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ON THE EXISTENCE OF MULTIPLE NONTRIVIAL SOLUTIONS FOR RESONANT NEUMANN PROBLEMS

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Abstract. We consider a semilinear second order elliptic problem with Neumann boundary conditions and a nonsmooth potential (hemivariational inequality). Using nonsmooth critical point theory, we establish the existence of at least two nontrivial smooth solutions, when double resonance occurs at the origin between any two distinct successive eigenvalues.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We consider the following semilinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\begin{cases} -\Delta x\,(z) \in \partial j\,(z, x\,(z)) & \text{a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 & \text{on } \partial Z. \end{cases} \tag{1.1}$$

Here the potential function $(z, x) \to j\,(z, x)$ is measurable with respect to z and locally Lipschitz (in general nonsmooth) with respect to x. By $\partial j\,(z, x)$ we denote the Clarke's *generalized subdifferential* of the locally Lipschitz function $x \to j\,(z, x)$. Also, $n\,(z)$ is the outward unit normal at $z \in \partial Z$ and $\frac{\partial x}{\partial n} = (Dx, n)_{\mathbb{R}^N}$ in the sense of traces.

Recently Tang-Wu [11] studied problem (1.1) with a smooth potential, that is, with $j(z, \cdot) \in C^1(\mathbb{R})$ and proved a multiplicity result for problems which are resonant at zero between two successive eigenvalues λ_k, λ_{k+1}. The resonance is complete with respect to λ_k, but incomplete (nonuniform nonresonance) with respect to λ_{k+1}. It was left as an open problem, whether their multiplicity result is actually valid when complete resonance occurs also with respect to λ_{k+1} (double resonance situation; see Remark 4 of Tang-Wu [11]).

In this paper we answer this open problem and prove a multiplicity result for semilinear Neumann problems which are doubly resonant at the origin with respect to any spectral interval $[\lambda_k, \lambda_{k+1}]$. We also relax the hypotheses of Tang-Wu [11] and we also allow the potential function to be nonsmooth.

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We should mention that existence theorems for semilinear resonant Neumann problems were proved by Iannacci-Nkashama [4], [5], Kuo [7], Mawhin-Ward-Willem [8] and Rabinowitz [9].

In Iannacci-Nkashama [4] the equation is an ordinary differential equation (i.e., $N = 1$). In Iannacci-Nkashama [5] and Kuo [7], the authors use variants of the Landesman-Lazer asymptotic conditions. In all three papers the approach is degree theoretic.

Mawhin-Ward-Willem [8] use the monotonicity condition, while Rabinowitz [9] uses the periodicity condition. In both papers the approach is variational based on critical point theory.

In all the aforementioned works, with the exception of Iannacci-Nkashama [5], the resonance is with respect to the principal eigenvalue $\lambda_0 = 0$. None of these works deals with the doubly resonant situation and also they do not address the question of existence of multiple solutions.

Our approach is variational based on the nonsmooth critical point theory.

2. Mathematical Background

The nonsmooth critical point theory, which will be used in our variational approach, is based on the subdifferential theory for locally Lipschitz functions.

We first recall some basic definitions and facts from this theory. Our main reference is the book of Clarke [1]. Let X be a Banach space, X^* its topological dual and by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*).

Given a locally Lipschitz $\varphi : X \to \mathbb{R}$, the generalized directional derivative $\varphi^0(x;h)$ of φ at $x \in X$ in the direction $h \in X$ is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $\varphi^0(x;h)$ is sublinear, so it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) = \left\{ x^* \in X^* : \langle x^*, h \rangle \le \varphi^0(x;h) \text{ for all } h \in X \right\}.$$

The multifunction $x \to \partial\varphi(x)$ is called the *generalized subdifferential* of φ.

If $\varphi : X \to \mathbb{R}$ is continuous and convex, then φ is locally Lipschitz and the generalized subdifferential coincides with the subdifferential $\partial\varphi_c(x)$ in the sense of convex analysis, defined by

$$\partial\varphi_c(x) = \{ x^* \in X^* : \langle x^*, h \rangle \le \varphi(x+h) - \varphi(x) \text{ for all } h \in X \}.$$

Also, if $\varphi \in C^1(X)$, then φ is locally Lipschitz and

$$\partial\varphi(x) = \left\{ \varphi'(x) \right\}.$$

If φ, $\psi : X \to \mathbb{R}$ are both locally Lipschitz functions and $\lambda \in \mathbb{R}$, then for all $x \in X$:

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \text{ and } \partial(\lambda\varphi)(x) = \lambda\partial\varphi(x).$$

We say that $x \in X$ is a *critical point* of a locally Lipschitz function $\varphi : X \to \mathbb{R}$, if

$$0 \in \partial\varphi(x).$$

If $x \in X$ is a critical point of φ, then $c = \varphi(x)$ is called a *critical value* of φ.

It is easy to see that, if $x \in X$ is a local extremum of φ (i.e., a local minimizer or a local maximizer), then x is a critical point of φ.

Recall that, a locally Lipschitz function $\varphi : X \to \mathbb{R}$ is said to satisfy the *Palais-Smale condition* (*PS-condition* for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that

$$\{\varphi(x_n)\}_{n\geq 1} \text{ is bounded}$$

and

$$m(x_n) = \inf\{\|x^*\| : x^* \in \partial\varphi(x_n)\} \to 0 \text{ as } n \to \infty,$$

has a strongly convergent subsequence.

Our multiplicity result is based on a nonsmooth version of the *local linking theorem*, due to Kandilakis-Kourogenis-Papageorgiou [6] (see also Gasinski-Papageorgiou [2], p.178).

Theorem 2.1. *If X is a Banach space, $X = Y \oplus V$ with $\dim Y < +\infty$ and if $\varphi : X \to \mathbb{R}$ is Lipschitz continuous on bounded sets, satisfies the PS-condition, $\varphi(0) = 0$, φ is bounded below,*

$$\inf_X \varphi < 0$$

and there exists $r > 0$ such that

$$\begin{cases} \varphi(y) \leq 0 & \text{if } y \in Y,\ \|y\| \leq r \\ \varphi(v) \geq 0 & \text{if } v \in V,\ \|v\| \leq r \end{cases},$$

then φ has at least two nontrivial critical points.

Let us recall now a few basic things about the spectrum of the negative Laplacian with Neumann boundary conditions, i.e., of $(-\triangle, H^1(Z))$.

So, we consider the following linear eigenvalue problem

$$\begin{cases} -\triangle x(z) = \lambda x(z) & \text{a.e. on } Z, \\ \frac{\partial x}{\partial n} = 0 & \text{on } \partial Z,\ \lambda \in \mathbb{R}. \end{cases} \tag{2.1}$$

By an *eigenvalue* of (2.1), we mean a number $\lambda \in \mathbb{R}$ for which problem (2.1) has a nontrivial solution $u \in H^1(Z)$, which is called an *eigenfunction* corresponding to the eigenvalue $\lambda \in \mathbb{R}$.

It is easy to see, that a necessary condition for $\lambda \in \mathbb{R}$ to be an eigenvalue, is that $\lambda \geq 0$. Note that $\lambda = 0$ is an eigenvalue and the corresponding eigenspace (the linear subspace spanned by the eigenfunctions corresponding to $\lambda = 0$) is $\mathbb{R}$ (the space of constant functions).

Using the spectral theorem for compact self-adjoint operators on a Hilbert space, we show that (2.1) has a sequence $\{\lambda_k\}_{k\geq 0}$ of distinct eigenvalues, $\lambda_k \to +\infty$ as $k \to +\infty$ and $\lambda_0 = 0$.

We can also choose a corresponding sequence of eigenfunctions, which form an orthonormal basis for $L^2(Z)$ and an orthogonal basis for $H^1(Z)$. If $E(\lambda_k)$ denotes the eigenspace corresponding to the eigenvalue λ_k, $k \geq 0$, then we have the orthogonal direct sum decomposition

$$H^1(Z) = \overline{\underset{k\geq 0}{\oplus} E(\lambda_k)}.$$

Moreover, we have the following variational characterizations of the eigenvalues

$$\lambda_k = \min\left\{ \frac{\|Dx\|_2^2}{\|x\|_2^2} : x \in \overline{\underset{i\geq k}{\oplus} E(\lambda_i)}, x \neq 0 \right\} \tag{2.2}$$

$$= \max\left\{ \frac{\|Dx\|_2^2}{\|x\|_2^2} : x \in \overset{k}{\underset{i=0}{\oplus}} E(\lambda_i), x \neq 0 \right\}, \; k \geq 0.$$

By linear regularity theory, we have that every eigenfunction $u \in H^1(Z)$ belongs in $C^1(\overline{Z})$. Moreover, the eigenfunctions for λ_k, $k \geq 1$, change sign (nodal functions).

3. Multiplicity result

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z and let $|.|_N$ be the Lebesgue measure on $\mathbb{R}^N$.The hypotheses on the nonsmooth potential function $j(.,.)$ are the following:

H(j): The function $j : Z \times \mathbb{R} \to \mathbb{R}$ is such that $j(z,0) = 0$ a.e. on Z and:

(*i*) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(*ii*) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(*iii*) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a(z) + c|x|^{r-1},$$

where

$$a \in L^\infty(Z)_+, c > 0, 1 < r < 2^* := \begin{cases} \frac{2N}{N-2} & \text{if } N > 2 \\ +\infty & \text{if } N \in \{1,\ 2\} \end{cases};$$

(*iv*) $j(z,x) \to -\infty$ as $|x| \to \infty$ for almost all $z \in C \subseteq Z$, with $|C|_N > 0$ and $j(z,x) \leq \eta(z)$ for almost all $z \in Z$, all $x \in \mathbb{R}$, with $\eta \in L^1(Z)_+$.

(*v*) there exist $\delta > 0$ and an integer $m \geq 0$, such that for almost all $z \in Z$, all $0 < |x| \leq \delta$ and all $u \in \partial j(z,x)$

$$\lambda_m \leq \frac{u}{x} \leq \lambda_{m+1}.$$

Remark. Note that in hypothesis $H(j)(iv)$ the convergence to $-\infty$ occurs only for $z \in C$ and not for almost all $z \in Z$ as in Tang-Wu [11]. Moreover, the convergence need not to be uniform in $z \in C$, while in Tang-Wu [11] it is uniform for a.a. $z \in Z$ (see Theorem 2 in [11]). Hypothesis $H(j)(v)$ is the double resonance condition at $x = 0$ with respect to the spectral interval $[\lambda_m, \lambda_{m+1}]$. Complete resonance is possible at both ends of the interval. In contrast, Tang-Wu [11] allow complete resonance with respect to λ_m and they assume nonuniform nonresonance with respect to λ_{m+1}. If the potential $j(z,x)$ is z-independent, then in the setting of Tang-Wu [11] the quotient $\frac{u}{x}$ stays strictly below λ_{m+1} near zero. Finally in Tang-Wu [11], $j(z,\cdot) \in C^1(\mathbb{R})$ for all $z \in Z$.

Example. The following locally Lipschitz function $j(x)$ satisfies hypotheses $H(j)$ (for simplicity, we drop the z-dependence):

$$j(x) = \begin{cases} \frac{\lambda_m}{2}x^2 & \text{if } |x| \leq 1 \\ -|x| + \frac{c}{x^2} + \frac{\lambda_m}{2} + 1 - c & \text{if } |x| > 1 \end{cases},$$

with $m \geq 0$ and $c \in \mathbb{R}$. Note that if $c = -\frac{\lambda_m + 1}{2}$, then $j \in C^1(\mathbb{R})$.

Example. The following function satisfies hypotheses $H(j)$, but not those in Theorem 2 of Tang-Wu [11]:

$$j(z,x) = \int_0^x f(z,s)\,ds, \;\; f(z,x) = \lambda_m x - \chi_C(z)\left(|x|-1\right)^+ - \lambda_m (x-1)^+$$

where, $C \subseteq Z$ is measurable with $|C|_N > 0$ and, for $u \in \mathbb{R}$, we denote $u^+ := \max\{u, 0\}$.

Let $\varphi : H^1(Z) \to \mathbb{R}$ be the Euler functional for problem (1.1) defined by

$$\varphi(x) = \frac{1}{2}\|Dx\|_2^2 - \int_Z j(z, x(z))\,dz$$

for all $x \in H_0^1(Z)$. We know that φ is Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [1], p. 83).

Proposition 3.1. *If hypotheses $H(j)$ hold, then φ is coercive.*

Proof. We argue indirectly. So, suppose that the proposition is not true. We can find $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ such that $\|x_n\| \to \infty$ and

$$\varphi(x_n) = \frac{1}{2}\|Dx_n\|_2^2 - \int_Z j(z, x_n(z))\,dz \leq M, \tag{3.1}$$

for some $M > 0$, all $n \geq 1$. We consider the orthogonal direct sum decomposition

$$H^1(Z) = E(\lambda_0) \oplus V,$$

with $E(\lambda_0) = \mathbb{R}$, $V = E(\lambda_0)^\perp$. For each $n \geq 1$ we write in an unique way

$$x_n = \overline{x}_n + \widehat{x}_n,$$

with $\overline{x}_n \in E(\lambda_0) = \mathbb{R}$ and $\widehat{x}_n \in V$.

Because of hypothesis $H(j)(iv)$ and Lemmata 2 and 3 of Tang-Wu [10], given $\varepsilon > 0$, we can find $D_\varepsilon \subseteq C$ measurable set with $|C \backslash D_\varepsilon|_N < \varepsilon$ and functions $g \in C(\mathbb{R}_+)$, $g \geq 0$, $h \in L^1(C)_+$ such that

$$g(x+y) \leq g(x) + g(y) \;\text{ for all } x,\, y \in \mathbb{R}); \tag{3.2}$$

$$g \text{ is coercive (i.e., } g(x) \to +\infty \text{ as } |x| \to \infty); \tag{3.3}$$

$$g(x) \leq 4 + |x| \;\text{ for all } x \in \mathbb{R}; \tag{3.4}$$

and

$$j(z,x) \leq h(z) - g(x) \;\text{ for a.a. } z \in D_\varepsilon \text{ and all } x \in \mathbb{R}. \tag{3.5}$$

Then, by (3.2),

$$g(\overline{x}_n) = g(x_n(z) - \widehat{x}_n(z)) \leq g(x_n(z)) + g(-\widehat{x}_n(z))$$

hence

$$g(\overline{x}_n) - g(-\widehat{x}_n(z)) \leq g(x_n(z)) \tag{3.6}$$

for all $z \in Z$ and all $n \geq 1$. Therefore, by (3.5),

$$j(z, x_n(z)) \leq h(z) - g(x_n(z)) \leq h(z) - g(\overline{x}_n) + g(-\widehat{x}_n(z)) \tag{3.7}$$

for all $n \geq 1$, a.a. $z \in D_\varepsilon$ (see (3.6)).

We return to (3.1) and use (3.7). Then in view of $H(j)(iv)$, for all $n \geq 1$,

$$\begin{aligned} \varphi(x_n) &= \frac{1}{2}\|D\widehat{x}_n\|_2^2 - \int_{D_\varepsilon} j(z, x_n(z))\,dz - \int_{Z\setminus D_\varepsilon} j(z, x_n(z))\,dz \\ &\geq \frac{1}{2}\|D\widehat{x}_n\|_2^2 + g(\overline{x}_n)|D_\varepsilon|_N - \int_{D_\varepsilon} g(-\widehat{x}_n(z))\,dz - \|h\|_1 - \|\eta\|_1 \\ &\geq \frac{1}{2}\|D\widehat{x}_n\|_2^2 + g(\overline{x}_n)|D_\varepsilon|_N - \int_Z g(-\widehat{x}_n(z))\,dz - c_1, \\ &\geq \frac{1}{2}\|D\widehat{x}_n\|_2^2 + g(\overline{x}_n)|D_\varepsilon|_N - c_2\|D\widehat{x}_n\|_2 - c_3, \end{aligned} \tag{3.8}$$

for

$$c_1 = \|h\|_1 + \|\eta\|_1, \text{ and some } c_2, c_3 > 0.$$

In the last inequality we have used (3.4) and the Poincaré-Wirtinger inequality. Since $\|x_n\| \to \infty$, by the Poincaré-Wirtinger inequality we have

$$|\overline{x}_n| \to \infty \text{ and/or } \|D\widehat{x}_n\|_2 \to \infty,$$

so from (3.8) and since g is coercive (see (3.3)), we deduce that

$$\varphi(x_n) \to \infty,$$

a contradiction to the fact that

$$\varphi(x_n) \leq M \text{ for all } n \geq 1.$$

This proves the coercivity of all φ. □

Corollary 3.2. *If hypotheses $H(j)$ hold, then φ is bounded below and satisifies the PS-condition.*

Proof. Because φ is coercive (see Proposition 3.1), it is bounded below. Also, let $\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ be a sequence such that

$$|\varphi(x_n)| \leq \widehat{M} \text{ for some } \widehat{M} > 0, \text{ all } n \geq 1 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty \tag{3.9}$$

Since $\partial\varphi(x_n) \subseteq H^1(Z)^*$ is w-compact and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$.

We denote by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair $\left(H^1(Z)^*, H^1(Z)\right)$ and let $A \in \mathcal{L}\left(H^1(Z), H^1(Z)^*\right)$ be the operator defined by

$$\langle A(x), y\rangle = \int_Z (Dx, Dy)_{\mathbb{R}^N}\,dz \text{ for all } x, y \in H^1(Z).$$

We know that

$$x_n^* = A(x_n) - u_n,$$

with $u_n \in L^{r'}(Z)$ ($\frac{1}{r} + \frac{1}{r'} = 1$), $u_n(z) \in \partial j(z, x_n(z))$ a.e. on Z (see Clarke [1] and Gasinski-Papageorgiou [2]). Because of (3.9) and Proposition 3.1, we deduce that

$\{x_n\}_{n\geq 1} \subseteq H^1(Z)$ is bounded. Therefore, by passing to a subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } H^1(Z) \text{ and } x_n \to x \text{ in } L^2(Z) \text{ as } n \to \infty.$$

From (3.9), we have

$$|\langle x_n^*, x_n - x\rangle| \leq \varepsilon_n \|x_n - x\| \text{ with } \varepsilon_n \downarrow 0,$$

hence

$$\left|\langle A(x_n), x_n - x\rangle - \int_Z u_n (x_n - x)\, dz\right| \leq \varepsilon_n \|x_n - x\|. \tag{3.10}$$

Clearly

$$\int_Z u_n (x_n - x)\, dz \to 0 \text{ as } n \to \infty.$$

So, from (3.10), it follows that

$$\langle A(x_n), x_n - x\rangle \to 0 \text{ as } n \to \infty. \tag{3.11}$$

Note that $A(x_n) \xrightarrow{w} A(x)$ in $H^1(Z)^*$. So, from (3.11), we have

$$\|Dx_n\|_2^2 = \langle A(x_n), x_n\rangle \to \langle A(x), x\rangle = \|Dx\|_2^2.$$

Since

$$Dx_n \xrightarrow{w} Dx \text{ in } L^2\left(Z, \mathbb{R}^N\right)$$

from the Kadec-Klee property of Hilbert spaces, we infer that

$$Dx_n \to Dx \text{ in } L^2\left(Z, \mathbb{R}^N\right)$$

hence

$$x_n \to x \text{ in } H^1(Z).$$

Therefore φ satisfies the PS-condition. □

Now, we are ready for the multiplicity result.

Theorem 3.3. *If hypotheses $H(j)$ hold, then problem* (1.1) *has at least two nontrivial solutions x_0, $y_0 \in C^1\left(\overline{Z}\right)$.*

Proof. By virtue of hypothesis $H(j)(v)$, we have

$$\lambda_m x \leq u \leq \lambda_{m+1} x \text{ for a.a. } z \in Z, \text{ all } x \in (0, \delta], \text{ all } u \in \partial j(z, x) \tag{3.12}$$

and

$$\lambda_{m+1} x \leq u \leq \lambda_m x \text{ for a.a. } z \in Z, \text{ all } x \in [-\delta, 0), \text{ all } u \in \partial j(z, x). \tag{3.13}$$

From hypotheses $H(j)(i)$, (ii) and Rademacher's theorem, we know that for all $z \in Z\backslash D$, with $|D|_N = 0$, the function $r \to j(z, r)$ is differentiable a.e. on $\mathbb{R}$ and at a point of differentiability, we have

$$\frac{d}{dr} j(z, r) \in \partial j(z, r)$$

(see Clarke [1]). So, from (3.12) and (3.13) we have

$$\lambda_m r \leq \frac{d}{dr} j(z, r) \leq \lambda_{m+1} r \text{ for a.a. } z \in Z\backslash D \text{ and a.a. } r \in (0, \delta] \tag{3.14}$$

and

$$\lambda_{m+1} r \leq \frac{d}{dr} j(z,r) \leq \lambda_m r \text{ for a.a. } z \in Z \backslash D \text{ and a.a. } r \in [-\delta, 0). \tag{3.15}$$

Integrating (3.14) and (3.15), we obtain

$$\frac{1}{2}\lambda_m x^2 \leq j(z,x) \leq \frac{1}{2}\lambda_{m+1} x^2 \text{ for a.a. } z \in Z \text{ and all } |x| \leq \delta. \tag{3.16}$$

We consider the orthogonal direct sum decomposition

$$H^1(Z) = Y \oplus V,$$

with

$$Y = \overset{m}{\underset{k=0}{\oplus}} E(\lambda_k) \text{ and } V = \overline{\underset{k \geq m+1}{\oplus} E(\lambda_k)} = Y^{\perp}.$$

Since Y is finite dimensional, all norms are equivalent and, because $Y \subseteq C(\overline{Z})$, we can find $c_4 > 0$ such that

$$\|y\|_\infty \leq c_4 \|y\| \text{ for all } y \in Y. \tag{3.17}$$

Therefore, if $y \in Y$ satisfies $\|y\| \leq \frac{\delta}{c_4}$ with $\delta > 0$, as in hypothesis $H(j)(v)$, then from (3.17) we have

$$|y(z)| \leq \delta \text{ for all } z \in \overline{Z}.$$

Hence (3.16) implies

$$\frac{\lambda_m}{2} y(z)^2 \leq j(z, y(z)) \leq \frac{\lambda_{m+1}}{2} y(z)^2 \text{ a.e. on } Z. \tag{3.18}$$

Thus, for $y \in Y$ with $\|y\| \leq \frac{\delta}{c_4}$, we have

$$\varphi(y) = \frac{1}{2}\|Dy\|_2^2 - \int_Z j(z, y(z))\,dz \leq \frac{1}{2}\|Dy\|_2^2 - \frac{\lambda_m}{2}\|y\|_2^2 \leq 0. \tag{3.19}$$

(see (2.2)). On the other hand, by virtue of hypothesis $H(j)(iii)$, we can find $c_5 > 0$ such that

$$|u| \leq c_5 |x|^{r-1} \text{ for a.a. } z \in Z, \text{all } |x| > \delta \text{ and all } u \in \partial j(z,x). \tag{3.20}$$

Moreover, without any loss of generality, we can always assume $2 < r < 2^*$. Then, as above, from (3.20) and Rademacher's theorem, after integration we obtain

$$j(z,x) \leq c_6 |x|^r \text{ for a.a. } z \in Z, \text{all } |x| > \delta \text{ and some } c_6 > 0. \tag{3.21}$$

Let $v \in V$. We have

$$v = u + w, \text{ with } u \in E(\lambda_{m+1}) \text{ and } w \in W = \overline{\underset{k \geq m+2}{\oplus} E(\lambda_k)}.$$

Let

$$Z_\delta = \{z \in Z : |v(z)| > \delta\}.$$

Then, for $z \in Z_\delta$, we have (since $u \in C^1(\overline{Z})$)

$$\begin{aligned} |w(z)| &= |v(z) - u(z)| \geq |v(z)| - |u(z)| \\ &\geq |v(z)| - \|u\|_\infty \geq |v(z)| - c_7 \|u\|, \text{ for some } c_7 > 0) \end{aligned} \tag{3.22}$$

(since all norms are equivalent on the finite dimensional eigenspace $E(\lambda_{m+1}) \subseteq C^1(\overline{Z})$).

Suppose that $\|v\| \leq \frac{\delta}{2c_7}$. If by p_{m+1} we denote the orthogonal projection operator onto the eigenspace $E(\lambda_{m+1})$, we have

$$\|u\| = \|p_{m+1}(v)\| \leq \|v\| \leq \frac{\delta}{2c_7}. \tag{3.23}$$

From (3.22) and (3.23), we have

$$|w(z)| \geq |v(z)| - \frac{\delta}{2} \geq |v(z)| - \frac{1}{2}|v(z)| = \frac{1}{2}|v(z)|. \tag{3.24}$$

Now, for $v \in V$, with $\|v\| \leq \frac{\delta}{2c_7}$, we have

$$\begin{aligned} \varphi(v) &= \frac{1}{2}\|Dv\|_2^2 - \int_Z j(z, v(z))\,dz \\ &= \frac{1}{2}\|Dv\|_2^2 - \int_{Z_\delta} j(z, v(z))\,dz - \int_{Z\setminus Z_\delta} j(z, v(z))\,dz. \end{aligned} \tag{3.25}$$

Note that

$$\int_{Z\setminus Z_\delta} j(z, v(z))\,dz = \int_{\{|v(z)|\leq\delta\}} j(z, v(z))\,dz \leq \frac{\lambda_{m+1}}{2}\|v\|_2^2 \tag{3.26}$$

(see (3.16)). Also, in view of (3.21) and (3.24) and for $c_8 := 2^r c_6$ we have

$$\begin{aligned} \int_{Z_\delta} j(z, v(z))\,dz &\leq c_6 \int_{Z_\delta} |v(z)|^r\,dz \ \leq c_8 \int_{Z_\delta} |w(z)|^r\,dz, \\ &\leq c_8\|w\|_r^r \leq c_9\|w\|^r, \text{ for some } c_9 > 0 \end{aligned} \tag{3.27}$$

(since $H^1(Z)$ is embedded continuously in $L^r(Z)$).Using (3.26) and (3.27) in (3.25), we obtain

$$\varphi(v) \geq \frac{1}{2}\|Dv\|_2^2 - \frac{\lambda_{m+1}}{2}\|v\|_2^2 - c_9\|w\|^r.$$

Exploiting the orthogonality of the component spaces in the decomposition $V = E(\lambda_{m+1}) \oplus W$ and since

$$\|Du\|_2^2 = \lambda_{m+1}\|u\|_2^2, \text{ for } u \in E(\lambda_{m+1}),$$

we have

$$\varphi(v) \geq \frac{1}{2}\|Dw\|_2^2 - \frac{\lambda_{m+1}}{2}\|w\|_2^2 - c_9\|w\|^r \geq c_{10}\|w\|^2 - c_9\|w\|^r, \tag{3.28}$$

for some $c_{10} > 0$.

Since $2 < r$, from (3.28) and if

$$\|w\| \leq \|v\| \leq r \leq \min\left\{\frac{\delta}{2c_7}, \frac{\delta}{c_4}\right\}$$

with $r > 0$ small enough, we have

$$\varphi(v) \geq 0, \text{ for all } v \in V,\ \|v\| \leq r. \tag{3.29}$$

If

$$\inf \varphi = 0 = \varphi(0),$$

then, from (3.19), we see that all $y \in Y$, with $0 < \|y\| \leq \frac{\delta}{c_4}$, are minimizers of φ, hence critical points of φ. Using Green's identity, we check that the critical points of φ are solutions of (1.1) and regularity theory implies that they belong in $C^1\left(\overline{Z}\right)$.

If $\inf \varphi < 0$, then we can apply the Theorem 2.1 and obtain two nontrivial critical points x_0, $y_0 \in H^1(Z)$ of φ. Again, using Green's identity (see, for example, Gasinski-Papageorgiou [3], p.209), we verify that both x_0, y_0 are solutions of (1.1) and from regularity theory, we have x_0, $y_0 \in C^1\left(\overline{Z}\right)$. $\square$

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