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# A NOTE ON THE APPROXIMATION OF FIXED POINTS IN THE HILBERT BALL

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Abstract. We establish a strong convergence theorem for an iterative scheme which approximates fixed points of $\rho$-nonexpansive self-mappings of the Hilbert ball.

## 1. Introduction

In a recent paper we have established a strong convergence theorem [5, Theorem 3.12] for an implicit continuous scheme which approximates fixed points of $\rho$-nonexpansive self-mappings of the Hilbert ball. In the present note we complement this result by proving a corresponding strong convergence theorem (Theorem 4.1 below) for an explicit discrete scheme. This theorem may be considered a possible Hilbert ball analogue of the Hilbert space theorems in [8] and [13]. Another such analogue can be found in [6].

## 2. Preliminaries

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $|\cdot|$, and let $\mathbb{B} := \{x \in H : |x| < 1\}$ be its open unit ball. We denote the set of natural numbers, the interval $[0,\infty)$ and the complex plane by $\mathbb{N}$, $\mathbb{R}^+$ and $\mathbb{C}$, respectively. The *hyperbolic metric* $\rho : \mathbb{B}\times\mathbb{B} \mapsto \mathbb{R}^+$ [3, page 98] is defined by

$$\rho(x,y) := \operatorname{argtanh}\big(1-\sigma(x,y)\big)^{\frac{1}{2}}, \tag{2.1}$$

where

$$\sigma(x,y) := \frac{(1-|x|^2)(1-|y|^2)}{|1-\langle x,y\rangle|^2}, \quad x,y \in \mathbb{B}. \tag{2.2}$$

This metric is the infinite-dimensional analogue of the Poincaré metric on the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$. We let $B(a,r) := \{x \in \mathbb{B} : \rho(a,x) < r\}$ stand for the $\rho$-ball of center $a$ and radius $r$. A subset of $\mathbb{B}$ is called *$\rho$-bounded* if it is contained in a $\rho$-ball. We say that a mapping $c : \mathbb{R} \mapsto \mathbb{B}$ is a *metric embedding* of the real line $\mathbb{R}$ into $\mathbb{B}$ if $\rho(c(s),c(t)) = |s-t|$ for all real $s$ and $t$. The image of $\mathbb{R}$ under a metric embedding is called a *metric line*. The image of a real interval $[a,b] = \{t \in \mathbb{R} : a \le t \le b\}$ under such a mapping is called a *metric segment*. It is known [3, page 102] that for any two distinct points $x$ and $y$ in $\mathbb{B}$, there is a unique metric line (also called a *geodesic*) which passes through $x$ and $y$. This metric line determines a unique metric segment joining $x$ and $y$. We denote this

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segment by $[x, y]$. For each $0 \le t \le 1$, there is a unique point $z \in [x, y]$ such that $\rho(x, z) = t\rho(x, y)$ and $\rho(z, y) = (1-t)\rho(x, y)$. This point will be denoted by $(1-t)x \oplus ty$. Similarly, for $r \ge 0$, we let $(1+r)x \ominus ry$ stand for the unique point $z \in \mathbb{B}$ that satisfies $\rho(z, x) = r\rho(x, y)$ and $\rho(z, y) = (1+r)\rho(x, y)$. This point lies on the unique geodesic determined by $x$ and $y$. The following inequality [3, page 104] shows that the metric space $(\mathbb{B}, \rho)$ is *hyperbolic* in the sense of [11].

**Lemma 2.1.** *For any four points $a, b, x$ and $y$ in $\mathbb{B}$, and any number $t \in [0, 1]$,*

$$\rho((1-t)a \oplus tx, (1-t)b \oplus ty) \le (1-t)\rho(a, b) + t\rho(x, y). \tag{2.3}$$

Next, we mention another useful property of the hyperbolic metric.

**Lemma 2.2.** *For any two points $x$ and $y$ in $\mathbb{B}$, and any number $t \in [0, 1]$,*

$$\rho(tx, ty) \le t\rho(x, y). \tag{2.4}$$

*Proof.* It is clear that we may assume without any loss of generality that $|x| \le |y|$ and that $0 < t < 1$. For a fixed $0 < t < 1$, the function $g : (0, 1) \mapsto \mathbb{R}^+$ defined by

$$g(r) := \frac{\operatorname{argtanh}(tr)}{\operatorname{argtanh}(r)}, \quad 0 < r < 1, \tag{2.5}$$

is decreasing and $\lim_{r \to 0^+} g(r) = t$. Therefore inequality (2.4) does hold for $x = 0$ and we may also assume in the sequel that $x \ne 0$. There are numbers $0 < p < 1$ and $0 < s < 1$ such that $tx = (1-p)0 \oplus px$ and $ty = (1-s)0 \oplus sy$. Since the function $g$ is decreasing and its right limit at zero is $t$, we have $s \le p \le t$. Let $z := (1+r)(ty) \ominus r0$, where $r := 1/p - 1 > 0$. Then $ty = (1-p)0 \oplus pz$ and $|x| \le |z| \le |y|$. Hence $\rho(x, z) \le \rho(x, y)$ and

$$\rho(tx, ty) = \rho((1-p)0 \oplus px, (1-p)0 \oplus pz) \le p\rho(x, z) \le p\rho(x, y) \le t\rho(x, y),$$

as claimed. □

Recall (see [11] and [12]) that a set-valued operator $T \subset \mathbb{B} \times \mathbb{B}$ with domain $D(T)$ and range $R(T)$ is said to be *coaccretive* if

$$\rho(x_1, x_2) \le \rho((1+r)x_1 \ominus ry_1, (1+r)x_2 \ominus ry_2) \tag{2.6}$$

for all $y_1 \in Tx_1$, $y_2 \in Tx_2$, and $r > 0$. Such operators are the Hilbert ball analogues of the operators of the form $T = I - A$, where $A$ is an accretive operator on a Banach space. In this case, the operator $T$ is also said to be pseudo-contractive [2, page 876]. Let $D$ be a subset of $\mathbb{B}$. A mapping $T : D \mapsto \mathbb{B}$ is called *$\rho$-nonexpansive* if $\rho(Tx_1, x_2) \le \rho(x_1, x_2)$ whenever $x_1$ and $x_2$ belong to $D$. It is known (see, for example, [3, page 91]) that each holomorphic self-mapping of $\mathbb{B}$ is $\rho$-nonexpansive. Using Lemma 2.1, one can check that all $\rho$-nonexpansive mappings are coaccretive. An interesting family of (possibly set-valued) coaccretive operators is described in [12, page 641]. These operators are analogues of sub-differentials of convex functions in Hilbert space. In particular, if $R_K : \mathbb{B} \mapsto K$ is the nearest point projection of $\mathbb{B}$ onto an arbitrary $\rho$-closed and $\rho$-convex subset $K$ of $\mathbb{B}$, then the operator $\{(R_K z, 2R_K z \ominus z) : z \in \mathbb{B}\} \subset \mathbb{B} \times \mathbb{B}$ is coaccretive.

When the operator $T$ is coaccretive, one can define for each positive $r$, a single-valued $\rho$-nonexpansive mapping $J_r : R((1+r)I \ominus rT) \mapsto D(T)$, the *resolvent* of $T$, by

$$J_r((1+r)x \ominus ry) = x, \tag{2.7}$$

where $x \in D(T)$ and $y \in Tx$. These mappings (which in normed linear spaces are indeed the resolvents of the accretive operator $A = I - T$) satisfy the following resolvent identity for all $t \geq s > 0$ and $x \in D(J_t)$:

$$J_t x = J_s((s/t)x \oplus (1 - s/t)J_t x). \tag{2.8}$$

Recall that a mapping $T : D \mapsto \mathbb{B}$ is said to be *firmly nonexpansive of the first kind* [3, page 124] if for each $x$ and $y$ in $D$, the function $\phi : [0,1] \mapsto [0,\infty)$ defined by

$$\phi(s) := \rho((1-s)x \oplus sTx, (1-s)y \oplus sTy), \ 0 \leq s \leq 1, \tag{2.9}$$

is decreasing. The set of all firmly nonexpansive mappings of the first kind will be denoted by $FN_1$.

A proof of our next lemma (based on the resolvent identity (2.8)) can be found in [5, Section 2].

**Lemma 2.3.** *Any resolvent of a coaccretive operator is firmly nonexpansive of the first kind.*

We say that a coaccretive operator $T \subset \mathbb{B} \times \mathbb{B}$ is *m-coaccretive* if

$$R((1+r)I \ominus rT) = \mathbb{B} \tag{2.10}$$

for all positive $r$.

Actually, given a coaccretive operator $T$, the assumption that (2.10) holds when $r = 1$ already implies that it holds for all $r > 0$. Any $\rho$-nonexpansive mapping $T : \mathbb{B} \mapsto \mathbb{B}$ is $m$-coaccretive.

**Lemma 2.4.** *If $0 \leq \alpha < 1$ and the mapping $f : \mathbb{B} \mapsto \alpha\mathbb{B}$ is holomorphic, then $f$ is a strict $\rho$-contraction with a $\rho$-Lipschitz constant $\alpha$.*

*Proof.* Since our claim is obviously true when $\alpha = 0$, we may assume that $\alpha$ is positive. In this case, the mapping $g = f/\alpha$ is a holomorphic self-mapping of $\mathbb{B}$. Hence it is $\rho$-nonexpansive and we have $\rho(f(x), f(y)) = \rho(\alpha g(x), \alpha g(y)) \leq \alpha\rho(g(x), g(y)) \leq \alpha\rho(x,y)$ by Lemma 2.2. □

We conclude this section with a simple consequence of inequality (2.3).

**Lemma 2.5.** *Let $f$ and $g$ be two $\rho$-Lipschitz self-mappings of $\mathbb{B}$ with Lipschitz constants $L$ and $M$, respectively, and let $\beta \in [0,1]$. Then the mapping $h : \mathbb{B} \mapsto \mathbb{B}$ defined by $h(x) := (1-\beta)f(x) \oplus \beta g(x)$, $x \in \mathbb{B}$, is also $\rho$-Lipschitz, with Lipschitz constant $(1-\beta)L + \beta M$.*

## 3. Approximating Curves

Given a $\rho$-nonexpansive self-mapping $T$ of $\mathbb{B}$, a holomorphic mapping $f : \mathbb{B} \to \alpha\mathbb{B}$, where $0 \leq \alpha < 1$, and a number $0 \leq t < 1$, we define the point $z_t \in \mathbb{B}$ as the unique fixed point of the strict $\rho$-contraction $S : \mathbb{B} \mapsto \mathbb{B}$ defined by

$$Sx := (1-t)f(x) \oplus tTx, \quad x \in \mathbb{B}. \tag{3.1}$$

Note that $S$ is indeed a strict $\rho$-contraction by Lemmata 2.4 and 2.5. It has a unique fixed point because the metric space $(\mathbb{B}, \rho)$ is complete. In other words,

$$z_t = (1-t)f(z_t) \oplus tTz_t, \quad 0 \le t < 1. \tag{3.2}$$

In this section we recall a few facts regarding the behavior of the *approximating curve* $\{z_t : 0 \le t < 1\}$. See [5, Section 3] for more information regarding this curve and [6, Section 3] for a study of a related, but different approximating curve.

We can also write

$$z_t = F_t(f(z_t)), \tag{3.3}$$

where $F_t : \mathbb{B} \mapsto \mathbb{B}$ is the mapping defined on page 123 of [3]. This mapping is, in fact, the resolvent $J_{r(t)}$ of the $m$-coaccretive operator $T$, where $r(t) = t/(1-t)$. In view of Lemma 2.3, it is firmly nonexpansive of the first kind. It may be defined by the equation

$$F_t(x) = (1-t)x \oplus tTF_t(x), \quad x \in \mathbb{B}. \tag{3.4}$$

Next, we recall [3, Theorem 24.1, page 122] (see also [12, Theorem 3.4, page 642]). Note (see [3, pages 110 and 120]) that the fixed point set $F(T)$ of a $\rho$-nonexpansive self-mapping $T$ of $\mathbb{B}$ is both $\rho$-closed and $\rho$-convex, and that the nearest point projection $R_K$ of $\mathbb{B}$ onto a $\rho$-closed and $\rho$-convex subset $K$ of $\mathbb{B}$ is $\rho$-nonexpansive (and belongs to $FN_1$). The retraction $R_K$ is also strongly nonexpansive [10, 1] and sunny [4, Proposition 5.4].

**Proposition 3.1.** *Let $T : \mathbb{B} \to \mathbb{B}$ be $\rho$-nonexpansive and let $F_t$, $0 \le t < 1$, be the family of mappings defined by (3.4). If $T$ has a fixed point, then for each $x \in \mathbb{B}$, the strong $\lim_{t \to 1^-} F_t(x) = R_{F(T)}x$.*

Finally, we recall Theorem 3.12 of [5] (the proof of which makes use of Proposition 3.1). We say that a mapping $f : \mathbb{B} \mapsto \mathbb{B}$ is compact if the closure of its image $\overline{f(\mathbb{B})}$ is a compact subset of $H$.

**Proposition 3.2.** *Let $T$ be a $\rho$-nonexpansive self-mapping of $\mathbb{B}$, $f : \mathbb{B} \mapsto \alpha\mathbb{B}$ a holomorphic mapping, where $0 \le \alpha < 1$, and let $z_t$, $0 \le t < 1$, be defined by (3.2). If $T$ has a fixed point and $f$ is compact, then the strong $\lim_{t \to 1^-} z_t = v$, where $v$ is the unique solution of the equation $z = R_{F(T)}(f(z))$.*

## 4. An Iterative Scheme

In this section we study a discrete iterative scheme for approximating fixed points of $\rho$-nonexpansive self-mappings of $\mathbb{B}$. The proof of our convergence theorem (Theorem 4.1 below) depends on Proposition 3.2.

Let a sequence $\{\alpha_n \in [0,1) : n \in \mathbb{N}\}$ satisfy the following three conditions:

$$\lim_{n \to \infty} \alpha_n = 1; \tag{4.1}$$

$$\sum_{n=1}^{\infty} (1-\alpha_n) = \infty; \tag{4.2}$$

$$\lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{(1-\alpha_n)^2} = 0. \tag{4.3}$$

These conditions, which originate with P.-L. Lions [7], are satisfied, for instance, when for each $n \in \mathbb{N}$, $\alpha_n = 1 - n^{-\beta}$, where $0 < \beta < 1$. They had already been used in [9].

Given a $\rho$-nonexpansive self-mapping of $\mathbb{B}$, a holomorphic $f : \mathbb{B} \mapsto \alpha\mathbb{B}$, where $0 \le \alpha < 1$, and a point $x_0 \in \mathbb{B}$, we consider in this section the iterative scheme

$$x_n = (1-\alpha_n) f(x_{n-1}) \oplus \alpha_n T x_{n-1}, \quad n \in \mathbb{N}. \tag{4.4}$$

**Theorem 4.1.** *Let $T$ be a $\rho$-nonexpansive self-mapping of $\mathbb{B}$, $f : \mathbb{B} \mapsto \alpha\mathbb{B}$ a holomorphic mapping, where $0 \le \alpha < 1$, $\{\alpha_n \in [0,1) : n \in \mathbb{N}\}$ a sequence satisfying (4.1)–(4.3), and $x_0$ a point in $\mathbb{B}$. If $T$ has a fixed point and $f$ is compact, then the sequence $\{x_n : n \in \mathbb{N}\}$ defined by (4.4) converges strongly to the unique solution $v \in \mathbb{B}$ of the equation $z = R_{F(T)}(f(z))$, where $R_{F(T)} : \mathbb{B} \mapsto F(T)$ is the nearest point projection of $\mathbb{B}$ onto the fixed point set $F(T)$ of $T$.*

*Proof.* Fix $n \in \mathbb{N}$ and consider the mapping $S_n : \mathbb{B} \mapsto \mathbb{B}$ defined by

$$S_n z := (1-\alpha_n) f(z) \oplus \alpha_n T z, \quad z \in \mathbb{B}. \tag{4.5}$$

In view of Lemmata 2.4 and 2.5, this mapping is a strict $\rho$-contraction with Lipschitz constant

$$p_n := (1-\alpha_n)\alpha + \alpha_n < 1.$$

Since the metric space $(\mathbb{B}, \rho)$ is complete, $S_n$ has a unique fixed point $y_n \in \mathbb{B}$. In other words,

$$y_n = (1-\alpha_n) f(y_n) \oplus \alpha_n T y_n, \quad n \in \mathbb{N}. \tag{4.6}$$

Note that $y_n = z_{\alpha_n}$ in the notation of equation (3.2) and Section 3. Since we already know by Proposition 3.2 and (4.1) that $y_n \to v$ strongly as $n \to \infty$, it is sufficient to show that $(x_n - y_n) \to 0$ strongly as $n \to \infty$. To this end, we first note that

$$\rho(x_n, y_n) = \rho(S_n x_{n-1}, S_n y_n) \le p_n \rho(x_{n-1}, y_n) \le p_n \rho(x_{n-1}, y_{n-1}) + \rho(y_{n-1}, y_n)$$

for all $n \in \mathbb{N}$. Setting

$$A(m) = \sup_{n \ge m+1} \frac{\rho(y_{n-1}, y_n)}{1 - p_n} \tag{4.7}$$

for all $m \in \mathbb{N}$, we conclude that

$$\rho(x_n, y_n) \le \rho(x_m, y_m) \prod_{j=m+1}^{n} p_j + A(m) \tag{4.8}$$

for all $n \ge m+1$. Since for each $n \ge 2$, the points $S_{n-1} y_{n-1}$ and $S_n y_{n-1}$ lie on the metric segment joining $f(y_{n-1})$ and $T y_{n-1}$, and since the sequences $\{y_n : n \in \mathbb{N}\}$

and $\{f(y_n) : n \in \mathbb{N}\}$ are $\rho$-bounded, there is a number $M \in \mathbb{R}^+$ such that

$$\begin{aligned}\rho(y_{n-1}, y_n) &= \rho(S_{n-1}y_{n-1}, S_n y_n)\\ &\le \rho(S_{n-1}y_{n-1}, S_n y_{n-1}) + \rho(S_n y_{n-1}, S_n y_n)\\ &= |\alpha_n - \alpha_{n-1}|\rho(f(y_{n-1}), Ty_{n-1}) + \rho(S_n y_{n-1}, S_n y_n)\\ &\le M|\alpha_n - \alpha_{n-1}| + p_n \rho(y_{n-1}, y_n)\end{aligned}$$

for all $n \geq 2$. Hence

$$A(m) \le \frac{M}{(1-\alpha)^2} \sup_{n \ge m+1} \frac{|\alpha_n - \alpha_{n-1}|}{(1-\alpha_n)^2} \tag{4.9}$$

for all $m \in \mathbb{N}$. Combining (4.2)–(4.3) with (4.8) and (4.9), we now see that $\rho(x_n, y_n) \to 0$ as $n \to \infty$. Since the sequences $\{x_n\}$ and $\{y_n\}$ are $\rho$-bounded, it follows that $(x_n - y_n) \to 0$ and $x_n \to v$ strongly, as asserted. □

This theorem seems to be new even in the special case where the mapping $f$ is a constant. It holds, in particular, when the (complex) Hilbert space $H$ is finite dimensional. It remains an open question whether it continues to hold when $H$ is infinite dimensional and $f$ is no longer assumed to be compact. Note that although $f$ is not assumed to be compact in [6, Theorem 4.1] (which concerns a related, but different iterative scheme), the self-mapping $T$ is assumed to be holomorphic there. It would also be of interest to determine the behavior of the sequence $\{x_n : n \in \mathbb{N}\}$ when other conditions are imposed on the sequence of parameters $\{\alpha_n : n \in \mathbb{N}\}$ and when the mapping $T$ is fixed point free, and to find out if Theorem 4.1 can be extended to other hyperbolic spaces in the sense of [11].

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