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WEIGHTED EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce a weighted equilibrium problem over product of sets and a system of weighted equilibrium problems for vector-valued bifunctions and show that both have the same solution set. Further, we introduce the concept of normalized solution of the system of weighted equilibrium problems and give its relationship with the solutions of systems of vector equilibrium problems. Furthermore, several kinds of weighted monotonicities are defined for the family of vector-valued bifunctions. Using fixed-point theorems, we establish some existence theorems for these problems. The concepts and results presented in this paper extend and unify a number of previously known corresponding concepts and results in the literature.

1. Introduction

The equilibrium problems theory has emerged as an interesting and fascinating branch of applicable mathematics. This theory has become a rich source of inspiration and motivation for the study of a large number of problems arising in economics, optimization, operation research in a general and unified way. Equilibrium problems include variational inequalities, complementarity problems, convex optimization problems, saddle point problems, and Nash equilibrium problems as special cases.

An important generalization of equilibrium problem is the vector equilibrium problem which has applications in multiobjective optimization problems. In recent past, various classes of vector equilibrium problems have been introduced and studied by many authors, see for example [5,6,10-18] and the references therein.

Recently, some systems of vector equilibrium problems, that is, the families of equilibrium problems with vector-valued bifunctions, defined on product of sets have been studied by Ansari *et al.* [3,4] which include the systems of vector variational(-like) inequality problems; the systems of vector optimization problems; Nash equilibrium problems for vector-valued functions, relative monotone variational inequalities, see for example [1,5,6,14,17] and the references therein.

Very recently, Ansari *et al.* [2] introduced the weighted variational inequalities over product of sets and system of weighted variational inequalities and establish some existence results for these problems.

Motivated and inspired by the recent work going in this direction, in this paper, we introduce a weighted equilibrium problem over product of sets and a system of weighted equilibrium problems for vector-valued bifunctions and show that both

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have the same solution set. Further, we introduce the concept of normalized solution of the system of weighted equilibrium problems and give its relationship with the solutions of systems of vector equilibrium problems. Furthermore, several kinds of weighted monotonicities are defined for the family of vector-valued bifunctions. Using fixed-point theorems, we establish some existence theorems for these problems. The concepts and results presented in this paper, extend and unify a number of previously known corresponding concepts and results in the literature, see for example [2] and the references therein.

2. Preliminaries

Throughout this paper unless otherwise stated, we use the following notations and assumptions. Let for each given $m \in \mathbb{N}$, $(\mathbb{R}^m, \mathbb{R}^m_+)$ be an ordered Banach space where $\mathbb{R}^m_+ = \{u = (u_1, \dots, u_m) \in \mathbb{R}^m : u_j \geq 0 \text{ for } j = 1, \dots, m\}$ is a pointed cone. We denote by $\mathbb{T}^m_+$ and $\operatorname{int} \mathbb{T}^m_+$, the simplex of $\mathbb{R}^m_+$ and its relative interior, respectively, that is,

$$\mathbb{T}^m_+ = \{u = (u_1, \dots, u_m) \in \mathbb{R}^m_+ : \sum_{j=1}^{m} u_j = 1\},$$

$$\operatorname{int} \mathbb{T}^m_+ = \{u = (u_1, \dots, u_m) \in \operatorname{int} \mathbb{R}^m_+ : \sum_{j=1}^{m} u_j = 1\}.$$

Let $I = \{1, 2, \dots, n\}$ be an index set and for each $i \in I$, let l_i be a positive integer. Let, for each $i \in I$, X_i be a real topological vector space (not necessarily Hausdorff) and let K_i be a nonempty convex subset of X_i, with $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$.

Let $\{\phi_i\}_{i \in I}$ be a family of bifunctions $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ such that $\phi_i(x, x_i) = 0, \forall i \in I,\ x \in K$.

We consider the following system of vector equilibrium problems (for short, SVEP): Find $\bar{x} \in K$ such that, for each $i \in I$,

$$\phi_i(\bar{x}, y_i) \notin -\mathbb{R}^{l_i}_+ \backslash \{\mathbf{0}\}, \quad \forall y_i \in K_i \tag{2.1}$$

and the system of weak vector equilibrium problems (for short, SVEP$_w$): Find $\bar{x} \in K$ such that, for each $i \in I$,

$$\phi_i(\bar{x}, y_i) \notin -\operatorname{int} \mathbb{R}^{l_i}_+, \quad \forall y_i \in K_i. \tag{2.2}$$

Throughout rest of the paper unless otherwise stated, we assume that

$$W = (W_1, \dots, W_n) \in \prod_{i \in I} (\mathbb{R}^{l_i}_+ \backslash \{\mathbf{0}\})$$

is a given weight vector.

Now, we introduce the following weighted equilibrium problem over product of sets (for short, WEP): Find $\bar{x} \in K$ with respect to (for short, wrt) the weight vector W such that

$$\sum_{i \in I} W_i . \phi_i(\bar{x}, y_i) \geq 0, \quad \forall y_i \in K_i \tag{2.3}$$

and the following system of weighted equilibrium problems (for short, SWEP): Find $\bar{x} \in K$ wrt the weight vector W such that, for each $i \in I$,

$$W_i.\phi_i(\bar{x}, y_i) \geq 0, \quad \forall y_i \in K_i \tag{2.4}$$

where dot '.' denotes the inner product on $\mathbb{R}^{l_i}$. If, for each $i \in I, W_i \in \mathbb{T}_+^{l_i}$, then the solutions of WEP (2.3) and SWEP (2.4) are called normalized. We denote E^w (respectively E_s^w) the solution set of WEP (2.3) (respectively SWEP (2.4)) and by E_n^w (respectively E_{sn}^w) the normalized solution set of WEP (2.3) (respectively SWEP (2.4)).

Finally, we define the following problem which is closely related to WEP (2.3) and can be termed as Minty weighted equilibrium problem (for short, MWEP): Find $x \in K$ wrt weight vector W such that

$$\sum_{i \in I} W_i.\phi_i(y, x_i) \leq 0, \quad \forall y_i \in K_i,\ i \in I. \tag{2.5}$$

The solution set of MWEP (2.5) is denoted by E_M^w.

First, we show that both WEP (2.3) and SWEP (2.4) have the same solution set.

Lemma 2.1. *For a given weight vector W (respectively $W = (W_1, \ldots, W_n) \in \prod_{i \in I} \mathbb{T}_+^{l_i},\ i \in I$), $E^w = E_s^w$ (respectively $E^w = E_{sn}^w$).*

Proof. Evidently, $E_s^w \subseteq E^w$. Let $\bar{x} \in E^w$. Then

$$\sum_{i \in I} W_i.\phi_i(\bar{x}, y_i) \geq 0, \quad \forall y_i \in K_i,\ i \in I.$$

For each $j \neq i$, let $y_j = \bar{x}_j$. Then, from the preceding inequality, it follows that, for each $i \in I$,

$$W_i.\phi_i(\bar{x}, y_i) \geq 0, \quad \forall y_i \in K_i.$$

Hence, $\bar{x} \in E_s^w$ and this completes the proof. □

Next, by making use of SWEP (2.4), we solve SVEP (2.1) (or SVEP$_w$ (2.2)).

Lemma 2.2. *Each normalized solution $\bar{x} \in K$ with weight vector $W \in \prod_{i \in I} \mathbb{T}_+^{l_i}$ (respectively $W \in \prod_{i \in I} \operatorname{int} \mathbb{T}_+^{l_i}$) of SWEP (2.4) is a solution of SVEP$_w$ (2.2) (respectively SVEP (2.1)).*

Proof. Let $\bar{x} \in K$ be a normalized solution of SWEP (2.4) with weight vector $W \in \prod_{i \in I} \mathbb{T}_+^{l_i}$ (respectively $W \in \prod_{i \in I} \operatorname{int} \mathbb{T}_+^{l_i}$). Suppose that $\bar{x} \in K$ is not a solution of SVEP$_w$ (2.2) (respectively SVEP (2.1)). Then, there exist some $i \in I$ and $y_i \in K_i$ such that

$$\phi_i(\bar{x}, y_i) \in -\operatorname{int} \mathbb{R}_+^{l_i}, \text{ (respectively } \phi_i(\bar{x}, y_i) \in -\mathbb{R}_+^{l_i} \backslash \{\mathbf{0}\}).$$

Since $W \in \mathbb{T}_+^{l_i}$ (respectively $W \in \operatorname{int} \mathbb{T}_+^{l_i}$), for each $i \in I$, we have $W_i.\phi_i(\bar{x}, y_i) < 0$, for each $i \in I$ which contradicts our assumption that $\bar{x} \in K$ is a normalized solution of SWEP (2.4). Hence, $\bar{x} \in K$ is a solution of SVEP$_w$ (2.2) (respectively SVEP (2.1)) and this completes the proof. □

From Lemma 2.1 and Lemma 2.2, we deduce the following result.

Lemma 2.3. *Each normalized solution $\bar{x} \in K$ with weight vector $W \in \prod_{i\in I} \mathbb{T}_+^{l_i}$ (respectively $W \in \prod_{i\in I} \operatorname{int} \mathbb{T}_+^{l_i}$) of WEP* (2.3) *is a solution of $SVEP_w$* (2.2) *(respectively SVEP* (2.1)*).*

Now, we recall the following fixed point theorems which are important in establishing the existence theorem for WEP (2.3).

For every nonempty set A, we denote by 2^A (respectively $\mathcal{F}(A)$) the family of all subsets (respectively, finite subsets) of A.

Theorem 2.1 ([8]). *Let K be a nonempty and convex subset of a topological vector space (not necessarily Hausdorff) X and let $T : K \to 2^K$ be a set-valued mapping. Assume that the following conditions hold:*

(i) *For all $x \in K$, $T(x)$ is convex;*
(ii) *For each $A \in \mathcal{F}(K)$ and for all $y \in CoA$, $T^{-1}(y) \bigcap CoA$ is open in CoA, where CoA denotes the convex hull of set A;*
(iii) *For each $A \in \mathcal{F}(K)$ and all $x, y \in CoA$ and every net $\{x_\alpha\}_{\alpha\in\Lambda}$ in K converging to x such that $ty + (1-t)x \notin T(x_\alpha)$, for all $\alpha \in \Lambda$ and for all $t \in [0,1]$, we have $y \notin T(x)$;*
(iv) *There exists a nonempty compact subset D of K and an element $\tilde{y} \in D$ such that $\tilde{y} \in T(x)$ for all $x \in K \backslash D$;*
(v) *For all $x \in D$, $T(x)$ is nonempty.*

Then, there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

Theorem 2.2 ([9]). *Let K be nonempty convex subset of a topological vector space (not necessarily a Hausdorff) E and let $S, T : K \to 2^K$ be set-valued mappings. Assume that the following conditions hold:*

(i) *For all $x \in K$, $S(x) \subseteq T(x)$;*
(ii) *For all $x \in K$, $T(x)$ is convex and $S(x)$ is nonempty;*
(iii) *For all $y \in K$, $S^{-1}(y) := \{x \in K : y = S(x)\}$ is compactly open;*
(iv) *There exists a nonempty closed, compact (not necessarily convex) subset D of K and $\tilde{y} \in D$ such that $K \backslash D \subset T^{-1}(\tilde{y})$.*

Then, there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

3. Existence theorems for WEP (2.3)

First, we give the following definitions.

Definition 3.1. A family $\{\phi_i\}_{i\in I}$ of bifunctions $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ is said to be

(i) *Weighted monotone* wrt the weight vector W if, for all $x, y \in K$, we have
$$\sum_{i\in I} W_i.\phi_i((x, y_i) + \phi(y, x_i)) \le 0;$$
(ii) *Weighted pseudomonotone* wrt the weight vector W if, for all $x, y \in K$, we have
$$\sum_{i\in I} W_i.\phi_i(x, y_i) \ge 0 \;\Rightarrow \sum_{i\in I} W_i.\phi_i(y, x_i) \le 0;$$
(iii) *Weighted strictly pseudomonotone* wrt the weight vector W if, the second inequality in (ii) is strict for all $x \neq y$;

(iv) *Weighted convex in second argument* wrt weight vector W if for all $x, y, z \in K$ and $\lambda \in [0,1]$, we have

$$\sum_{i\in I} W_i.f_i(z, \lambda x_i + (1-\lambda)y_i) \leq \lambda \sum_{i\in I} W_i.f_i(z, x_i) + (1-\lambda) \sum_{i\in I} W_i.f_i(z, y_i).$$

Definition 3.2. A family $\{\phi_i\}_{i\in I}$ of bifunctions $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ is said to be *weighted hemicontinuous* wrt the weight vector W if, for all $x, y \in K$ and $\lambda \in [0,1]$, the mapping $\lambda \to \sum_{i\in I} W_i.\phi_i(y + \lambda(x-y), y_i)$ is continuous.

It is remarked that the concepts given in Definitions 3.1–3.2 are the natural extensions of the corresponding concepts given in [1,2,17].

Now, we prove the following Minty type lemma for WEP (2.3).

Lemma 3.1. *If family $\{\phi_i\}_{i\in I}$ of bifunctions $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ is weighted pseudomonotone and weighted hemicontinuous wrt the weight vector W and convex in second argument, then $E^w = E^w_M$.*

Proof. $E^w \subseteq E^w_M$ is directly followed by weighted pseudomonotonicity of the family $\{\phi_i\}_{i\in I}$. Let $x \in E^w_M$, we have

$$\sum_{i\in I} W_i.\phi_i(y, x_i) \leq 0, \ \forall y_i \in K_i, \ i \in I.$$

Since, for each $i \in I, K_i$ is convex, $]x_i, y_i[:= \alpha y_i + (1-\alpha)x_i \in K \quad \forall \alpha \in (0,1)$, and hence, we have

$$\sum_{i\in I} W_i.\phi_i(z, x_i) \leq 0, \quad \forall z_i \in]x_i, y_i[. \tag{3.1}$$

Again, since $\phi_i(z, .)$ is convex, we have

$$0 = \sum_{i\in I} W_i.\phi_i(z, z_i) \leq \alpha \sum_{i\in I} W_i.\phi_i(z, y_i) + (1-\alpha) \sum_{i\in I} W_i.\phi_i(z, x_i). \tag{3.2}$$

From inclusions (3.1) and (3.2), we have

$$\sum_{i\in I} W_i.\phi_i(z, y_i) \geq 0.$$

By weighted hemicontinuity of the family $\{\phi_i\}_{i\in I}$, the preceding inequality implies that

$$\sum_{i\in I} W_i.\phi_i(x, y_i) \geq 0, \ \forall y_i \in K_i, \ i \in I.$$

That is, $x \in E^w$, and this completes the proof. □

Theorem 3.1. *Let the family $\{\phi_i\}_{i\in I}$ of bifunctions $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ be such that $\phi_i(x, x_i) = 0, \ \forall x \in K$ and let $\{\phi_i\}_{i\in I}$ be weighted convex and weighted continuous in second argument and weighted pseudomonotone, weighted hemicontinuous wrt the weight vector W. Assume that there exists a nonempty, closed and compact subset of D of K and $\tilde{y} \in D$ such that, for each $x \in K \backslash D$, $\sum_{i\in I} W_i.\phi_i(x, \tilde{y}_i) < 0$. Then, there exists a solution $\bar{x} \in K$ of WEP* (2.3) *and hence it is a solution SWEP* (2.4)*. Furthermore, if $W \in \prod_{i\in I} \mathbb{T}^{l_i}_+$, then there exists a normalized solution $\bar{x} \in K$ of WEP* (2.3) *and hence it is a solution of SVEP$_w$* (2.2)*. Furthermore, if $W \in \prod_{i\in I} \operatorname{int} \mathbb{T}^{l_i}_+$, then $\bar{x} \in K$ is a solution of SVEP* (2.1)*.*

Proof. For each $x \in K$, define the set-valued mappings $S, T : K \to 2^K$ by

$$S(x) = \{y \in K : \sum_{i \in I} W_i.\phi_i(y, x_i) > 0\}$$

and

$$T(x) = \{y \in K : \sum_{i \in I} W_i.\phi_i(x, y_i) < 0\}.$$

Now, for each $x \in K$, we claim that $T(x)$ is convex. Indeed, let $y_1, y_2 \in T(x)$, $\alpha, \beta \geq 0$ such that $\alpha + \beta = 1$, $\alpha y_1 + \beta y_2 \in K$ as K is convex. Hence

$$\sum_{i \in I} W_i.\phi_i(x, y_{1,i}) < 0 \quad \text{and} \quad \sum_{i \in I} W_i.\phi_i(x, y_{2,i}) < 0.$$

Since $\{\phi_i\}_{i \in I}$ is convex in second argument, we have

$$\sum_{i \in I} W_i.\phi_i(x, (\alpha y_{1,i} + \beta y_{2,i})) < 0 \Rightarrow \alpha y_1 + \beta y_2 \in T(x).$$

Hence our claim is verified.

Further, it follows from weighted pseudomonotonicity of the family $\{\phi_i\}_{i \in I}$, that $S(x) \subseteq T(x)$ for each $x \in K$. Since

$$S^{-1}(y) = \{x \in K : y \in S(x)\},$$

i.e.,

$$S^{-1}(y) = \{x \in K : \sum_{i \in I} W_i.\phi_i(y, x_i) > 0\},$$

then

$$(S^{-1}(y))^c = \{x \in K : \sum_{i \in I} W_i.\phi_i(y, x_i) \leq 0\}.$$

It is easy observed from weighted hemicontinuity of the family $\{\phi_i\}_{i \in I}$, that $(S^{-1}(y))^c$ is closed, for each $y \in K$ and hence $S^{-1}(y)$ is open in K. Therefore, $S^{-1}(y)$ is compactly open.

Assume that, for all $x \in K$, $S(x)$ is nonempty. Then all the conditions of Theorem 2.2 are satisfied and therefore there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$.

It follows that

$$0 = \sum_{i \in I} W_i.\phi_i(\hat{x}, \hat{x}_i) < 0,$$

which is impossible.

Hence, there exists $x \in K$ such that $S(x) = \emptyset$. This implies that, for all $y \in K$, $\sum_{i \in I} W_i.\phi_i(y, x_i) \leq 0$, that is, there exists $\bar{x} \in K$ wrt the weight vector W such that

$$\sum_{i \in I} W_i.\phi_i(y, \bar{x}_i) \leq 0, \quad \forall y_i \in K_i,\ i \in I.$$

By Lemma 3.1, $\bar{x} \in K$ is a solution of WEP (2.3) and so by Lemma 2.1, it is a solution of SWEP (2.4). If $W \in \prod_{i \in I} \mathbb{T}_+^{l_i}$, then $\bar{x} \in K$ is a normalized solution of SWEP (2.4) and hence by Lemma 2.2, it is a solution of SVEP$_w$ (2.2). Further, if $W \in \prod_{i \in I} \operatorname{int} \mathbb{T}_+^{l_i}$, then again by Lemma 2.2, $\bar{x} \in K$ is a solution of SVEP (2.1), and this completes the proof. □

Theorem 3.2. *Let the family* $\{\phi_i\}_{i\in I}$ *of bifunctions* $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ *be such that* $\phi_i(x, x_i) = 0,\ \forall x \in K$ *and let* $\{\phi_i\}_{i\in I}$ *be weighted convex and weighted continuous in second argument and weighted strictly pseudomonotone, weighted hemicontinuous wrt weight vector* W. *Assume that there exists a nonempty, closed and compact subset of* D *of* K *and* $\tilde{y} \in D$ *such that for each* $x \in K\backslash D$, $\sum_{i\in I} W_i.\phi_i(x, \tilde{y}_i) < 0$. *Then there exists a unique solution* $\bar{x} \in K$ *of WEP* (2.3) *and hence it is a solution SWEP* (2.4). *Furthermore, if* $W \in \prod_{i\in I} \mathbb{T}^{l_i}_+$, *then there exists a unique normalized solution* $\bar{x} \in K$ *of WEP* (2.3) *and hence it is a solution of* $SVEP_w$ (2.2). *Furthermore, if* $W \in \prod_{i\in I} \operatorname{int} \mathbb{T}^{l_i}_+$, *then* $\bar{x} \in K$ *is a unique solution of SVEP* (2.1).

Proof. In view of Theorem 3.1, it is sufficient to show that WEP (2.3) has at most one solution. Suppose that there exist two solutions $x^{'}, x^{''}$ of WEP (2.3). Then, we have

$$\sum_{i\in I} W_i.\phi_i(x^{''}, x_i^{'}) \geq 0.$$

By the weighted strictly pseudomonotonicity of the family $\{\phi_i\}_{i\in I}$ of bifunctions, we have

$$\sum_{i\in I} W_i.\phi(x^{'}, x_i^{''}) \leq 0,$$

that is, $x^{'}$ is not a solution of WEP(2.3), a contradiction. This completes the proof. □

Now, we extend the notion of B-pseudomonotonicity given by Brezis [5] and Ansari *et al.* [1, 2].

Definition 3.3. A family $\{\phi_i\}_{i\in I}$ of bifunctions $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ is said to be *weighted B-pseudomonotone* wrt weight vector W, if for each $x \in K$ and every net $\{x^\alpha\}_{\alpha\in\Lambda}$ in K converging to x with $\liminf_\alpha[\sum_{i\in I} W_i.\phi_i(x^\alpha, x_i)] \geq 0$, we have

$$\limsup_\alpha \sum_{i\in I} W_i.\phi_i(x^\alpha, y_i) \leq \sum_{i\in I} W_i.\phi_i(x, y_i), \quad \forall y_i \in K_i.$$

Theorem 3.3. *Let the family* $\{\phi_i\}_{i\in I}$ *of bifunctions* $\phi_i : K \times K_i \to \mathbb{R}^{l_i}$ *be such that* $\phi_i(x, x_i) = 0,\ \forall x \in K$ *and let* $\{\phi\}_{i\in I}$ *be weighted convex in second argument and weighted B-pseudomonotone wrt weight vector* W *such that, for each* $A \in \mathcal{F}(K)$, $x \longmapsto \sum_{i\in I} W_i.\phi_i(x, y_i)$ *is lower semicontinuous on CoA. Assume that there exists a nonempty compact subset* D *of* K *and* $\tilde{y} \in D$ *such that for all* $x \in K\backslash D$, $\sum_{i\in I} W_i.\phi_i(x, \tilde{y}_i) < 0$. *Then there exists a solution* $\bar{x} \in K$ *of WEP* (2.3) *and hence it a solution of SWEP* (2.4). *Furthermore, if* $W \in \prod_{i\in I} \mathbb{T}^{l_i}_+$, *then there exists a normalized solution* $\bar{x} \in K$ *of WEP* (2.3) *which is also a unique solution of* $SVEP_w$ (2.4). *Furthermore, if* $W \in \prod_{i\in I} \operatorname{int} \mathbb{T}^{l_i}_+$, *then* $\bar{x} \in K$ *is a unique solution of SVEP* (2.1).

Proof. For each $x \in K$, let $T : K \to 2^K$ be same defined in the proof of Theorem 3.1, then for all $x \in K$, $T(x)$ is convex. Let $A \in \mathcal{F}(K)$. Then for all $y \in CoA$,

$$[(T^{-1}(y))^c] \bigcap CoA = \{x \in CoA : \sum_{i\in I} W_i.\phi_i(x, y_i) \geq 0\}$$

is closed in CoA by the lower semicontinuity of the mapping $x \longmapsto \sum_{i\in I} W_i.\phi_i(x, y_i)$ on CoA. Hence $(T^{-1}(y)) \bigcap CoA$ is open in CoA. Suppose that $x, y \in CoA$ and $\{x^\alpha\}_{\alpha\in\Lambda}$ is a net in K converging to x such that $\sum_{i\in I} W_i.\phi_i(x^\alpha, ty_i + (1-t)x_i) \geq 0$ for all $\alpha \in \Lambda$ and all $t \in [0,1]$. For $t = 0$, we have $\sum_{i\in I} W_i.\phi(x^\alpha, x_i) \geq 0$, $\forall \alpha \in \Lambda$ and therefore $\liminf_\alpha [\sum_{i\in I} W_i.\phi_i(x^\alpha, x_i)] \geq 0$.

By weighted B-pseudomonotonicity of ϕ, we have

$$\sum_{i\in I} W_i.\phi_i(x, y_i) \geq \limsup_\alpha \sum_{i\in I} W_i.\phi_i(x^\alpha, y_i). \tag{3.3}$$

For $t = 1$, we have $\sum_{i\in I} W_i.\phi_i(x^\alpha, y_i) \geq 0$ for all $\alpha \in \Lambda$ and therefore

$$\liminf_\alpha [\sum_{i\in I} W_i.\phi_i(x^\alpha, x_i)] \geq 0. \tag{3.4}$$

From inclusions (3.3) and (3.4), we have $\sum_{i\in I} W_i.\phi_i(x, y_i) \geq 0$ which implies $y \notin T(x)$.

Assume that, for all $x \in K$, $T(x)$ is nonempty. Then, all the conditions of Theorem 2.1 are satisfied. Hence, there exists $\hat{x} \in K$ such that $\hat{x} \in T(\hat{x})$, that is,

$$0 = \sum_{i\in I} W_i.\phi_i(\hat{x}, \hat{x}_i) < 0.$$

This is a contradiction.

Thus, there exists $\bar{x} \in K$ such that $T(\bar{x}) = \emptyset$, that is,

$$\sum_{i\in I} W_i.\phi_i(x, y_i) \geq 0, \ \forall y_i \in K_i, \ i \in I.$$

Hence, $\bar{x}$ is a solution of WEP (2.3) and so by Lemma 2.1, it is a solution of SWEP (2.4).

If $W_i \in \prod_{i\in I} \mathbb{T}_+^{l_i}$, then $\bar{x} \in K$ is a normalized solution of SVEP$_w$ (2.2). Further, if $W_i \in \prod_{i\in I}(\text{int}\, \mathbb{T}_+^{l_i})$ then again by Lemma 2.2, $\bar{x} \in K$ is a solution of SVEP (2.1). This completes the proof. □

It is of further research interest to generalize and extend the concepts and theorems presented in this paper for the system of weighted equilibrium problems involving set-valued bifunctions and set-valued mappings.

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