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NEW FOUNDATIONS OF THE KKM THEORY

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Abstract. A KKM space is an abstract convex space satisfying an abstract form of the KKM theorem and its 'open' version. We give several characterizations of KKM spaces as abstract convex spaces satisfying one of the properties of matching, intersection, geometric or section, Fan-Browder type fixed point, or existence of maximal elements. We deduce fundamental results on KKM spaces; for example, several whole intersection properties, analytic alternatives, minimax inequalities, variational inequalities, etc. These results are all abstract versions of known corresponding ones for convex subsets of topological vector spaces, convex spaces due to Lassonde, C-spaces due to Horvath, G-convex spaces due to the author, and their variations. Some earlier applications of those results are indicated. Moreover, it is noted that many of the results are mutually equivalent.

1. Introduction

It is well-known that the Brouwer fixed point theorem has numerous equivalent formulations and applications in various fields of mathematics such as topology, nonlinear analysis, various equilibria theory, mathematical economics, game theory, and others. One of the equivalent forms is the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) [19], which was deduced from the Sperner lemma [48, 54]. The KKM theorem provides the foundations for many of the modern essential results in diverse areas of mathematical sciences; see [27, 55].

The KKM theory, first called by the author [24], is the study on equivalent formulations of the KKM theorem and their applications. At the beginning, the theory was mainly concerned with convex subsets of topological vector spaces as in the works of Fan [7-13]. Later, it has been extended to convex spaces by Lassonde [20], and to spaces having certain families of contractible subsets (simply, C-spaces or H-spaces) by Horvath [14-17]. This line of generalizations of earlier works is followed by the author for generalized convex spaces or G-convex spaces; see [27-29, 32, 38-43]. Moreover, there have appeared several variations of such spaces; see [34].

Recently, in [30, 31, 33-36], the author introduced the concepts of abstract convex spaces and KKM spaces, which seem to be more adequate to establish the KKM theory for various purposes. A KKM space is an abstract convex space satisfying an abstract form of the original KKM theorem and its "open" version. In fact, our new concept of KKM spaces is a common generalization of many of known abstract convexities without any linear structure developed in connection mainly with the fixed point theory and the KKM theory.

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In the present paper, we introduce several characterizations of KKM spaces as abstract convex spaces satisfying one of the properties of KKM, matching, intersection, geometric or section, Fan-Browder type fixed point, maximal element, and others. We also introduce fundamental results in the KKM theory for KKM spaces. Some of them characterize abstract convex spaces satisfying an abstract form of the KKM theorem, and some of their particular forms are equivalent to the Brouwer theorem, the Sperner lemma, and the KKM theorem. Our new results are all abstract versions of known corresponding ones for convex subsets of topological vector spaces mainly due to Fan, convex spaces due to Lassonde, C-spaces due to Horvath, G-convex spaces due to the author, and their variations. Some earlier applications of such characterizations are indicated in each section. It is also noted that many of the results are mutually equivalent.

In Sections 3–7, we obtain several characterizations of KKM spaces as abstract convex spaces satisfying one of the properties of matching, intersection, geometric or section, Fan-Browder type fixed point, or existence of maximal elements. Sections 8–11 deal with fundamental results on abstract convex spaces satisfying an abstract form of the KKM theorem, for example, the KKM type whole intersection properties, analytic alternatives, minimax inequalities, variational inequalities, etc. These results are all abstract versions of known corresponding ones for various particular types of abstract convex spaces.

2. The KKM spaces

In this paper, multimaps are simply called maps.

Let $\langle D\rangle$ denote the set of all nonempty finite subsets of a set D.

Definition 2.1. An *abstract convex space* $(E, D; \Gamma)$ consists of nonempty sets E, D, and a map $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$.

Let $(E, D; \Gamma)$ be an abstract convex space. For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. This means that $(X, D'; \Gamma|_{\langle D'\rangle})$ itself is an abstract convex space called a *subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

Many examples of abstract convex spaces were given in [30,33,34]. One of the typical examples is the following:

Definition 2.2. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D\rangle \multimap X$ such that for each $A \in \langle D\rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A\rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

We have established a large number of literature on G-convex spaces; see [27-29, 32, 38-43] and references therein.

Example 2.3. The following are typical examples of G-convex spaces:

1. Any nonempty convex subset of a t.v.s.
2. A convex space due to Lassonde [20].
3. A C-space (or an H-space) due to Horvath [14-17]. Hyperconvex metric spaces are particular C-spaces.
4. Hyperbolic spaces due to Reich and Shafrir [44].
5. An L-space due to Ben-El-Mechaiekh et al. The so-called FC-spaces are L-spaces; see [34].
6. Other major examples of G-convex spaces can be seen in [32, 38, 41].
7. A ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$ consisting of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D\rangle$ with $|A| = n + 1$ and $n \in \mathbb{N} \cup \{0\}$, can be made into a G-convex space [34].

Example 2.4. The following are typical examples of abstract convex spaces:

1. A convexity space $(E, \mathcal{C})$ in the classical sense; see [47], where the bibliography lists 283 papers.
2. A generalized convex space.
3. According to Horvath [18], a convexity on a set X is an algebraic closure operator $A \mapsto [[A]]$ from $\mathcal{P}(X)$ to $\mathcal{P}(X)$ such that $[[\{x\}]] = \{x\}$ for all $x \in X$, or equivalently, a family $\mathcal{C}$ of subsets of X, the convex sets, which contains the whole space and the empty set as well as singletons and which is closed under arbitrary intersections and updirected unions.

Definition 2.5. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y\in A} G(y) \qquad \text{for all} \;\; A \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

There are a large number of examples of KKM maps. The following is a new one:

Example 2.6. For a ϕ_A-space $(X, D; \{\phi_A\}_{A\in\langle D\rangle})$, any map $T : D \multimap X$ satisfying

$$\phi_A(\Delta_J) \subset T(J) \;\; \text{for each} \;\; A \in \langle D\rangle \text{ and } J \in \langle A\rangle$$

is a KKM map on a G-convex space $(X, D; \Gamma)$.

The following KKM theorem for G-convex spaces and its proof are just simple modification of the one in [28, 29]:

Theorem 2.7. *Let $(X, D; \Gamma)$ be a G-convex space and $G : D \multimap X$ a multimap such that*

(2.7.1) *G has closed [resp., open] values; and*
(2.7.2) *G is a KKM map.*

Then $\{G(z)\}_{z\in D}$ *has the finite intersection property.* (*More precisely, for each* $N \in \langle D\rangle$ *with* $|N| = n+1$, *we have* $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.)

Further, if

(2.7.3) $\bigcap_{z\in M} \overline{G(z)}$ *is compact for some* $M \in \langle D\rangle$,

then we have $\bigcap_{z\in D} \overline{G(z)} \neq \emptyset$.

Proof. Let $N = \{z_0, z_1, \dots, z_n\}$. Since G is a KKM map, for each vertex e_i of Δ_n, we have $\phi_N(e_i) \in \Gamma(\{z_i\}) \subset G(z_i)$ for $0 \le i \le n$. Then $e_i \mapsto \phi_N^{-1} G(z_i)$ is a closed [resp., open] valued map such that $\Delta_k = \mathrm{co}\{e_{i_0}, e_{i_1}, \dots, e_{i_k}\} \subset \bigcup_{j=0}^{k} \phi_N^{-1} G(z_{i_j})$ for each face Δ_k of Δ_n. Therefore, by the original KKM theorem and its 'open' version [45], we have $\Delta_n \supset \bigcap_{i=0}^{n} \phi_N^{-1} G(z_i) \neq \emptyset$ and hence $\phi_N(\Delta_n) \cap \bigcap_{z\in N} G(z) \neq \emptyset$.

The second part is clear. □

Remarks. (1) Instead of (2.7.1), we may assume that, for each $a \in D$ and $N \in \langle D\rangle$, $G(a) \cap \phi_N(\Delta_n)$ is closed [resp., open] in $\phi_N(\Delta_n)$. This is said by some authors that G has finitely closed [resp., open] values. This generalizes nothing; see [28].

(2) For $X = \Delta_n$, if D is the set of vertices of Δ_n and $\Gamma = \mathrm{co}$, the convex hull operation, Theorem 2.7 reduces to the celebrated KKM theorem [19] and its open version [37, 45]. The theorem was first used in [19] to obtain one of the most direct proofs of the Brouwer fixed point theorem, and later applied to topological results on Euclidean spaces in [1,2]; see [37].

(3) If D is a nonempty subset of a topological vector space X (not necessarily Hausdorff), Theorem 2.7 extends Fan's KKM lemma [7]. Fan applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space. Later, Fan [8] also applied his lemma to an intersection theorem (concerning sets with convex sections) which implies the Sion minimax theorem and the Tychonoff fixed point theorem.

(4) For another forms of the KKM theorem for various G-convex spaces and their applications, see [21-30, 38-43, 46, 48-50, 54].

Definition 2.8. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. A map $F : E \multimap Z$ is said to have the *KKM property* and called *a* $\mathfrak{K}$*-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E, Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E, Z) \subset \mathfrak{KC}(E, Z) \cap \mathfrak{KO}(E, Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors use the notation $\mathrm{KKM}(E, Z)$ instead of $\mathfrak{KC}(E, Z)$.

Definition 2.9. The *partial KKM principle* for an abstract convex topological space $(E, D; \Gamma)$ is the statement that, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property; that is, $1_E \in \mathfrak{KC}(E, E)$.

The *KKM principle* is the statement that the same property also holds for any open-valued KKM maps.

An abstract convex topological space $(E, D; \Gamma)$ is called a *KKM space* if it satisfies the KKM principle $1_E \in \mathfrak{KC}(E, E) \cap \mathfrak{KO}(E, E)$.

In our recent work [33], we studied elements or foundations of the KKM theory on abstract convex spaces and noticed there that many important results therein are related to KKM spaces.

Example 2.10. We give examples of KKM spaces:

1. Every G-convex space is a KKM space; see Theorem 2.7.
2. A connected linearly ordered space $(X, \leq)$ can be made into an abstract convex topological space $(X \supset D; \Gamma)$ for any nonempty $D \subset X$ by defining $\Gamma_A := [\min A,\ \max A] := \{x \in X \mid \min A \leq x \leq \max A\}$ for each $A \in \langle D \rangle$. Further, it is a KKM space; see [31, Theorem 5(i)].
3. The extended long line L^* can be made into a KKM space $(L^* \supset D; \Gamma)$; see [31]. In fact, L^* is constructed from the ordinal space $D := [0, \Omega]$ consisting of all ordinal numbers less than or equal to the first uncountable ordinal Ω, together with the order topology. Recall that L^* is a generalized arc obtained from $[0, \Omega]$ by placing a copy of the interval $(0, 1)$ between each ordinal α and its successor $\alpha + 1$ and we give L^* the order topology. Now let $\Gamma : \langle D \rangle \multimap L^*$ be the one as in 2.

   But L^* is not a G-convex space. In fact, since $\Gamma\{0, \Omega\} = L^*$ is not path connected, for $A := \{0, \Omega\} \in \langle L^* \rangle$ and $\Delta_1 := [0, 1]$, there does not exist a continuous function $\phi_A : [0, 1] \to \Gamma_A$ such that $\phi_A\{0\} \subset \Gamma\{0\} = \{0\}$ and $\phi_A\{1\} \subset \Gamma\{\Omega\} = \{\Omega\}$. Therefore $(L^* \supset D; \Gamma)$ is not G-convex.
4. For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the corresponding abstract convex space $(X; \Gamma)$ is a KKM space, where $\Gamma_A := [[A]] = \bigcap\{C \in \mathcal{C} \mid A \subset C\}$ is metrizable for each $A \in \langle X \rangle$; see [18, Proposition 5.1].

Example 2.11. We give examples of abstract convex spaces satisfying the partial KKM principle:

1. All KKM spaces.
2. For Horvath's convex space $(X, \mathcal{C})$ with the weak Van de Vel property, the $(X; \Gamma)$ is a partial KKM space, where $\Gamma_A := [[A]]$ for each $A \in \langle X \rangle$; see [18, Theorem 5.1].

Now we have the following diagram for triples $(E, D; \Gamma)$:

$$\begin{gathered}\text{Simplex} \Longrightarrow \text{Convex subset of a t.v.s.} \Longrightarrow \text{Lassonde type convex space} \\ \Longrightarrow H\text{-space} \Longrightarrow G\text{-convex space} \Longleftrightarrow \phi_A\text{-space} \Longrightarrow \text{KKM space} \\ \Longrightarrow \text{Space satisfying the partial KKM principle} \Longrightarrow \text{Abstract convex space.}\end{gathered}$$

It is not known yet whether there is a space satisfying the partial KKM principle which is not a KKM space.

3. Matching property

From now on, a triple $(X, D; \Gamma)$ denotes an abstract convex space such that X is a topological space unless explicitly stated otherwise. Recall that $(X, D; \Gamma)$ is a KKM space iff for any KKM map $G : D \multimap X$ with closed [resp., open] values, $\{G(z)\}_{z\in D}$ has the finite intersection property.

The (partial) KKM principle is equivalent to the Fan type matching property:

Theorem 3.1. *An abstract convex space $(X, D; \Gamma)$ satisfies the partial KKM principle iff for any map $S : D \multimap X$ satisfying*

(3.1.1) *$S(z)$ is open for each $z \in D$; and*

(3.1.2) *$X = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$,*

there exists an $N \in \langle D\rangle$ such that

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

An abstract convex space $(X, D; \Gamma)$ is a KKM space iff the above condition also holds for closed-valued map S.

Proof. (Necessity) Let $G : D \multimap X$ be a map given by $G(z) := X \setminus S(z)$ for $z \in D$. Then G has closed [resp., open] values. Suppose, on the contrary to the conclusion, that for any $N \in \langle D\rangle$, we have $\Gamma_N \cap \bigcap_{z\in N} S(z) = \emptyset$; that is, $\Gamma_N \subset X \setminus \bigcap_{z\in N} S(z) = \bigcup_{z\in N}(X \setminus S(z)) = G(N)$. Therefore G is a KKM map. Since $(X, D; \Gamma)$ satisfies the (partial) KKM principle, there exists a $\widehat{y} \in \bigcap_{z\in N} G(z) = \bigcap_{z\in N}(X \setminus S(z))$. Hence $\widehat{y} \notin S(z)$ for all $z \in N$. This violates condition (3.1.2).

(Sufficiency) Let $G : D \multimap X$ be a KKM map with closed [resp., open] values. Suppose $\bigcap_{z\in M} G(z) = \emptyset$ for some $M \in \langle D\rangle$. Then $\bigcup_{z\in M} G^c(z) = \bigcup_{z\in M}(X\setminus G(z)) = X$. Therefore, by the sufficiency assumption, there exists an $N \in \langle D\rangle$ such that $\Gamma_N \cap \bigcap_{z\in N} G^c(z) \neq \emptyset$. Since G is a KKM map, $\Gamma_N \subset G(N)$. Therefore, we have a contradiction $G(N) \cap (G(N))^c \neq \emptyset$.

This completes our proof. □

Corollary 3.2. *Let $(X, D; \Gamma)$ satisfy the partial KKM principle and $S : D \multimap X$ be a map such that*

(1) *$S(z)$ is open for each $z \in D$;*

(2) *$S^-(y)$ is nonempty for each $y \in X$ (that is, S is surjective); and*

(3) *$X\setminus S(z_0)$ is compact for some $z_0 \in X$.*

Then there exists an $N \in \langle D\rangle$ such that

$$\Gamma_N \cap \bigcap_{z\in N} S(z) \neq \emptyset.$$

Proof. Note that (2) and (3) imply (3.1.2). □

Remark. The origin of Corollary 3.2 goes back to Fan [12, 13] for a convex set $X = D$. For applications, see also [21-23].

4. Another intersection property

The (partial) KKM principle is equivalent to another intersection property:

Theorem 4.1. *An abstract convex space $(X, D; \Gamma)$ satisfies the partial KKM principle iff for any maps $S : D \multimap X$, $T : X \multimap X$ satisfying*

(4.1.1) *S has closed values;*

(4.1.2) *for each* $x \in X$, $\mathrm{co}_\Gamma(D \setminus S^-(x)) \subset X \setminus T^-(x)$*; and*
(4.1.3) $x \in T(x)$ *for each* $x \in X$,

$\{S(z)\}_{z\in D}$ *has the finite intersection property.*

An abstract convex space $(X, D; \Gamma)$ *is a KKM space iff the above condition also holds for any open-valued map* S.

Proof. (Necessity) By the definition of the partial KKM principle, it suffices to show that S is a KKM map. Otherwise, there exists a subset $N \in \langle D\rangle$ such that $\Gamma_N \not\subset S(N)$; that is, there exists an $x \in \Gamma_N$ such that $x \notin S(z)$ for all $z \in N$. Hence, $N \in \langle D \setminus S^-(x)\rangle$ and, by (4.1.2), we have $\Gamma_N \subset X \setminus T^-(x)$. Therefore, $x \in X \setminus T^-(x)$ or $x \notin T^-(x)$, which contradicts (4.1.3).

(Sufficiency) For any KKM map $S : D \multimap X$ with closed [resp., open] values, we have to show $\{S(z)\}_{z\in D}$ has the finite intersection property. Consider the particular case where $\Gamma_N := \mathrm{co}_\Gamma N$ for all $N \in \langle D\rangle$. Let $T : X \multimap X$ be defined by

$$X \setminus T^-(x) := \mathrm{co}_\Gamma(D \setminus S^-(x)) \quad \text{for } x \in X.$$

Then clearly (4.1.2) holds. We claim that (4.1.3) holds. Suppose, on the contrary, that $x \notin T(x)$ for some $x \in X$. Then $x \notin T^-(x)$ and hence $x \in X \setminus T^-(x)$. This implies $x \in \Gamma_N$ for some $N \in \langle D \setminus S^-(x)\rangle$ by the definition of T. Then, for all $z \in N$, we have $z \in D \setminus S^-(x) \iff z \notin S^-(x) \iff x \notin S(z)$ and hence $x \notin S(N)$. Therefore $\Gamma_N \not\subset S(N)$, which contradicts that S is a KKM map. Now all of the requirements of Theorem 4.1 are satisfied, and hence $\{S(z)\}_{z\in D}$ has the finite intersection property. □

From Theorem 4.1 or Corollary 3.2, we immediately have another whole intersection property:

Corollary 4.2. *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle and* $S : D \multimap X$, $T : X \multimap X$ *be maps such that*
(1) S *has closed values;*
(2) *for each* $x \in X$, $\mathrm{co}_\Gamma(D \setminus S^-(x)) \subset X \setminus T^-(x)$*;*
(3) $x \in T(x)$ *for each* $x \in X$*; and*
(4) $\bigcap_{z\in M} S(z)$ *is compact for some* $M \in \langle D\rangle$.

Then

$$\bigcap_{z\in D} S(z) \neq \emptyset.$$

Remark. The first particular form of Corollary 4.2 is due to Tarafdar [52] for a convex space $X = D$. Another forms of Corollary 2.1 also appear in [14, 25] and others.

5. Geometric or section properties

In this section, we show that the (partial) KKM principle is equivalent to two geometric forms. The following is usually called the section property:

Theorem 5.1. *An abstract convex space* $(X, D; \Gamma)$ *satisfies the partial KKM principle iff for any sets* $A \subset D \times X$, $B \subset X \times X$ *satisfying*
(5.1.1) $\{y \in X \mid (z, y) \in A\}$ *is closed for each* $z \in D$*;*

(5.1.2) *for each* $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid (z,y) \notin A\} \subset \{x \in X \mid (x,y) \notin B\}$*; and*
(5.1.3) $(x,x) \in B$ *for each* $x \in X$,

and for each $N \in \langle D \rangle$*, there exists an* $x_0 \in X$ *such that* $N \times \{x_0\} \subset A$.

An abstract convex space $(X, D; \Gamma)$ *is a KKM space iff the above condition also holds for any set* $A \subset D \times X$ *satisfying*

(5.1.1)′ $\{y \in X \mid (z,y) \in A\}$ *is open for each* $z \in D$

instead of (5.1.1).

Proof of Theorem 5.1 using Theorem 4.1. For each $z \in D$, let $S(z) := \{y \in X \mid (z,y) \in A\}$. Then (5.1.1) $\Longrightarrow$ (4.1.1). Moreover, for each $x \in X$, let $T(x) := \{y \in X \mid (x,y) \in B\}$. Then (5.1.2) $\Longrightarrow$ (4.1.2). Further (5.1.3) $\Longrightarrow$ (4.1.3). Therefore, by Theorem 4.1, for each $N \in \langle D \rangle$, we have

$$\bigcap_{z \in N} S(z) = \bigcap_{z \in N} \{y \in X \mid (z,y) \in A\} \neq \emptyset.$$

Hence there exists an $x_0 \in X$ such that $(z, x_0) \in A$ for all $z \in N$; that is, $N \times \{x_0\} \subset A$. □

Proof of Theorem 4.1 using Theorem 5.1. Let $A :=\mathrm{Gr}(S)$ and $B :=\mathrm{Gr}(T)$. Then (4.1.1)-(4.1.3) $\Longrightarrow$ (5.1.1)-(5.1.3). Therefore, by Theorem 5.1, for each $N \in \langle D \rangle$, there exists an $x_0 \in X$ such that $N \times \{x_0\} \subset A$. Hence $\{S(z)\}_{z \in D}$ has the finite intersection property. □

Corollary 5.2. *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle and* $A \subset D \times X$, $B \subset X \times X$ *be two sets satisfying* (5.1.1), (5.1.2), *and* (5.1.3). *Further if*

(5.1.4) $\{y \in X \mid (z_0, y) \in A\}$ *is compact for some* $z_0 \in D$,

then there exists an $x_0 \in X$ *such that* $D \times \{x_0\} \subset A$.

Proof of Corollary 5.2 using Corollary 4.2. As in the proof of Theorem 6.1, define maps S and T. Then (5.1.1)-(5.1.4) imply (1)-(4) in Corollary 4.2. Therefore, by Corollary 4.2, we have

$$\bigcap_{z \in D} S(z) = \bigcap_{z \in D} \{y \in X \mid (z,y) \in A\} \neq \emptyset.$$

Hence there exists an $x_0 \in X$ such that $(z, x_0) \in A$ for all $z \in D$; that is, $D \times \{x_0\} \subset A$. □

Remark. If $X = D$ is a convex subset of a topological vector space and if $A = B$, Corollary 5.2 reduces to Fan's 1961 Lemma [7, Lemma 4]. He obtained his result from his own generalization of the KKM theorem and applied it to a direct proof of the Tychonoff fixed point theorem. Other interesting applications of his useful lemma to fixed points, minimax theorems, equilibrium points, extension of monotone sets, potential theory, etc. have been made by Fan [9] and others; see [27].

The following shows that some geometric property is equivalent to the (partial) KKM principle:

Theorem 5.3. *An abstract convex space* $(X, D; \Gamma)$ *satisfies the partial KKM principle iff for any sets* $A \subset D \times X$, $B \subset X \times X$ *satisfying*

(5.3.1) *$\{y \in X \mid (z,y) \in A\}$ is open for each $z \in D$;*
(5.3.2) *for each $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid (z,y) \in A\} \subset \{x \in X \mid (x,y) \in B\}$; and*
(5.3.3) *there exists an $M \in \langle D\rangle$ such that for any $x \in X$, $(z,x) \in A$ for some $z \in M$,*

there exists an $x_0 \in X$ such that $(x_0,x_0) \in A$.

An abstract convex space $(X,D;\Gamma)$ is a KKM space iff the above condition also holds when the set in (5.3.1) *is closed.*

Proof of Theorem 5.3 using Theorem 5.1. Consider Theorem 5.1 replacing (A,B) by their respective complements (A^c,B^c). Then (5.1.1) and (5.1.2) are satisfied by (5.3.1) and (5.3.2). Since (5.3.3) is the negation of the conclusion of Theorem 5.1, we should have the negation of (5.1.3). Therefore, the conclusion follows. □

Proof of Theorem 5.1 using Theorem 5.3. Similar. □

Corollary 5.4. *Let $(X,D;\Gamma)$ satisfy the partial KKM principle and $A \subset D \times X$, $B \subset X \times X$ be two sets satisfying* (5.3.1) *and* (5.3.2). *Further if*

(1) *for each $y \in X$, there exists a $z \in D$ such that $(z,y) \in A$; and*
(2) *$\{y \in X \mid (z_0,y) \notin A\}$ is compact for some $z_0 \in D$,*

then there exists an $x_0 \in X$ such that $(x_0,x_0) \in B$.

Proof of Corollary 5.4 using Corollary 5.2. Consider Corollary 5.2 replacing (A,B) by their respective complements (A^c,B^c). Then (5.1.1) and (5.1.2) are satisfied by (5.3.1) and (5.3.2). Moreover, (2) implies (5.1.4). Since (1) is the negation of the conclusion of Corollary 5.2, we should have the negation of (5.1.3). Therefore, the conclusion follows. □

Proof of Corollary 5.2 using Corollary 5.4. Similar. □

Remark. If $X = D$ is a convex subset of a topological vector space and if $A = B$, Corollary 5.4 reduces to Fan [11, Theorem 2]. In this case, (5.3.2) merely tells that $\{x \in X \mid (x,y) \in A\}$ is convex.

6. The Fan-Browder type fixed point theorems

The (partial) KKM principle is equivalent to the Fan-Browder type fixed point theorem:

Theorem 6.1. *An abstract convex space $(X,D;\Gamma)$ satisfies the partial KKM principle iff for any maps $S : D \multimap X$, $T : X \multimap X$ satisfying*

(6.1.1) *$S(z)$ is open for each $z \in D$;*
(6.1.2) *for each $y \in X$, $\mathrm{co}_\Gamma S^-(y) \subset T^-(y)$; and*
(6.1.3) *$X = \bigcup_{z\in M} S(z)$ for some $M \in \langle D\rangle$,*

T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.

An abstract convex space $(X,D;\Gamma)$ is a KKM space iff the above condition also holds for any map $S : D \multimap X$ satisfying

(6.1.1)′ *$S(z)$ is closed for each $z \in D$*

instead of (6.1.1).

Proof of Theorem 6.1 using Theorem 5.3. Let A and B be the graphs of S and T, respectively. Then (6.1.1) - (6.1.3) imply (5.3.1) - (5.3.3). Therefore, by Theorem 5.3, there exists an $x_0 \in X$ such that $(x_0, x_0) \in B$, that is, T has a fixed point $x_0 \in X$. □

Proof of Theorem 5.3 using Theorem 6.1. Define $S(z) := \{y \in X \mid (z, y) \in A\}$ and $T(x) := \{y \in X \mid (x, y) \in B\}$. Apply Theorem 6.1. □

Corollary 6.2. *Let $(X, D; \Gamma)$ satisfy the partial KKM principle and $S : D \multimap X$, $T : X \multimap X$ be two maps satisfying* (6.1.1) *and* (6.1.2). *If*

(1) *for each $y \in X$, $S^-(y) \neq \emptyset$; and*
(2) *$X \setminus S(z_0)$ is compact for some $z_0 \in D$,*

then T has a fixed point $x_0 \in X$; that is $x_0 \in T(x_0)$.

Proof of Corollary 6.2 using Corollary 5.4. Let A and B be the graphs of S and T, respectively. Then (6.1.1), (6.1.2) and (1), (2) imply (5.3.1) - (5.3.4). Then, by Corollary 5.4, there exists an $x_0 \in X$ such that $(x_0, x_0) \in B$, that is, T has a fixed point $x_0 \in X$. □

Proof of Corollary 5.4 using Corollary 6.2. Define $S(z) := \{y \in X \mid (z, y) \in A\}$ and $T(x) := \{y \in X \mid (x, y) \in B\}$. Apply Theorem 6.1. □

Corollary 6.3. *Let $(X, D; \Gamma)$ satisfy the partial KKM principle, X be compact, and $S : X \multimap D$, $T : X \multimap X$ two maps satisfying*

(1) *for each $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and*
(2) $X = \bigcup\{\mathrm{Int}\, S^-(z) \mid z \in D\}$.

Then T has a fixed point $x_0 \in X$.

Proof. Replacing S and T in Theorem 6.1 by $\mathrm{Int}\, S^-$ and T^-, respectively, observe the following:

(i) $\mathrm{Int}\, S^-(z)$ is open for each $z \in D$;
(ii) for each $y \in X$, $N \in \langle(\mathrm{Int}\, S^-)^-(y)\rangle \subset \langle S(y)\rangle$ implies $\Gamma_N \subset T(y)$ by (1);
(iii) for each $y \in X$, by (2), there exists a $z \in D$ such that $y \in \mathrm{Int}\, S^-(z)$, and hence $(\mathrm{Int}\, S^-)^-(y) \neq \emptyset$; and
(iv) since X itself is compact, $X \setminus \mathrm{Int}\, S^-(z_0)$ is compact for any $z_0 \in D$.

Therefore, by Corollary 6.2, T^- has a fixed point $x_0 \in X$; that is, $x_0 \in T^-(x_0)$ or $x_0 \in T(x_0)$. □

Corollary 6.4. *Let $(X, D; \Gamma)$ satisfy the partial KKM principle, X be compact, and $S : X \multimap D$ a map satisfying*

(1) *for each $x \in X$, $S(x)$ is nonempty; and*
(2) *for each $z \in D$, $S^-(z)$ is open.*

Then there exists an $\widehat{x} \in X$ such that $\widehat{x} \in \mathrm{co}_\Gamma\, S(\widehat{x})$.

The following simplified form of Corollary 6.3 or 6.4 is also a Fan-Browder type fixed point theorem:

Corollary 6.5. *Let $(X; \Gamma)$ satisfy the partial KKM principle, X be compact, and $T : X \multimap X$ a map satisfying*

(1) *for each* $x \in X$, $T(x)$ *is* Γ*-convex; and*
(2) $X = \bigcup\{\mathrm{Int}\, T^-(y) \mid y \in X\}$.

Then T *has a fixed point.*

Proof. Replacing $(S^-, \mathrm{co}_\Gamma S)$ in Corollary 6.4 by $(\mathrm{Int}\, T^-,\ T)$, we have the conclusion immediately. □

Remarks. (1) For a convex subset X of a topological vector space E, if $T^-(y)$ itself is open, then Corollary 6.5 reduces to Browder's result [6]. Condition (2) was first considered by Tarafdar [51].

(2) Note that Browder's result is a reformulation of Fan's geometric lemma [7] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it is known as the Fan-Browder fixed point theorem.

(3) Browder [6] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. For further developments on generalizations and applications of the Fan-Browder theorem, we refer to [21, 25, 27].

7. The existence theorems of maximal elements

Any binary relation R in a set X can be regarded as a map $T : X \multimap X$ and conversely by the following obvious way:

$$y \in T(x) \quad \text{if and only if} \quad (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a map T if $T(x_0) = \emptyset$.

In this section, we give another equivalent form of the (partial) KKM principle:

Theorem 7.1. *An abstract convex space* $(X, D; \Gamma)$ *satisfies the partial KKM principle iff for any maps* $S : X \multimap D$, $T : X \multimap X$ *satisfying*

(7.1.1) $S^-(z)$ *is open for each* $z \in D$;
(7.1.2) *for each* $x \in X$, $\mathrm{co}_\Gamma S(x) \subset T(x)$*; and*
(7.1.3) *for each* $x \in X$, $x \notin T(x)$,

X *can not be covered by a finite number of* $S^-(z)$*'s,* $z \in D$.

An abstract convex space $(X, D; \Gamma)$ *is a KKM space iff the above condition also holds for any map* $S : X \multimap D$ *satisfying*

(7.1.1)′ $S^-(z)$ *is closed for each* $z \in D$

instead of (7.1.1).

Proof of Theorem 7.1 using Theorem 6.1. Suppose that X is covered by a finite number of $S^-(z)$'s, $z \in D$. Consider Theorem 6.1 replacing S, T by S^-, T^-, respectively. Then all of the requirements of Theorem 6.1 are satisfied. Therefore, there exists an $x_0 \in X$ such that $x_0 \in T^-(x_0)$ or $x_0 \in T(x_0)$. But this violates (7.1.3). □

Proof of Theorem 6.1 using Theorem 7.1. Replacing S, T in Theorem 6.1 by S^-, T^-, respectively, follow the above proof. □

Remark. Theorems 3.1, 4.1, 5.1, 5.3, 6.1, and 7.1 are all characterizations of the KKM spaces. This means that there are no spaces other than KKM spaces satisfying any of the properties of matching, intersection, geometric or section, Fan-Browder type fixed point, or maximal element. Similarly the theorems and their Corollaries also characterizes abstract convex spaces satisfying the partial KKM principle.

From Theorem 7.1, we can deduce some results on maximal elements as follows:

Corollary 7.2. *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle and* $S : X \multimap D$, $T : X \multimap X$ *be two maps satisfying* (7.1.1)–(7.1.3). *If*

(7.1.4) $X \setminus S^-(z_0)$ *is compact for some* $z_0 \in D$,

then there exists an $\widehat{x} \in X$ *such that* $S(\widehat{x}) = \emptyset$.

Proof of Corollary 7.2 using Corollary 6.2. Suppose that $S(x) \neq \emptyset$ for each $x \in X$. Consider Corollary 6.2 replacing S, T by S^-, T^-, respectively. Then all of the requirements of Corollary 6.2 are satisfied. Therefore, there exists an $x_0 \in X$ such that $x_0 \in T^-(x_0)$ or $x_0 \in T(x_0)$. But this violates (7.1.3). □

Proof of Corollary 6.2 using Corollary 7.2. Replacing S, T in Corollary 6.2 by S^-, T^-, respectively, follow the above proof. □

Corollary 6.4 is equivalent to the following simple consequence of Corollary 7.2:

Corollary 7.3. *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle,* X *be compact, and* $S : X \multimap D$ *a map satisfying*

(1) $x \notin \mathrm{co}_\Gamma S(x)$ *for each* $x \in X$*; and*
(2) $S^-(z)$ *is open for each* $z \in D$.

Then there exists an $\widehat{x} \in X$ *such that* $S(\widehat{x}) = \emptyset$.

Corollary 7.3 is used by Borglin and Keiding [5] and Yannelis and Prabhakar [53] to the existence of maximal elements in mathematical economics.

Until recently, all conditions of Theorems 3.1, 4.1, 5.1, 5.3, 6.1, 7.1 and their Corollaries are known for G-convex spaces only. Now those Theorems characterizes the KKM spaces and all of Corollaries hold for any abstract convex spaces satisfying the partial KKM principle.

8. The KKM type theorems

From the partial KKM principle we have a whole intersection property:

Theorem 8.1. *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle and* $G : D \multimap X$ *be a closed-valued KKM map. If*

(8.1.1) $\bigcap_{z \in M} G(z)$ *is compact for some* $M \in \langle D \rangle$,

then we have

$$\bigcap_{z \in D} G(z) \neq \emptyset.$$

Note that Theorem 8.1 properly generalizes the second part of Theorem 2.7. Corollary 3.2 can be stated in its contrapositive form and in terms of the complement $G(z)$ of $S(z)$ in X. Then we obtain Theorem 8.1 with $M = \{z_0\}$. Conversely, we can deduce Corollary 3.2 from Theorem 8.1.

The following can be deduced from Theorem 8.1, as in [28, Theorem 5]:

Theorem 8.2. *Let $(X, D; \Gamma)$ satisfy the partial KKM principle, K be a nonempty compact subset of X, and $G : D \multimap X$ a map such that*

(8.2.1) $\bigcap_{z\in D} G(z) = \bigcap_{z\in D} \overline{G(z)}$ *[that is, G is transfer closed-valued];*
(8.2.2) $\overline{G}$ *is a KKM map; and*
(8.2.3) *either*
 (i) $\bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ *for some $M \in \langle D\rangle$; or*
 (ii) *for each $N \in \langle D\rangle$, there exists a compact Γ-convex subset L_N of X relative to some $D' \subset D$ such that $N \subset D'$ and*

$$L_N \cap \bigcap\{\overline{G(z)} \mid z \in D'\} \subset K.$$

Then $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$.

Proof. Suppose that $K \cap \bigcap\{G(z) \mid z \in D\} = K \cap \bigcap\{\overline{G(z)} \mid z \in D\} = \emptyset$; that is, $K \subset \bigcup\{X \setminus \overline{G(z)} \mid z \in N\}$ for some $N \in \langle D\rangle$.

Case (i): By Theorem 8.1, we have an $x \in \bigcap\{\overline{G(z)} \mid z \in D\} \subset \bigcap\{\overline{G(z)} \mid z \in M\} \subset K$ by (i). But, we have $x \in K \subset \bigcup\{X \setminus \overline{G(z)} \mid z \in D\}$, a contradiction.

Case (ii): Let L_N be the compact Γ-convex subspace of X in (ii). Define $G' : D' \multimap L_N$ by $G'(z) := G(z) \cap L_N$ for $z \in D'$. Then $A \in \langle D'\rangle$ implies $\Gamma'_A := \Gamma_A \cap L_N \subset \overline{G}(A) \cap L_N = \overline{G'}(A)$ by (8.2.2); and hence $\overline{G'} : D' \multimap L_N$ is a KKM map on $(L_N, D'; \Gamma')$ with closed values. Since $(X, D; \Gamma)$ is a KKM space, so is $(L_N, D'; \Gamma')$. Hence, $\{\overline{G'(z)} \mid z \in D'\}$ has the finite intersection property and $\bigcap\{\overline{G'(z)} \mid z \in D'\} \neq \emptyset$, by Theorem 8.1. For any $y \in \bigcap\{\overline{G'(z)} \mid z \in D'\}$, we have $y \in K$ by (ii). However, since $y \in K \subset \bigcup\{X \setminus \overline{G(z)} \mid z \in N\}$, we have $y \notin \overline{G(z)}$ for some $z \in N \subset D'$. This is a contradiction.

Therefore, we must have $K \cap \bigcap\{G(z) \mid z \in D\} \neq \emptyset$. □

Remarks. (1) The reader might prefer to assume that each $\overline{G(z)}$ is "compactly" closed, but this does not generalize anything; see [28].

(2) Conditions (8.1.1) or (8.2.3) are usually called the "compactness" or "coercivity" conditions. A large number of such requirements appeared in the literature to generalize known results and condition (ii) is one of them. As we have seen that Theorem 8.1 implies Theorem 8.2 as above, such general conditions are not essential. Therefore, for the simplicity, we adopt only (8.1.1) or a more simple form for a singleton $M = \{z_0\}$; see Corollaries 3.2, 4.2, 5.2, 5.4, 6.2, 7.2 and Theorems 9.1, 9.2, 10.1, 10.2.

(3) From now on, in many cases, we consider the case $M = \{z_0\}$ for simplicity.

9. Analytic alternatives

From Theorem 8.1, we deduced Theorem 8.2 and Corollaries 3.2, 4.2, 5.2, 5.4, 6.2, 7.2, from any of which the following analytic alternative follows:

Theorem 9.1. *Let $(X, D; \Gamma)$ satisfy the partial KKM principle and $A, B \subset C$ be sets. Let $f : D \times X \to C$ and $g : X \times X \to C$ be functions satisfying*

(9.1.1) $\{y \in X \mid f(z, y) \in A\}$ *is open for each $z \in D$;*
(9.1.2) *for each $y \in X$,* $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) \in A\} \subset \{x \in X \mid g(x, y) \in B\}$*; and*
(9.1.3) $\{y \in X \mid f(z_0, y) \notin A\}$ *is compact for some $z_0 \in D$.*

Then either

(a) *there exists a* $\widehat{y} \in X$ *such that* $f(z, \widehat{y}) \notin A$ *for all* $z \in D$*; or*
(b) *there exists a* $\widehat{x} \in X$ *such that* $g(\widehat{x}, \widehat{x}) \in B$.

Proof. It is immediate that Theorem 9.1 follows from Corollary 5.4 by replacing A, B by

$$A' = \{(z, y) \in D \times X \mid f(z, y) \in A\}, \quad B' = \{(x, y) \in X \times X \mid g(x, y) \in B\},$$

respectively. Similarly, we can obtain the converse. □

Remark. The first form of Theorem 9.1 is due to Lassonde [20]. For another form, see [25, 41].

From Theorem 9.1, we obtain the following analytic alternative which is a basis of various equilibrium problems:

Theorem 9.2. *Let* $(X, D; \Gamma)$ *satisfy the partial KKM principle,* $f : D \times X \to \overline{\mathbb{R}}$ *and* $g : X \times X \to \overline{\mathbb{R}}$ *be two extended real valued functions, and* $\alpha, \beta \in \overline{\mathbb{R}}$. *Suppose that*

(9.2.1) $\{y \in X \mid f(z, y) > \alpha\}$ *is open for each* $z \in D$*;*
(9.2.2) *for each* $y \in X$, $\mathrm{co}_\Gamma\{z \in D \mid f(z, y) > \alpha\} \subset \{x \in X \mid g(x, y) > \beta\}$*; and*
(9.2.3) $\{y \in X \mid f(z_0, y) \leq \alpha\}$ *is compact for some* $z_0 \in D$.

Then either

(a) *there exists a* $\widehat{y} \in X$ *such that* $f(z, \widehat{y}) \leq \alpha$ *for all* $z \in D$*; or*
(b) *there exists a* $\widehat{x} \in X$ *such that* $g(\widehat{x}, \widehat{x}) > \beta$.

Proof. Put $C = \overline{\mathbb{R}}$, $A = (\alpha, \infty]$, and $B = (\beta, \infty]$ in Theorem 9.1. □

Remark. If $X = D$ is a compact convex space, Theorem 9.2 reduces to the principal result of Ben-El-Mechaiekh et al. [3, 4], where this result is applied to variational inequalities of Hartman-Stampacchia and Browder, and a generalization of the Fan minimax inequality.

10. Minimax inequalities

From Theorem 9.2, we immediately have the following generalized form of the Fan minimax inequality [11]:

Theorem 10.1. *Under the hypothesis of Theorem 9.2, if* $\alpha = \beta = \sup\{g(x, x) \mid x \in X\}$*, then*

(c) *there exists a* $\widehat{y} \in X$ *such that*

$$f(z, \widehat{y}) \leq \sup_{x \in X} g(x, x) \quad \text{for all } z \in D; \text{ and}$$

(d) *we have the minimax inequality*

$$\inf_{y \in X} \sup_{z \in D} f(z, y) \leq \sup_{x \in X} g(x, x).$$

In order to show Theorem 10.1 is equivalent to any of Corollaries 3.2, 4.2, 5.2, 5.4, 6.2, 7.2, and Theorems 8.1, 8.2, 9.1, 9.2, we give the following:

Proof of Corollary 4.2 using Theorem 10.1. Define functions $f : D \times X \to \mathbb{R}$ and $g : X \times X \to \mathbb{R}$ by

$$f(z, y) = \begin{cases} 0 & \text{if } y \in S(z) \\ 1 & \text{otherwise} \end{cases}$$

for $(z, y) \in D \times X$ and

$$g(x, y) = \begin{cases} 0 & \text{if } y \in T(x) \\ 1 & \text{otherwise} \end{cases}$$

for $(x, y) \in X \times X$. Then $\alpha = \beta = 0$ by (3). Note that, for each $z \in D$, since $\{y \in X \mid f(z, y) > 0\} = \{y \in X \mid y \notin S(z)\} = X \setminus S(z)$ is open by (1), we have (9.2.1). And, since $S(z_0) = \{y \in X \mid f(z_0, y) \le 0\}$ is compact for some $z_0 \in D$, we have (9.2.3). Moreover, (2) implies (9.2.2). Therefore, by Theorem 10.1(c), there exists a $\widehat{y} \in X$ such that

$$f(z, \widehat{y}) \le \sup_{x \in X} g(x, x) = 0 \quad \text{for all } z \in D;$$

whence $\widehat{y} \in S(z)$ for all $z \in D$. This completes our proof of Corollary 4.2. □

Until now, we observe the following:

Proposition 1. (1) *Theorems 3.1, 4.1, 5.1, 5.3, 6.1, and 7.1 are mutually equivalent and characterize the KKM spaces and abstract convex spaces satisfying the partial KKM principle.*

(2) *Corollaries 3.2, 4.2, 5.2, 5.4, 6.2, 7.2 and Theorems 8.1, 8.2, 9.1, 9.2, 10.1 are mutually equivalent.*

For an abstract convex space $(X \supset D; \Gamma)$ satisfying the partial KKM principle, the KKM Theorem 8.1 can be reformulated to another minimax inequality as follows:

Theorem 10.2. *Let $(X \supset D; \Gamma)$ satisfy the partial KKM principle, $\phi : D \times X \to \overline{\mathbb{R}}$ be an extended real valued function, and $\gamma \in \overline{\mathbb{R}}$ such that*

(10.2.1) *$\{y \in X \mid \phi(z, y) \le \gamma\}$ is closed for each $z \in D$;*
(10.2.2) *for each $N \in \langle D \rangle$ and for each $y \in \Gamma_N$, we have $\min_{z \in N} \phi(z, y) \le \gamma$; and*
(10.2.3) *$\{y \in X \mid \phi(z_0, y) \le \gamma\}$ is compact for some $z_0 \in D$.*

Then

(a) *there exists a $\widehat{y} \in X$ such that*

$$\phi(z, \widehat{y}) \le \gamma \quad \textit{for all } z \in D;$$

and

(b) *if $\gamma = \sup_{x \in D} \phi(x, x)$, then we have the minimax inequality:*

$$\min_{y \in X} \sup_{z \in D} \phi(z, y) \le \sup_{x \in D} \phi(x, x).$$

Proof of Theorem 10.2 using Theorem 8.1. Let $G(z) := \{y \in X \mid \phi(z, y) \le \gamma\}$ for $z \in D$. Then, by (10.2.1) and (10.2.3), $G : D \multimap X$ has closed values and $G(z_0)$ is compact for some $z_0 \in D$. Moreover, by (10.2.2), G is a KKM map: Indeed, suppose that there exists an $N \in \langle D \rangle$ such that $\Gamma_N \not\subset G(N)$. Choose a $y \in \Gamma_N$ such that $y \notin G(N)$; that is, $y \notin G(z)$ or $\phi(z, y) > \gamma$ for all $z \in N$. Then $\min_{z \in N} \phi(z, y) > \gamma$, contradicting (10.2.2). Therefore, by Theorem 8.1, there exists a $\widehat{y} \in X$ such that $\widehat{y} \in \bigcap_{z\in D} G(z) \neq \emptyset$; that is, $\phi(z,\widehat{y}) \leq \gamma$ for all $z \in D$. This completes the proof of (a). Note that (b) immediately follows from (a). $\square$

Proof of Theorem 8.1 for $(X \supset D; \Gamma)$ *using Theorem 10.2.* Define $\phi : D \times X \to \mathbb{R}$ by

$$\phi(z,y) = \begin{cases} 0 & \text{if } y \in G(z) \\ 1 & \text{otherwise} \end{cases}$$

for $(z,y) \in D \times X$ and put $\gamma = 0$ in Theorem 10.2. Since $\{y \in X \mid \phi(z,y) \leq 0\} = G(z)$ is closed, (10.2.1) follows. Moreover, condition (8.1.1) for $M = \{z_0\}$ implies (10.2.3). [Note that we are still working on the particular case of (8.1.1) for a singleton M.] Furthermore, since G is a KKM map, condition (10.2.2) follows: Indeed, suppose that there exist an $N \in \langle D\rangle$ and a $y \in \Gamma_N$ such that $\min_{z\in N} \phi(z,y) > 0$. Then $y \notin G(z)$ for all $z \in N$; that is, $y \in \Gamma_N \not\subset G(N)$, a contradiction. Therefore, by Theorem 10.2, there exists a $\widehat{y} \in X$ such that $\phi(z,\widehat{y}) = 0$ for all $z \in D$; that is, $\widehat{y} \in \bigcap_{z\in D} F(z)$. This completes our proof of Theorem 8.1. $\square$

Remark. The first particular form of Theorem 10.2 is due to Zhou and Chen [56], who applied it to a variation of the Fan minimax inequality, a saddle point theorem, and a quasi-variational inequality.

Proposition 2. *For an abstract convex space* $(X \supset D; \Gamma)$ *satisfying the partial KKM principle, Corollaries 3.2, 4.2, 5.2, 5.4, 6.2, 7.2 and Theorems 8.1, 8.2, 9.1, 9.2, 10.1, 10.2 are mutually equivalent.*

Recall that an extended real valued function $f : X \to \overline{\mathbb{R}}$, where X is a topological space, is *lower* [resp., *upper*] *semicontinuous* (l.s.c.) [resp., u.s.c.] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is open for each $r \in \overline{\mathbb{R}}$.

For an abstract convex space $(X;\Gamma)$, a real function $f : X \to \overline{\mathbb{R}}$ is said to be *quasiconcave* [resp., *quasiconvex*] if $\{x \in X \mid f(x) > r\}$ [resp., $\{x \in X \mid f(x) < r\}$] is Γ-convex for each $r \in \overline{\mathbb{R}}$.

From now on, we mainly consider abstract convex compact spaces $(X;\Gamma)$ satisfying the partial KKM principle for simplicity.

Theorem 10.3. *Let* $(X;\Gamma)$ *satisfy the partial KKM principle,* X *be compact, and* $f, g : X \times X \to \mathbb{R} \cup \{+\infty\}$ *functions satisfying*

(10.3.1) $f(x,y) \leq g(x,y)$ *for each* $(x,y) \in X \times X$*;*
(10.3.2) *for each* $x \in X$, $g(x,\cdot)$ *is quasiconcave on* X*; and*
(10.3.3) *for each* $y \in X$, $f(\cdot,y)$ *is l.s.c. on* X*.*

Then we have

$$\min_{y\in X} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} g(x,x).$$

Proof. Observe that $\sup_{x\in X} f(x,y)$ is by (10.3.3) a l.s.c. function of y on the compact space X, and therefore its minimum exists. If $\sup_{x\in X} g(x,x) = +\infty$, then the inequality in the conclusion holds automatically. If $\alpha = \beta = \sup_{x\in X} g(x,x) < +\infty$, then by Theorem 10.1, we have the conclusion. $\square$

Remarks. (1) For $f = g$, Theorem 10.3 reduces to Fan's minimax inequality [11]. Fan obtained his inequality from his own generalization of the original KKM theorem, and applied it to deduce fixed point theorems, theorems on sets with convex sections, a fundamental existence theorem in potential theory, and so on.

(2) Later, the inequality has been an important tool in nonlinear analysis, game theory, and economic theory; see [27].

In particular, we have the following:

Corollary 10.4. *Under the hypothesis of Theorem 10.3, if $g(x,x) \le 0$ for all $x \in X$, then there exists a $y_0 \in X$ such that $f(x, y_0) \le 0$ for all $x \in X$. Thus in particular*

$$\min_{y\in X} \sup_{x\in X} f(x,y) \le 0.$$

11. Variational inequalities

Theorem 10.3 can be applied to the existence of solutions of certain variational inequalities:

Theorem 11.1. *Let $(X;\Gamma)$ satisfy the partial KKM principle, X be compact, and $p, q : X \times X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ functions satisfying*

(11.1.1) *$p(x,y) \le q(x,y)$ for each $(x,y) \in X \times X$, and $q(x,x) \le 0$ for all $x \in X$;*
(11.1.2) *for each $x \in X$, $q(x,\cdot) + h(\cdot)$ is quasiconcave on X; and*
(11.1.3) *for each $y \in X$, $p(\cdot, y) - h(\cdot)$ is l.s.c. on X.*

Then there exists a $y_0 \in X$ such that

$$p(x, y_0) + h(y_0) \le h(x) \quad \text{for all} \;\; x \in X.$$

Proof. Let

$$f(x,y) := p(x,y) + h(y) - h(x), \quad g(x,y) := q(x,y) + h(y) - h(x)$$

for $(x,y) \in X \times Y$. Then f and g satisfy the requirements of Theorem 10.3. Furthermore, $g(x,x) = q(x,x) \le 0$ for all $x \in X$. Therefore, by Corollary 10.4, the conclusion follows. □

Remarks. (1) Putting $h = 0$, Theorem 11.1 reduces to Corollary 10.4.

(2) Theorem 11.1 is a basis of existence theorems of many results concerning variational inequalities; see [25] and references therein.

Theorem 11.2. *Let $(X;\Gamma)$ satisfy the partial KKM principle, X be compact, and $p, q : X \times X \to \boldsymbol{R}$ functions satisfying*

(11.2.1) *$p \le q$ on the diagonal $\Delta := \{(x,x) \mid x \in X\}$ and $q \le p$ on $(X \times X) \setminus \Delta$;*
(11.2.2) *for each $x \in X$, $y \mapsto q(y,y) - q(x,y)$ is quasiconcave on X; and*
(11.2.3) *for each $y \in X$, $x \mapsto p(x,y)$ is u.s.c. on X.*

Then there exists a $y_0 \in X$ such that

$$p(y_0, y_0) \le p(x, y_0) \quad \text{for all} \;\; x \in X.$$

Proof. Define $f, g : X \times X \to \boldsymbol{R}$ by

$$f(x,y) := p(y,y) - p(x,y), \quad g(x,y) := q(y,y) - q(x,y)$$

Then f and g satisfy the hypothesis of Theorem 10.3. Since $g(x, x) = 0$ for all $x \in X$, Corollary 10.4 implies that $f(x, y_0) \leq 0$ for all $x \in X$. This implies the conclusion. □

Remark. For a convex space X and $p = q$, Theorem 11.2 reduces to a result of Fan [11], which was shown to be very useful in nonlinear functional analysis. In fact, the Tychonoff (and hence, the Brouwer) fixed point theorem, Browder's variational inequality, and many other applications follow from his result.

Since Theorem 11.2 implies the Brouwer fixed point theorem, in view of Theorem 2.7, we have the following:

Proposition 3. *For an abstract convex compact space $(X; \Gamma)$ satisfying the partial KKM principle, Theorems 10.3, 11.1, 11.2 follow from any of Theorems 3.1, 4.1, 5.1, 5.3, 6.1, 7.1, 8.1, 8.2, 9.1, 9.2, 10.1, and 10.2. For a compact G-convex space $(X; \Gamma)$, each of Theorems in this paper and Corollaries is equivalent to the original KKM theorem.*

12. Further applications

Note that Theorems 10.3, 11.1, and 11.2 are repeated in our previous work [35], where further applications of our theory on the partial KKM principle in the present paper are given as follows:

(1) Variational inequalities [35, Theorem 7.1];
(2) Best approximations [35, Corollary 7.3];
(3) The von Neumann type minimax theorem [35, Theorem 8.2];
(4) The von Neumann type intersection theorem [35, Theorem 9.1];
(5) The Nash type equilibrium theorem [35, Theorem 9.2];
(6) The Himmelberg fixed point theorem for KKM spaces [35, Theorem 10.1].

Moreover, in our forthcoming work [36] on fixed point theory, we will generalize the results on G-convex spaces in [32] and others to the ones on the KKM spaces or abstract convex spaces.

Recall that there are several hundred published works on the KKM theory, most of results in them are consequences of the ones given in this paper, where we cover only an essential part of the theory. For the more historical background, the reader can consult with [27]. For more involved or generalized versions of the results in this paper, see [25] for convex spaces, [39] for H-spaces, [29, 32, 38, 40-43] for G-convex spaces, and [30, 31, 33-35] for abstract convex spaces and references therein.

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