* \right\rangle \geq 0,$$

which implies

$$\langle z - u, -u^* \rangle \geq 0$$

for each $(u, u^*) \in E \times E^*$ with $u^* \in Au$. As A is a maximal monotone operator, we obtain $z \in A^{-1}0$. □

Let C be a nonempty closed convex subset of a strictly convex and reflexive Banach space E. Then, for arbitrarily fixed $x \in E$, a function $C \ni y \mapsto \|x - y\| \in \mathbb{R}$

has a unique minimizer $y_x \in C$. Using such a point, we define the metric projection $P_C : E \to C$ by $P_C x = y_x$ for every $x \in E$.

For a nonempty closed convex subset C of E, we define the indicator function i_C of C as follows:

$$i_C(x) = \begin{cases} 0, & (x \in C) \\ \infty. & (x \notin C) \end{cases}$$

Since $i_C : E \to (-\infty, \infty]$ is proper lower semicontinuous and convex, we may define the subdifferential ∂i_C of i_C by

$$\partial i_C(x) = \{x^* \in E^* : i_C(y) \geq i_C(x) + \langle y - x, x^* \rangle \text{ for all } y \in E\}$$

for $x \in E$. Then, by [14] we have ∂i_C is a maximal monotone operator of E into E^*. Further we have the following result.

Lemma 2.2. *Let C be a nonempty closed convex subset of a strictly convex, smooth, and reflexive Banach space E. Then, $(\partial i_C)^{-1}0 = C$ and $J_r x = P_C x$ for every $r > 0$ and $x \in E$, where J_r is the resolvent of ∂i_C.*

Proof. From the definitions of the indicator function and its subdifferential, we have

$$\partial i_C(x) = \begin{cases} N_C(x), & (x \in C) \\ \emptyset, & (x \notin C) \end{cases}$$

where $N_C(x) = \{x^* \in E^* : \langle y - x, x^* \rangle \leq 0 \text{ for all } y \in C\}$. Thus $(\partial i_C)^{-1}0 = C$. Next, let $r > 0$ and $x, y \in E$. We may easily get that $y = J_r x$ if and only if $y \in C$ and $\langle y - z, J(x - y) \rangle \geq 0$ for all $z \in C$. Hence we have $J_r x = P_C x$ for every $r > 0$ and $x \in E$; see [17, p. 196]. □

Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of a reflexive Banach space E. We define a subset $\text{s-Li}_n C_n$ of E as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, a subset $\text{w-Ls}_n C_n$ of E is defined by the following: $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If $C_0 \subset E$ satisfies that $C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n$, it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [9] and we write $C_0 = \text{M-lim}_n C_n$. For more details, see [2].

Tsukada [20] proved the following theorem for the metric projection in a Banach space.

Theorem 2.3 (Tsukada [20])**.** *Let E be a reflexive and strictly convex Banach space and $\{C_n\}$ a sequence of nonempty closed convex subsets of E. If $C_0 = \text{M-lim}_n C_n$ exists and nonempty, then, for each $x \in E$, $P_{C_n}x$ converges weakly to $P_{C_0}x$. Moreover, if E has the Kadec-Klee property, the convergence is in the strong topology.*

3. Main results

Let C be a nonempty closed convex subset of a smooth Banach space E. Let $\{T_n\}$ be a countable family of mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$, which satisfies the following condition:

(I) There exists $\{a_n\} \subset (-\infty, 0)$ such that

$$\langle x - z, J(T_n x - x)\rangle \leq a_n \|x - T_n x\|^2$$

for every $n \in \mathbb{N}$, $x \in C$, and $z \in F(T_n)$.

There are various examples that the countable family $\{T_n\}$ satisfying this condition. For Hilbert spaces, see [10] and references therein. We also give another example of mappings in Example 4.2 in the next section. For Banach spaces, the resolvents of maximal monotone operators and metric projections for closed convex subsets of E have this property under some appropriate conditions for the underlying space E; see the next section.

We know that, if $\{T_n\}$ satisfies the condtion (I), then $F = \bigcap_{n=1}^{\infty} F(T_n)$ is closed and convex. Indeed, let $n \in \mathbb{N}$ and let $\{z_m\} \subset F(T_n)$ such that $z_m \to z$. We have

$$\langle z - z_m, J(T_n z - z)\rangle \leq a_n \|z - T_n z\|^2$$

for all $m \in \mathbb{N}$, which implies

$$a_n \|z - T_n z\|^2 \geq 0.$$

Since $a_n < 0$, we obtain $z \in F(T_n)$. Thus $F(T_n)$ is closed. Next, let $z_1, z_2 \in F(T_n)$, $0 \leq \alpha \leq 1$ and $x = \alpha z_1 + (1 - \alpha) z_2$. We get

$$\langle x - z_1, J(T_n x - x)\rangle \leq a_n \|x - T_n x\|^2 \text{ and } \langle x - z_2, J(T_n x - x)\rangle \leq a_n \|x - T_n x\|^2,$$

which implies

$$\begin{aligned} 0 &= \langle x - x, J(T_n x - x)\rangle \\ &= \alpha \langle x - z_1, J(T_n x - x)\rangle + (1 - \alpha) \langle x - z_2, J(T_n x - x)\rangle \\ &\leq a_n \|x - T_n x\|^2 . \end{aligned}$$

Hence we obtain $x \in F(T_n)$ and therefore $F(T_n)$ is convex.

Let us define a sequence $\{x_n\}$ as follows:

$$(1) \qquad \begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = T_n x_n, \\ C_{n+1} = \{z \in C_n : \langle x_n - z,\ J(y_n - x_n)\rangle \leq a_n \|x_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for each $n \in \mathbb{N}$, where $P_{C_{n+1}}$ is the metric projection onto C_{n+1}. Now, using the method of [13, Theorem 3.1], we obtain the following by Theorem 2.3.

Theorem 3.1. *Let $\{T_n\}$ be a countable family of mappings of a nonempty closed convex subset C of a smooth Banach space E into itself such that $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ and suppose that $\{T_n\}$ satisfies the condition (I). Let $\{x_n\}$ and $\{C_n\}$ be defined by (1). Then, the following hold:*

(i) *$F \subset C_{n+1} \subset C_n$ for all $n \in \mathbb{N}$ and a sequence $\{x_n\}$ generated by (1) is well defined;*

(ii) *assume that for every sequence* $\{z_n\}$ *in* C *and* $z \in C$, $z_n \to z$ *and* $T_n z_n \to z$ *imply* $z \in F$. *If* E *is strictly convex and reflexive,* E *has the Kadec-Klee property, and* $\sup_{n\in\mathbb{N}} a_n < 0$, *then* $\{x_n\}$ *converges strongly to* $P_F x$, *where* P_F *is the metric projection onto* F*;*

(iii) *assume that for every bounded sequence* $\{z_n\}$ *in* C, $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ *imply* $\omega_w(z_n) \subset F$, *where* $\omega_w(z_n)$ *is the set of all weak cluster points of* $\{z_n\}$. *If* E *is strictly convex and reflexive,* E *has the Kadec-Klee property, and* $\sup_{n\in\mathbb{N}} a_n < 0$, *then* $\{x_n\}$ *converges strongly to* $P_F x$.

Proof. Let us prove (i). It is obvious that C_n is closed and convex for all $n \in \mathbb{N}$. By mathematical induction, we have $F \subset C_{n+1} \subset C_n$ for every $n \in \mathbb{N}$ from the condition (I) and $\{x_n\}$ is well defined.

Let us show (ii). It is obvious from the definition of $\text{s-Li}_n C_n$ that

$$\bigcap_{n=1}^{\infty} C_n \subset \operatorname*{s-Li}_n C_n.$$

Let $z \in \text{w-Ls}_n C_n$. Then, there exists a sequence $\{z_i\}$ such that $z_i \in C_{n_i}$ for all $i \in \mathbb{N}$ and $\{z_i\}$ converges weakly to z, where $\{C_{n_i}\}$ is a subsequence of $\{C_n\}$. Suppose that $z \notin \bigcap_{n=1}^{\infty} C_n$. Then, there exists $n_0 \in \mathbb{N}$ such that $z \notin C_{n_0}$. On the other hand, $z_{n_i} \in C_{n_i} \subset C_{n_0}$ for every $n_i \geq n_0$, hence $z \in C_{n_0}$. This is a contradiction. Hence

$$\operatorname*{w-Ls}_n C_n \subset \bigcap_{n=1}^{\infty} C_n.$$

Therefore, $\text{s-Li}_n C_n = \text{w-Ls}_n C_n = \bigcap_{n=1}^{\infty} C_n$, that is,

$$\operatorname*{M-lim}_n C_n = \bigcap_{n=1}^{\infty} C_n.$$

By Theorem 2.3, we have

$$x_n = P_{C_n} x \to P_{\bigcap_{n=1}^{\infty} C_n} x. \tag{2}$$

Since $x_{n+1} = P_{C_{n+1}} x$, it follows that $\|x_{n+1} - x\| \leq \|x - P_F x\|$ for every $n \in \mathbb{N}$. From (2), we have

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{3}$$

Let $a = \sup_n a_n < 0$. Since $x_{n+1} \in C_{n+1}$, we have

$$\begin{aligned} -\|x_n - x_{n+1}\| \|x_n - y_n\| &\leq \langle x_n - x_{n+1}, J(y_n - x_n)\rangle \\ &\leq a_n \|x_n - y_n\|^2 \leq a \|x_n - y_n\|^2, \end{aligned}$$

which implies $0 \leq -a \|x_n - y_n\| \leq \|x_n - x_{n+1}\|$ for every $n \in \mathbb{N}$. By (3), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4}$$

From (2), (4), and the assumption, we obtain $P_{\bigcap_{n=1}^{\infty} C_n} x \in F$, which implies

$$P_{\bigcap_{n=1}^{\infty} C_n} x = P_F x.$$

Thus we have $x_n \to P_F x$. Hence the proof of (ii) is complete. The assertion (iii) is easily obtained from (ii). □

4. Strong convergence theorems

By Theorem 3.1, we may get the following result, an improved version of the theorem shown in [11] without putting error terms. Using this theorem, we have various results; see [10, 11] for details.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}$ a countable family of mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$ which satisfies the following condition: There exists $\{b_n\} \subset (-1, \infty)$ such that $\|T_n x - z\|^2 \leq \|x - z\|^2 - b_n \|(I - T_n)x\|^2$ for every $n \in \mathbb{N}$, $x \in C$ and $z \in F(T_n)$. Let $\{x_n\}$ be a sequence generated by*

$$(5) \qquad \begin{cases} x_1 = x \in C, \\ C_1 = C, \\ y_n = T_n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 - b_n \|x_n - y_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for each $n \in \mathbb{N}$, where $P_{C_{n+1}}$ is the metric projection onto C_{n+1} and $\{b_n\}$ satisfies $\inf_n b_n > -1$. *Then, the following hold:*

(i) *$F \subset C_{n+1}$ for all $n \in \mathbb{N}$ and a sequence $\{x_n\}$ is well defined;*
(ii) *assume that for every sequence $\{z_n\}$ in C and $z \in C$, $z_n \to z$ and $T_n z_n \to z$ imply $z \in F$. Then, $\{x_n\}$ converges strongly to $z_0 = P_F x$.*

Proof. For every $n \in \mathbb{N}$, $x \in C$ and $z \in F(T_n)$, we have

$$\begin{aligned} &\|x - z\|^2 - b_n \|x - T_n x\|^2 - \|T_n x - z\|^2 \\ &= \|x - z\|^2 - b_n \|x - T_n x\|^2 - \|T_n x - x\|^2 - 2 \langle T_n x - x, x - z \rangle - \|x - z\|^2 \\ &= 2 \left(-\frac{1 + b_n}{2} \|x - T_n x\|^2 - \langle T_n x - x, x - z \rangle \right). \end{aligned}$$

Thus, putting $a_n = -(1 + b_n)/2$ for all $n \in \mathbb{N}$, we get that $\|T_n x - z\|^2 \leq \|x - z\|^2 - b_n \|x - T_n x\|^2$ if and only if $\langle T_n x - x, x - z \rangle \leq a_n \|x - T_n x\|^2$. Hence the condition (I) is satisfied. By Theorem 3.1 (i) and (ii), $\{x_n\}$ is well defined and converges strongly to z_0. □

The following example gives a sequence of mappings which satisfies the conditions assumed in the theorem above. Several results for similar types of mappings appeared in [8] and [21].

Example 4.2. Let C be a nonempty closed convex subset of a real Hilbert space H and T a quasipseudocontractive and Lipschitz mapping of C into itself with the Lipschitz constant $L > 0$ such that $F(T) \neq \emptyset$, where T is called quasipseudocontractive [4] if $\|Tx - z\|^2 \leq \|x - z\|^2 + \|(I - T)x\|^2$ holds for every $x \in C$ and $z \in F(T)$. Let $T_n x = \alpha_n T(\beta_n T x + (1 - \beta_n)x) + (1 - \alpha_n)x$ for all $n \in \mathbb{N}$ and $x \in C$, where

$0 < \alpha \le \alpha_n \le 1$ and $\beta\alpha_n \le \beta_n < 1/(\sqrt{1+L^2}+1)$ for some $\alpha \in (0,1)$ and $\beta > 0$ for every $n \in \mathbb{N}$. Then, we have

$$\begin{aligned}\|T_n x - z\|^2 &\le \|x-z\|^2 + \alpha_n\beta_n(L^2\beta_n^2 + 2\beta_n - 1)\|x - Tx\|^2 \\ &\quad + \alpha_n(\alpha_n - \beta_n)\|x - T(\beta_n Tx + (1-\beta_n)x)\|^2 \\ &\le \|x-z\|^2 + \alpha_n\beta_n(L^2\beta_n^2 + 2\beta_n - 1)\|x - Tx\|^2 \\ &\quad + \frac{\alpha_n - \beta_n}{\alpha_n}\|x - T_n x\|^2 \\ &\le \|x-z\|^2 + (1-\beta)\|(I - T_n)x\|^2\end{aligned}$$

for each $n \in \mathbb{N}$, $x \in C$, and $z \in F(T)$ since $0 < \alpha_n \le 1$ and $\beta\alpha_n \le \beta_n < 1/(\sqrt{1+L^2}+1)$; see [6]. We also have $F(T) = F(T_n)$ for all $n \in \mathbb{N}$. Indeed, it is trivial that $F(T) \subset F(T_n)$ for all $n \in \mathbb{N}$. Let $z \in F(T)$ and $u \in F(T_n)$ for some $n \in \mathbf{N}$. We get

$$\begin{aligned}\|z-u\|^2 &= \|z - T_n u\|^2 \\ &\le \|u-z\|^2 + \alpha_n\beta_n(L^2\beta_n^2 + 2\beta_n - 1)\|u - Tu\|^2 \\ &\quad + (1-\beta)\|(I - T_n)u\|^2\end{aligned}$$

which implies

$$\alpha_n\beta_n(L^2\beta_n^2 + 2\beta_n - 1)\|u - Tu\|^2 \ge 0.$$

Since $\alpha_n\beta_n(L^2\beta_n^2 + 2\beta_n - 1) < 0$, we have $u = Tu$, that is, $F(T_n) \subset F(T)$. Let $\{z_n\}$ be a sequence in C such that $z_n \to z$ and $T_n z_n \to z$. We obtain

$$\lim_{n\to\infty}\|z_n - T(\beta_n T z_n + (1-\beta_n)z_n)\| = 0$$

by using $0 < \alpha \le \alpha_n$ and $\|z_n - T_n z_n\| = \alpha_n\|z_n - T(\beta_n T z_n + (1-\beta_n)z_n)\|$. Further, we have

$$\begin{aligned}\|z_n - Tz_n\| &\le \|z_n - T(\beta_n T z_n + (1-\beta_n)z_n)\| + \|T(\beta_n T z_n + (1-\beta_n)z_n) - Tz_n\| \\ &\le \|z_n - T(\beta_n T z_n + (1-\beta_n)z_n)\| + L\beta_n\|z_n - Tz_n\|\end{aligned}$$

which implies

$$(1-\beta_n L)\|z_n - Tz_n\| \le \|z_n - T(\beta_n T z_n + (1-\beta_n)z_n)\|.$$

Since $\beta_n < 1/(\sqrt{1+L^2}+1) < 1/L$, we get $\|z_n - Tz_n\| \to 0$ and hence $Tz_n \to z$. Since T is Lipschitz continuous, it follows that $z \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Therefore $\{T_n\}$ satisfies Theorem 4.1 (ii).

Using (iii) instead of (ii) in Theorem 3.1, we obtain the following.

Corollary 4.3 (Nakajo-Takahashi [11]). *Let C be a nonempty closed convex subset of a real Hilbert space H and $\{T_n\}$ a countable family of mappings of C into itself with $F = \bigcap_{n=1}^{\infty} F(T_n) \ne \emptyset$ which satisfies the following condition: There exists $\{b_n\} \subset (-1,\infty)$ such that $\|T_n x - z\|^2 \le \|x-z\|^2 - b_n\|(I - T_n)x\|^2$ for every $n \in \mathbb{N}$, $x \in C$ and $z \in F(T_n)$. Let $\{x_n\}$ and $\{C_n\}$ be defined by* (5) *with $\{b_n\}$ satisfying* $\inf_n b_n > -1$. *Then, the following hold:*

(i) *$F \subset C_{n+1}$ for all $n \in \mathbb{N}$ and a sequence $\{x_n\}$ is well defined;*

(ii) *assume that for every bounded sequence* $\{z_n\}$ *in* C, $\lim_{n\to\infty} \|z_{n+1} - z_n\| = 0$ *and* $\lim_{n\to\infty} \|z_n - T_n z_n\| = 0$ *imply* $\omega_w(z_n) \subset F$. *Then,* $\{x_n\}$ *converges strongly to* $z_0 = P_F x$.

Motivated by the idea of [1, p. 256], we have the following result for a family of maximal monotone operators, which we prove by using Lemma 2.1 and Theorem 3.1.

Theorem 4.4. *Let* E *be a strictly convex, smooth, and reflexive Banach space having the Kadec-Klee property. Let* I *be a countable set and let* $\{A_i\}_{i\in I}$ *be a family of maximal monotone operators of* E *into* E^* *such that* $F = \bigcap_{i\in I} A_i^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in E, \\ C_1 = E, \\ y_n = J_{r_n}^{A_{i(n)}} x_n, \\ C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for each $n \in \mathbb{N}$, *where* $\{r_n\} \subset (0,\infty)$, i *is an index mapping* $i : \mathbb{N} \to I$, *and* $J_{r_n}^{A_{i(n)}}$ *is the resolvent of* $A_{i(n)}$. *Suppose that, for each* $i \in I$, *there exists a subsequence* $\{n_k\}$ *of* $\mathbb{N}$ *such that* $i(n_k) = i$ *for all* $k \in \mathbb{N}$ *and that* $\inf_k r_{n_k} > 0$. *Then,* $\{x_n\}$ *converges strongly to* $P_F x$.

Proof. Let $T_n = J_{r_n}^{A_{i(n)}}$ for every $n \in \mathbb{N}$. We have $T_n : E \to D(A_{i(n)})$ and $F(T_n) = A_{i(n)}^{-1}0$ for all $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, $x \in E$ and $z \in F(T_n)$. Since $(1/r_n)J(x - T_n x) \in A_{i(n)} T_n x$ and $A_{i(n)}$ is monotone, we get $\langle T_n x - z, J(x - T_n x)\rangle \geq 0$, that is,

$$\langle x - z, J(T_n x - x)\rangle \leq - \|x - T_n x\|^2 .$$

So, the condition (I) is satisfied with $a_n = -1$ for all $n \in \mathbb{N}$. Let $\{z_n\}$ be a sequence in E and $z \in E$ such that $z_n \to z$ and $T_n z_n \to z$. Fix $i \in I$. By assumption, there exists a subsequence $\{i(n_k)\}$ of $\{i(n)\}$ such that $i(n_k) = i$ for all $k \in \mathbb{N}$. It follows that

$$z_{n_k} \to z \text{ and } T_{n_k} z_{n_k} = J_{r_{n_k}}^{A_i} z_{n_k} \to z.$$

By Lemma 2.1, we get $z \in A_i^{-1}0$ and hence $z \in F$. From Theorem 3.1 (ii), we obtain the desired result. □

We have the following by Theorem 4.4; see [13, 16].

Theorem 4.5. *Let* E *be a strictly convex, smooth, and reflexive Banach space having the Kadec-Klee property and* A *a maximal monotone operator of* E *into* E^* *such that* $A^{-1}0 \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in E, \\ C_1 = E, \\ y_n = J_{r_n} x_n \, , \\ C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for each $n \in \mathbb{N}$, *where* $\{r_n\} \subset (0, \infty)$ *such that* $\limsup_n r_n > 0$, *and* J_{r_n} *is the resolvent of* A. *Then,* $\{x_n\}$ *converges strongly to* $P_{A^{-1}0}x$.

We also have the following result for a convex feasibility problem by Lemma 2.2 and Theorem 4.4.

Theorem 4.6. *Let* E *be a strictly convex, smooth, and reflexive Banach space having the Kadec-Klee property. Let* I *be a countable index set and let* D_i *be a nonempty closed convex subset of* E *for every* $i \in I$ *such that* $D = \bigcap_{i \in I} D_i \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 = x \in E, \\ C_1 = E, \\ y_n = P_{D_{i(n)}} x_n, \\ C_{n+1} = \{z \in C_n : \langle y_n - z, J(x_n - y_n) \rangle \geq 0\}, \\ x_{n+1} = P_{C_{n+1}} x \end{cases}$$

for each $n \in \mathbb{N}$, *where an index mapping* $i : \mathbb{N} \to I$ *satisfies that for every* $i \in I$, *there are infinitely many* $k \in \mathbb{N}$ *such that* $i(k) = i$. *Then,* $\{x_n\}$ *converges strongly to* $P_D x$.

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