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SEQUENTIAL FORMULA FOR SUBDIFFERENTIAL OF INTEGRAL SUM OF CONVEX FUNCTIONS

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Abstract. The paper is devoted to the description of the subdifferential of continuous sum of convex functions on a Banach space. Without any qualification condition, general sequential formulas are established when the Banach space is separable. It is also shown how results under qualification condition in the literature can be derived from sequential ones.

1. Introduction

The paper is devoted to the study of the subdifferential of the integral (or continuous) sum

$$I_f(x) = \int_T f(t,x)\, d\mu(t), \tag{1.1}$$

where $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ is a *normal convex* integrand, $(T, \mathcal{T}, \mu)$ is a measure space with a σ-finite positive measure μ, and X is a real separable Banach space. Depending on whether X is finite dimensional, reflexive or not, the established results and the required assumptions are different.

Essentially, if it is possible, under qualification condition, to obtain (see [8, 7]) the equality

$$\partial I_f(\bar{x}) = \int_T \partial f_t(\bar{x})\, d\mu(t) + N(\operatorname{dom} I_f, \bar{x}), \tag{1.2}$$

such point formula does not hold without qualification condition. We also refer to the first papers [15, 4, 11, 19, 20] concerning (1.2) for points $\overline{x}$ where I_f is finite and continuous. When no qualification condition is assumed and when X is reflexive (resp. not reflexive) it is natural (taking into account results concerning *finite* sum) to look whether any continuous linear functional $\bar{x}^*$ of the subdifferential $\partial I_f(\bar{x})$ can be approximated by an appropriate sequence $(x_n^*)_{n\in\mathbb{N}}$ (resp. a net $(x_i^*)_{i\in I}$). In other words, do there exist appropriate sequences $(x_n)_{n\in\mathbb{N}}$ and $(x_n^*)_{n\in\mathbb{N}}$ (resp. nets $(x_i)_{i\in I}$ and $(x_i^*)_{i\in I}$) such that

$$\begin{aligned} &\bar{x}^* = \lim_{n\to\infty} \int_T x_n^*(t)\, d\mu(t) \quad \text{in} \quad (X^*, \|\ \|_{X^*}) \\ &x_n^*(t) \in \partial f_t(x_n(t)) \quad \mu - a.e. \end{aligned} \tag{1.3}$$

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(resp.

$$\begin{aligned} \bar{x}^* = \lim_{i \in I} \int_T x_i^*(t)\, d\mu(t) \quad \text{in} \quad (X^*, w(X^*, X)) \\ x_i^*(t) \in \partial f_t(x_i(t)) \quad \mu - a.e.)? \end{aligned} \tag{1.4}$$

Sequential subdifferential formulas in Convex Analysis began in Hilbert space with Attouch-Baillon-Théra [1] via the Moreau envelope and in general Banach space with Thibault [16, 17] via calculus formulas established by Hiriart-Urruty and Phelps [5] in terms of ε-subdifferentials. The papers [16, 17, 18] provide, in the setting of reflexive (resp. non reflexive) Banach space and without any qualification condition, a general approximation by sequences (resp. nets) of any element of the subdifferential of finite sum or composition (see also [3, 10, 12]). Recently Ioffe [6] investigated the study of the above *continuous sum* of convex functions (1.1) and described its subdifferential when no qualification condition is assumed. In the case when the Banach space is reflexive (resp. not reflexive) he proved that the above approximation formula (1.3) (resp. (1.4)) holds. His method involves the use of a smooth renorm and of an infimum convolution regularization procedure. The purpose of this paper is to show how formulas (1.3) and (1.4) can be derived from the sequential formula of the subdifferential of the composition of a convex function with a continuous linear mapping.

2. Preliminaries

Throughout the paper, we assume that $(T, \mathcal{T}, \mu)$ is a measure space with a positive σ-finite measure μ and that X is a (real) *separable Banach space*. For any element $p \in [1, \infty]$ we denote by $L^p(T, X)$ the usual space of classes of measurable (with respect to $\mathcal{T}$) mappings $g : T \to X$ such that the function $\|g(\cdot)\|$ is in $L^p(T, \mathbb{R})$. Unless otherwise stated, measurability will be taken with respect to the σ-algebra $\mathcal{T}$.

The topological dual of $L^p(T, X)$ endowed with its usual norm is relied to another concept of measurability for mappings with values in the topological dual X^* of X. A mapping $h : T \to X^*$ is called w^*-*measurable* if for any $x \in X$ the function $t \mapsto \langle h(t), x\rangle$ is measurable. The separability of X then yields that the function $\|h(\cdot)\|$ is measurable. For any $p \in [1, \infty]$ we will denote by $L^p_{w^*}(T, X^*)$ the classes of w^*-measurable mappings $h : T \to X^*$ such that the function $\|h(\cdot)\|$ is in $L^p(T, \mathbb{R})$. Here classes are taken in the sense that two w^*-measurable mappings h_1, h_2 from T into X^* are considered to be equivalent when for any $x \in X$ the functions $\langle h_1(\cdot), x\rangle$ and $\langle h_2(\cdot), x\rangle$ are equal a.e. It is known (see e.g. [9] where the notation $L^p_{X^*}[X]$ is used in place of $L^p_{w^*}(T, X^*)$) that, for any real number $p \in [1, \infty[$ and for $q \in]1, \infty]$ given by $\frac{1}{p} + \frac{1}{q} = 1$, the topological dual of $L^p(T, X)$ endowed with its usual norm is identified with $L^q_{w^*}(T, X^*)$ under the pairing $\langle h, g\rangle = \int_T \langle h(t), g(t)\rangle\, d\mu(t)$ for any $g \in L^p(T, X)$ and $h \in L^q_{w^*}(T, X^*)$. Observe that it is easily seen that the function $t \mapsto \langle h(t), g(t)\rangle$ is measurable and hence summable for $|\langle h(t), g(t)\rangle| \le \|h(t)\| \cdot \|g(t)\|$. When the separable Banach space X is reflexive, the normed vector space X^* is also a (reflexive) separable Banach space and hence $L^q_{w^*}(T, X^*)$ is equal to the usual space $L^q(T, X^*)$.

Let $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a function such that $f_t := f(t, \cdot)$ is, for each $t \in T$, a proper lower semicontinuous (lsc) function and such that the set-valued mapping $t \mapsto \Gamma(t) := \operatorname{epi} f_t$ is measurable in the usual sense, that is,

$$\Gamma^{-1}(U) := \{t \in T : \Gamma(t) \cap U \neq \emptyset\} \in \mathcal{T}$$

for any open set U of $X \times \mathbb{R}$. (Recall that the properness of $f(t, \cdot)$ corresponds to the non-vacuity of the set $\operatorname{dom} f(t, \cdot) := \{x \in X : f(t, x) < \infty\}$ and that the epigraph $\operatorname{epi} f_t$ is the set $\operatorname{epi} f_t := \{(x, r) \in X \times \mathbb{R} : f_t(x) \leq r\}$). Such function f is usually called a *normal* integrand. When f_t is further convex for each $t \in T$, one says that f is a normal convex integrand . Under the lsc property of $f(t, \cdot)$ (i.e., the closedness of $\Gamma(t)$), the measurability of the set-valued mapping Γ is known (see e.g. [2]) to be equivalent to the measurability of the function $(t, x, r) \mapsto \operatorname{dist}((x, r), \Gamma(t))$ with respect to the σ-algebra $\mathcal{T} \otimes \mathcal{B}(X \times \mathbb{R})$. The latter easily implies the measurability of the function f with respect to the σ-algebra $\mathcal{T} \otimes \mathcal{B}(X)$. Further, the measurability of Γ being characterized by the existence of a sequence $(y_n, \alpha_n)_{n \in \mathbb{N}}$ of measurable mappings of T into $X \times \mathbb{R}$ such that $\Gamma(t) = \operatorname{cl}\{(y_n(t), \alpha_n(t)) : n \in \mathbb{N}\}$ for all $t \in T$ (a Castaing representation of Γ), we see as in [14, p.223] that, for any w^*-measurable mapping $y^* : T \to X^*$, the Fenchel conjugate $f^*(t, \cdot)$ of $f(t, \cdot)$ at $y^*(t)$ takes the form

$$f^*(t, y^*(t)) = \sup_{(x,r) \in \operatorname{epi} f_t} [\langle y^*(t), u\rangle - r] = \sup_{n \in \mathbb{N}} [\langle y^*(t), y_n(t)\rangle - \alpha_n(t)]$$

and this yields that the function $t \mapsto f^*(t, y^*(t))$ is $\mathcal{T}$-measurable. When f_t is further convex, the function $f^*(\cdot, \cdot)$ is (see [14, Proposition 2]) even a normal convex integrand whenever X is reflexive and the σ-algebra $\mathcal{T}$ is μ-complete.

For any measurable function $\varphi : T \to \mathbb{R} \cup \{-\infty, \infty\}$ the extended integral $\int_T \varphi(t)\, d\mu(t)$ is defined with the convention that $\int_T \varphi(t)\, d\mu(t) = \infty$ whenever the positive part of the function φ is not summable. So for any measurable mapping $y : T \to X$ the integral $\int_T f(t, y(t))\, d\mu(t)$ makes sense since the function $t \mapsto f(t, y(t))$ is measurable according to the normality of the integrand f. For any element $p \in [1, \infty]$ we may then consider the function $\mathfrak{I}_{f,p} : L^p(T, X) \to \mathbb{R} \cup \{-\infty, \infty\}$ defined for any $y \in L^p(T, X)$ by

$$\mathfrak{I}_{f,p}(y) := \int_T f(t, y(t))\, d\mu(t). \tag{2.1}$$

In the same way the above measurability of $t \mapsto f(t, y^*(t))$ for any w^*-measurable mapping $y^* : T \to X^*$ allows us to define the function $\mathfrak{I}_{f^*,p}$ on $L^p_{w^*}(T, X^*)$ by

$$\mathfrak{I}_{f^*,p}(y^*) := \int_T f^*(t, y^*(t))\, d\mu(t) \quad \text{for all } y^* \in L^p_{w^*}(T, X^*). \tag{2.2}$$

From now on, we will assume that f is a *normal convex integrand.* Besides the function $\mathfrak{I}_{f,p}$, is associated the continuous sum function I_f defined on X by

$$I_f(x) := \int_T f(t, x)\, d\mu(t) \quad \text{for all } x \in X. \tag{2.3}$$

This function is obviously convex. Our aim is to establish, via a composition procedure, the sequential formulas stated in the introduction for elements of its subdifferential. To do so, we start by observing the following Rockafellar's description of the Fenchel conjugate of the functional integral $\Im_{f,p}$. Adapting the proofs of [14, Theorem 2] and [2, Theorem VII-7], for any $p \in [1,\infty[$ assuming that $\Im_{f,p}$ is finite at some point of the space $L^p(T,X)$ one obtains that its Fenchel conjugate is the function $\Im_{f^*,q}$ defined on $L^q_{w^*}(T,X^*)$, that is,

$$(\Im_{f,p})^*(y^*) = \int_T f^*(t, y^*(t))\, d\mu(t) \quad \text{for all } y^* \in L^q_{w^*}(T,X^*), \tag{2.4}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. The following theorem (see [14, 2]) is then a consequence of (2.3).

Theorem 2.1. *Let X be a separable Banach space, $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand , and $p \in [1,\infty[$. Assume that $\Im_{f,p}$ is finite at some point in $L^p(T,X)$. Then for $y \in L^p(T,X)$ where $\Im_{f,p}$ is finite, an element $y^* \in L^q_{w^*}(T,X^*)$ is in the subdifferential $\partial\Im_{f,p}(y)$ if and only if*

$$y^*(t) \in \partial f(t, y(t)) \quad \text{for a.e. } t \in T. \tag{2.5}$$

The case $p = \infty$ will be considered later.

3. Subdifferential of integral sum on reflexive space

We begin this section by recalling the following theorem (see [18, Theorem 1]) which, in the case of composition with a linear mapping on a reflexive space, can be stated as follows (see also [3]). Other versions have been first established in [16, 17, 12] and another approach for composition formula can be found in [10].

Theorem 3.1. *Let Y be any Banach space, X be a reflexive Banach space, and $A : X \to Y$ be a coninuous linear mapping. Let $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Then for any $\overline{x} \in X$ with $A\overline{x} \in \operatorname{dom}\varphi$ one has $\overline{x}^* \in \partial(\varphi \circ A)(\overline{x})$ if and only if there exist sequences $(y_n^*)_{n\in\mathbb{N}}$ in Y^*, $(y_n)_{n\in\mathbb{N}}$ in Y, and $(x_n)_{n\in\mathbb{N}}$ in X such that*

(a) $y_n^* \in \partial\varphi(y_n)$ *for each* $n \in \mathbb{N}$;
(b) $\overline{x}^* = \lim_{n\to\infty} y_n^* \circ A$ *in* $(X^*, \|\ \|)$;
(c) $y_n \overset{\|\ \|}{\to} Ax$ *and* $\varphi(y_n) \to \varphi(Ax)$;
(d) $\langle y_n^*, y_n - A\overline{x}\rangle \to 0$;
(e) $x_n \overset{\|\ \|}{\to} \overline{x}$ *and* $\|y_n^*\| \cdot \|y_n - Ax_n\| \to 0$.

This theorem will allow us to establish, via a direct composition procedure, the first sequential formula in Ioffe [6] concerning the subdifferential of the function I_f. The formula as well as all the results in the rest of the paper is stated under the assumption that the measure μ is finite. In the case where μ is *σ-finite*, the corresponding results are obtained by replacing μ by the measure $\beta(\cdot)\mu$ (having $\beta(\cdot)$ as density with respect to μ), where β is any μ-summable function with $\beta(t) > 0$ for all $t \in T$. (See [6] for more details).

Theorem 3.2 (Ioffe [6]). *Let X be a separable reflexive Banach space, $p \in [1,\infty[$, and $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand for which there exist $\alpha(\cdot) \in L^1(T,\mathbb{R})$ and $a^*(\cdot) \in L^q(T,X^*)$ such that for a.e. t and all $x \in X$*

$$f(t,x) \geq \langle a^*(t), x\rangle + \alpha(t), \tag{3.1}$$

where $\frac{1}{p}+\frac{1}{q}=1$. *Assume that the measure* μ *is finite. Then for* $\overline{x}\in \operatorname{dom} I_f$ *one has* $\overline{x}^*\in\partial I_f(\overline{x})$ *if and only if there are sequences of mappings* $(u_n)_{n\in\mathbb{N}}$ *and* $(u_n^*)_{n\in\mathbb{N}}$ *in the spaces* $L^p(T,X)$ *and* $L^q(T,X^*)$ *respectively such that*

(a) $u_n^*(t)\in\partial f_t(u_n(t))$ *a.e.;*
(b) $\overline{x}^*=\lim\limits_{n\to\infty}\int_T u_n^*(t)\,d\mu(t)$ *in* $(X^*,\|\cdot\|)$*;*
(c) $\lim\limits_{n\to\infty}\int_T \|u_n(t)-\overline{x}\|^p\,d\mu(t)=0$ *and* $\lim\limits_{n\to\infty}\int_T f(t,u_n(t))\,d\mu(t)=I_f(\overline{x})$*;*
(d) $\lim\limits_{n\to\infty}\int_T < u_n^*(t),\overline{x}-u_n(t) > \,d\mu(t)=0$.

Further, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ *in* X *converging in norm to* $\overline{x}$ *such that*

(e_1) $\lim\limits_{n\to\infty}\int_T \|x_n-u_n(t)\|^p\,d\mu(t)=0$*;*
(e_2) $\lim\limits_{n\to\infty}\int_T \|u_n^*(t)\|\cdot\|x_n-u_n(t)\|\,d\mu(t)=0.$

Before proving Theorem 3.2 we establish the following semicontinuity lemma.

Lemma 3.1. *Let* X *be a separable Banach space and* $p,q\in[1,+\infty]$ *with* $\frac{1}{p}+\frac{1}{q}=1$. *Let* $f:T\times X\to\mathbb{R}\cup\{+\infty\}$ *be a normal convex integrand for which there exist* $\alpha(\cdot)\in L^1(T,\mathbb{R})$ *and* $a^*(\cdot)\in L^q_{w^*}(T,X^*)$ *such that* $\mathfrak{I}_{f,p}$ *is finite at some point in* $L^p(T,X)$ *and for a.e.* t *and all* $x\in X$

$$f(t,x)\geq\langle a^*(t),x\rangle+\alpha(t). \tag{3.2}$$

Then the functional integral $\mathfrak{I}_{f,p}$ *is proper, convex, and lsc on* $L^p(T,X)$.

Proof. It is not difficult to see that $\mathfrak{I}_{f,p}$ is proper and convex so, we only prove that $\mathfrak{I}_{f,p}$ is lsc. Fix any $u\in L^p(T,X)$ and take any sequence $(u_n)_{n\in\mathbb{N}}$ in $L^p(T,X)$ converging in L^p-norm to u. Taking subsequences if necessary we may suppose that $\liminf\limits_n \mathfrak{I}_{f,p}(u_n)=\lim\limits_n \mathfrak{I}_{f,p}(u_n)$ and that $(u_n)_n$ converges almost everywhere to u. Observing that

$$f(t,u_n(t))-\langle a^*(t),u_n(t)\rangle\geq\alpha(t)$$

we see that we may apply the Fatou lemma to obtain

$$\mathfrak{I}_{f,p}(u)-\int_T\langle a^*(t),u(t)\rangle\,d\mu(t)\leq\liminf_n\,[\mathfrak{I}_{f,p}(u_n)-\int_T\langle a^*(t),u_n(t)\rangle\,d\mu(t)],$$

which yields $\mathfrak{I}_{f,p}(u)\leq\liminf\limits_n\mathfrak{I}_{f,p}(u_n)$ and hence the lsc property of $\mathfrak{I}_{f,p}$. □

Proof of Theorem 3.2. It is not difficult to see that the assertions (a) , (b) , (c) , and (d) imply that $\overline{x}^*\in\partial I_f(\overline{x})$. To prove the reverse implication, suppose that $\overline{x}^*\in\partial I_f(\bar{x})$. We may also suppose that $\mu(T)=1$ (replace the measure μ by the measure $\widetilde{\mu}:=\frac{1}{\mu(T)}\cdot\mu$). Let $j:X\to L^p(T,X)$ be the mapping defined for all $x\in X$ by

$$jx:T\to X\quad\text{with}\quad(jx)(t)=x\quad\text{for all } t\in T.$$

It is clear that j is a continuous linear mapping and that $I_f=\mathfrak{I}_{f,p}\circ j$. Observe also by Lemma 3.1 that $\mathfrak{I}_{f,p}$ is lsc. Then according to Theorem 3.1, there exist $u_n^*\in L^q(T,X^*)$, $u_n\in L^p(T,X)$, and $x_n\in X$ such that

(a′) $u_n^*\in\partial\mathfrak{I}_{f,p}(u_n)$;
(b′) $u_n^*\circ j\overset{\|\cdot\|}{\to}x^*$;
(c′) $u_n\to j\overline{x}$ in norm in $L^p(T,X)$ and $\mathfrak{I}_{f,p}(u_n)\to\mathfrak{I}_{f,p}(jx)$;

(d′) $\langle u_n^*, u_n - j\overline{x}\rangle_{L^q(X^*),L^p(X)} \to 0$;

(e′) $x_n \overset{\|\cdot\|}{\to} \overline{x}$ and $\|u_n^*\|_{L^q(X^*)} \cdot \|u_n - x_n\|_{L^p(X)} \to 0$.

It follows from (2.5) in Theorem 2.1, from the finiteness of $\mathfrak{I}_{f,p}$ at $j\bar{x}$ and from (a′) that for each integer n

$$u_n^*(t) \in \partial f_t(u_n(t)) \quad a.e.$$

The condition (c′) gives us on the one hand

$$\mathfrak{I}_{f,p}(u_n) = \int_T f(t, u_n(t))\, d\mu(t) \to I_f(\overline{x})$$

and on the other hand

$$\int_T \|u_n(t) - \overline{x}\|^p \, d\mu(t) \to 0,$$

which becomes by using the first part of (e′)

$$\int_T \|u_n(t) - x_n\|^p \, d\mu(t) \to 0.$$

The assertion (d′) can be translated into

$$\lim_{n\to+\infty} \int_T \langle u_n^*(t), u_n(t) - \overline{x}\rangle \, d\mu(t) = 0.$$

The second part of assertion (e′) corresponds to

$$\lim_{n\to+\infty} \|u_n^*\|_{L^q(X^*)} \cdot (\int_T \|u_n(t) - (jx_n)(t)\|^p \, d\mu(t))^{\frac{1}{p}} = 0,$$

and by Hölder inequality it follows that

$$\lim_{n\to+\infty} \int_T \|u_n^*(t)\| \cdot \|u_n(t) - x_n\| \, d\mu(t) = 0.$$

It remains to prove the assertion (b) of Theorem 3.2 . Observe that for all $x \in X$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \langle u_n^* \circ j, x\rangle_{X^*,X} &= \langle u_n^*, jx\rangle_{L^q(X^*),L^p(X)} \\ &= \int_T \langle u_n^*(t), (jx)(t)\rangle_{X^*,X} \, d\mu(t) \\ &= \langle \int_T u_n^*(t)\, d\mu(t), x\rangle_{X^*,X} \end{aligned}$$

and hence

$$u_n^* \circ j = \int_T u_n^*(t)\, d\mu(t).$$

Applying (b′) we obtain

$$\overline{x}^* = \lim_{n\to\infty} \int_T u_n^*(t)\, d\mu(t) \quad \text{in} \quad (X^*, \|\cdot\|),$$

which completes the proof. □

Sometimes it may be convenient to use the following corollary in order to get elements in ∂I_f.

Corollary 3.1. *Let X be a separable reflexive Banach space, $p \in [1, \infty[$, and $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand satisfying (3.1). Then for $\overline{x} \in \operatorname{dom} I_f$ one has $\overline{x}^* \in \partial I_f(\overline{x})$ if and only if there are sequences of mappings $(u_n)_{n\in\mathbb{N}}$ and $(u_n^*)_{n\in\mathbb{N}}$ in the spaces $L^p(T, X)$ and $L^q(T, X^*)$ respectively such that*

(a) $u_n^*(t) \in \partial f_t(u_n(t))$ *a.e.;*

(b) $\overline{x}^* = \lim_{n\to\infty} \int_T u_n^*(t)\, d\mu(t)$ *in* $(X^*, w^*(X^*, X))$*;*

(c′) $\lim_{n\to\infty} \int_T \|u_n(t) - \overline{x}\|^p \, d\mu(t) = 0$*;*

(d) $\lim_{n\to\infty} \int_T < u_n^*(t), \overline{x} - u_n(t) > \, d\mu(t) = 0$ *.*

Proof. It is enough to show that any such $\overline{x}^*$ is in $\partial I_f \overline{x}$. For such element, writing for any $x \in X$

$$\langle u_n^*(t), x - u_n(t)\rangle \leq f(t, x) - f(t, u_n(t)),$$

we get

$$\langle \int_T u_n^*(t)\, d\mu(t), x - \overline{x}\rangle + \int_T \langle u_n^*(t), \overline{x} - u_n(t)\rangle \, d\mu(t) \leq I_f(x) - \int_T f(t, u_n(t))\, d\mu(t).$$

Taking the lower semicontinuity of $\mathfrak{I}_f$ over $L^p(T, X)$ into account and passing to the limit, we obtain

$$\langle \overline{x}^*, x - \overline{x}\rangle \leq I_f(x) - \mathcal{I}_f(j\overline{x}) = I_f(x) - I_f(\overline{x}).$$

This means that $\overline{x} \in \partial I_f(\overline{x})$. $\square$

4. The non-reflexive case

In this section we consider the case when X is a general (non-reflexive) separable Banach space. We state first the following form of [18, Theorem 1] in this case.

Theorem 4.1. *Let X and Y be Banach spaces and $A : X \to Y$ be a continuous linear mapping from X into Y. Let $\varphi : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lsc convex function. Then for any $\overline{x} \in X$ with $A\overline{x} \in \operatorname{dom} \varphi$ one has $\overline{x}^* \in \partial(\varphi \circ A)(\overline{x})$ if and only if there exist nets $(y_i^*)_{i\in I}$ in Y^*, $(y_i)_{i\in I}$ in Y, and $(x_i)_{i\in I}$ such that (a), (c), (d), (e) in Theorem 3.1 hold with I instead of $\mathbb{N}$ and such that one has the weak-star convergence in (b) of Theorem 3.1, that is,*

(b) $\overline{x}^* = \lim_{i\in I} y_i^* \circ A$ *with respect to the* $w(X^*, X)$*-topology.*

The lsc property in Lemma 3.1 being true in the context where the Banach space X is not necessarily reflexive, we may follow the proof of Theorem 3.2 with the use of Theorem 4.1 in place of Theorem 3.1. So after appropriate adaptations we obtain the following theorem of Ioffe [6].

Theorem 4.2. *Let X be a separable Banach space, $p \in [1, \infty[$, and $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand for which there exist $\alpha(\cdot) \in L^1(T, \mathbb{R})$ and $a^*(\cdot) \in L^q_{w^*}(T, X^*)$ such that for a.e. t and all $x \in X$*

$$f(t, x) \geq \langle a^*(t), x\rangle + \alpha(t), \tag{4.1}$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Then for $\overline{x} \in \operatorname{dom} I_f$ one has $\overline{x}^* \in \partial I_f(\overline{x})$ if and only if there are nets of mappings $(u_i)_{i\in I}$ and $(u_i^*)_{i\in I}$ in the spaces $L^p(T, X)$ and $L^q_{w^*}(T, X^*)$ respectively such that

(a) $u_i^*(t) \in \partial f_t(u_i(t))$ *a.e.;*
(b) $\overline{x}^* = \lim\limits_{i\in I} \int_T u_i^*(t)\, d\mu(t)$ *with respect to the $w(X^*, X)$-topology;*
(c) $\lim\limits_{i\in I} \int_T \|u_i(t) - \overline{x}\|^p\, d\mu(t) = 0$ *and* $\lim\limits_{i\in I} \int_T f(t, u_i(t))\, d\mu(t) = I_f(\overline{x})$*;*
(d) $\lim\limits_{i\in I} \int_T < u_i^*(t), \overline{x} - u_i(t) >\, d\mu(t) = 0$.

Further, there exists a net $(x_i)_{i\in I}$ in X converging in norm to $\overline{x}$ such that
(e$_1$) $\lim\limits_{i\in I} \int_T \|x_i - u_i(t)\|^p\, d\mu(t) = 0$*;*
(e$_2$) $\lim\limits_{i\in I} \int_T \|u_i^*(t)\| \cdot \|x_i - u_i(t)\|\, d\mu(t) = 0.$

5. The case $p = \infty$

Otherwise stated we assume henceforth that the measure space is *complete* and that X is a *separable reflexive* Banach space. (The case where the separable Banach space X is not reflexive is considered in the comments after the proof of Theorem 5.2 and the ones preceding the proof of Theorem 5.3). Recall that a continous linear functional s^* on $L^\infty(T, X)$ is said to be *singular* if there is an increasing sequence $(T_n)_n$ of measurable sets satisfying $T = \bigcup_{n\in\mathbb{N}} T_n$ such that, whenever $u \in L^\infty(T, X)$ is a mapping vanishing almost everywhere outside of some T_n , one has $\langle s^*, u\rangle = 0$. The set of these singular functionals forms a vector space denoted by $L^{\text{sing}}(T, X)$. It is known (see e.g. [2, 14]) that

$$(L^\infty(T, X))^* = L^1(T, X^*) \oplus L^{\text{sing}}(T, X). \tag{5.1}$$

Rockafellar (see [14, Theorem 4]) established the following expression of the Fenchel conjugate of $\mathfrak{I}_{f,\infty}$.

Theorem 5.1. *Let X be a separable reflexive Banach space and $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand. Assume that the functional integral $\mathfrak{I}_{f,\infty}$ on $L^\infty(T, X)$ is finite at some point in $L^\infty(T, X)$. Then the Fenchel conjugate of $\mathfrak{I}_{f,\infty}$ on $(L^\infty(T, X))^*$ is given for all $u^* \in (L^\infty(T, X))^*$ by*

$$(\mathfrak{I}_{f,\infty})^*(u^*) = \mathfrak{I}_{f^*,1}(v^*) + \sup_{u\in \operatorname{dom} \mathfrak{I}_{f,\infty}} \langle s^*, u\rangle_{(L^\infty(X))^*, L^\infty(X)},$$

where $u^* = v^* + s^*$, $v^* \in L^1(T, X^*)$ and $s^* \in L^{\text{sing}}(T, X)$.

This theorem yields the following description of the subdifferential of $\mathfrak{I}_{f,\infty}$.

Proposition 5.1. *Assume the hypotheses of Theorem 5.1. For any $u \in L^\infty(T, X)$ where $\mathfrak{I}_{f,\infty}$ is finite and $u^* = v^* + s^*$ in $(L^\infty(T, X))^*$ with $v^* \in L^1(T, X^*)$ and $s^* \in L^{\text{sing}}(T, X)$, one has $u^* \in \partial\mathfrak{I}_{f,\infty}(u)$ if and only if*

$$v^*(t) \in \partial f_t(u(t)) \quad \text{a.e.} \quad \text{and} \quad s^* \in N(\operatorname{dom} \mathfrak{I}_{f,\infty}, u),$$

where $N(\operatorname{dom} \mathfrak{I}_{f,\infty}, u)$ is the normal cone to $\operatorname{dom} \mathfrak{I}_{f,\infty}$ at $u \in \operatorname{dom} \mathfrak{I}_{f,\infty}$.

Proof. The implication $\Leftarrow$ being easy to verify, only the opposite one needs to be proved. Let $u^* \in \partial\mathfrak{I}_{f,\infty}(u)$ where $u^* = v^* + s^*$, $v^* \in L^1(T, X^*)$, $s^* \in L^{\text{sing}}(T, X)$. By the characterization of the subdifferential in terms of the Fenchel conjugate we have

$$\mathfrak{I}_{f,\infty}(u) + (\mathfrak{I}_{f,\infty})^*(u^*) \le \langle u^*, u\rangle_{(L^\infty)^*, L^\infty}$$

which yields according to Theorem 5.1

$$\mathfrak{I}_{f,\infty}(u) + \mathfrak{I}_{f^*,1}(v^*) + \sup_{y\in\operatorname{dom}\mathfrak{I}_{f,\infty}} \langle s^*, y\rangle_{(L^\infty)^*,L^\infty} \le \langle v^*, u\rangle_{(L^\infty)^*,L^\infty} + \langle s^*, u\rangle_{(L^\infty)^*,L^\infty}.$$

Rewriting this inequality in the form

$$\begin{aligned}(5.2)\quad \mathfrak{I}_{f,\infty}(u) + \mathfrak{I}_{f^*,1}(v^*) - \langle v^*, u\rangle_{(L^\infty)^*,L^\infty}& \\ \le \langle s^*, u\rangle_{(L^\infty)^*,L^\infty} - &\sup_{y\in\operatorname{dom}\mathfrak{I}_{f,\infty}} \langle s^*, y\rangle_{(L^\infty)^*,L^\infty},\end{aligned}$$

we obtain in particular

$$\mathfrak{I}_{f,\infty}(u) + \mathfrak{I}_{f^*,1}(v^*) - \langle v^*, u\rangle_{(L^\infty)^*,L^\infty} \le 0. \tag{5.3}$$

In other respects, since $\langle v^*, u\rangle_{(L^\infty)^*,L^\infty} = \int_T \langle v^*(t), u(t)\rangle_{X^*,X}\, d\mu(t)$, the Fenchel inequality ensures that

$$\begin{aligned}(5.4)\quad \mathfrak{I}_{f,\infty}(u) + \mathfrak{I}_{f^*,1}(v^*) - \langle v^*, u\rangle& \\ = \int_T [f(t,u(t)) + f^*(t,v^*(t)) - &\langle v^*(t), u(t)\rangle_{X^*,X}]\, d\mu(t) \ge 0,\end{aligned}$$

which entails that (5.3) is in fact an equality, that is,

$$\int_T [f(t,u(t)) + f^*(t,v^*(t)) - \langle v^*(t), u(t)\rangle_{X^*,X}]\, d\mu(t) = 0.$$

Since the integrand in the latter integral is nonnegative by the Fenchel inequality (as already seen in (5.4)), we have

$$f(t,u(t)) + f^*(t,v^*(t)) - \langle v^*(t), u(t)\rangle_{X^*,X} = 0 \quad \text{a.e.},$$

that is,

$$v^*(t) \in \partial f_t(u(t)) \quad \text{a.e..}$$

On the other hand , by (5.2) and (5.4) we also have

$$\langle s^*, u\rangle_{(L^\infty)^*,L^\infty} - \sup_{y\in\operatorname{dom}\mathfrak{I}_{f,\infty}} \langle s^*, y\rangle_{(L^\infty)^*,L^\infty} \ge 0,$$

which is equivalent to $s^* \in N(\operatorname{dom}\mathfrak{I}_{f,\infty}, u)$. The proof is then complete. □

Our composition approach allows us to establish the following theorem which is new and which provides a sequential formula in the case $p = \infty$.

Theorem 5.2. *Let X be a separable reflexive Banach space and $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand for which there exist $\alpha(\cdot) \in L^1(T,\mathbb{R})$ and $a^*(\cdot) \in L^1(T,X^*)$ such that for a.e. t and all $x \in X$*

$$f(t,x) \ge \langle a^*(t), x\rangle + \alpha(t). \tag{5.5}$$

Assume that the measure μ is finite. Then for $\overline{x} \in \operatorname{dom} I_f$ one has $\overline{x}^* \in \partial I_f(\overline{x})$ if and only if there are sequences $(u_n)_{n\in\mathbb{N}}$ and $(u_n^*)_{n\in\mathbb{N}}$ in the spaces $L^\infty(T,X)$ and $(L^\infty(T,X^*))^*$ respectively with $u_n^* = v_n^* + s_n^*$, $v_n^* \in L^1(T,X^*)$ and $s_n^* \in L^{\text{sing}}(T,X)$ such that

(a) *$v_n^*(t) \in \partial f_t(u_n(t))$ a.e. and $s_n^* \in N(\operatorname{dom}\mathfrak{I}_{f,\infty}, u_n)$;*
(b) *$\overline{x}^* = \lim\limits_{n\to\infty} \left[\int_T v_n^*(t)\, d\mu(t) + s_n^* \circ j\right]$ in $(X^*, \|\cdot\|)$, where j denotes the canonical embedding of X into $L^\infty(T,X)$;*

(c) $\lim_{n\to\infty} \|u_n(\cdot) - \overline{x}\|_{L^\infty(X)} = 0$ *and* $\lim_{n\to\infty} \int_T f(t, u_n(t))\, d\mu(t) = I_f(\overline{x})$;

(d) $\lim_{n\to\infty} \left[\int_T < u_n^*(t), u_n(t) - \overline{x} > d\mu(t) + \langle s_n^*, u_n(\cdot) - j\overline{x}\rangle\right] = 0$.

Further, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ *in* X *converging in norm to* $\overline{x}$ *such that*

(e_1) $\lim_{n\to\infty} \|u_n(\cdot) - x_n\|_{L^\infty(X)} = 0$;

(e_2) $\lim_{n\to\infty} \|u_n^*\| \cdot \|u_n(\cdot) - x_n\|_{L^\infty(X)} = 0$.

Proof. Let $\overline{x}^*$ in $\partial I_f(\overline{x})$. Writing $I_f = \mathfrak{I}_{f,\infty} \circ j$ and arguing as in the proof of Theorem 3.2, we obtain there exist $u_n^* \in (L^\infty(T, X^*))^*$, $u_n \in L^\infty(T, X)$, and $x_n \in X$ such that

(a$'$) $u_n^* \in \partial\mathfrak{I}_{f,\infty}(u_n)$;

(b$'$) $u_n^* \circ j \overset{\|\cdot\|}{\to} \overline{x}^*$;

(c$'$) $u_n(\cdot) \to j\overline{x}$ in norm in $L^\infty(T, X)$ and $\mathfrak{I}_{f,\infty}(u_n) \to \mathfrak{I}_{f,\infty}(j\overline{x})$;

(d$'$) $\langle u_n^*, u_n(\cdot) - j\overline{x}\rangle_{(L^\infty(X^*))^*, L^\infty(X)} \to 0$;

(e$'$) $x_n \overset{\|\cdot\|}{\to} \overline{x}$ and $\|u_n^*\|_{(L^\infty(X^*))^*} \cdot \|u_n(\cdot) - jx_n\|_{L^\infty(X)} \to 0$.

Taking into account the fact that $\mathfrak{I}_{f,\infty}$ is finite at some point (here $j\overline{x}$), (a$'$) and Proposition 5.1 entail that for each integer n

$$v_n^*(t) \in \partial f_t(u_n(t)) \quad a.e.$$

The condition (c$'$) gives us on the one hand

$$\mathfrak{I}_{f,p}(u_n) = \int_T f(t, u_n(t))\, d\mu(t) \to I_f(\overline{x})$$

and on the other hand $\|u_n(\cdot) - \overline{x}\|_{L^\infty(X)} \to 0$ and hence according to (e$'$) we have that $\|u_n(\cdot) - x_n\|_{L^\infty(X)} \to 0$. The assertion (d$'$) can be translated into

$$\lim_{n\to+\infty} \left[\int_T \langle u_n^*(t), u_n(t) - \overline{x}\rangle\, d\mu(t) + \langle s_n^*, u_n(\cdot) - j\overline{x}\rangle\right] = 0$$

and the second part of assertion (e$'$) into

$$\lim_{n\to+\infty} \|u_n^*\| \cdot \|u_n(\cdot) - x_n\|_{L^\infty(X)} = 0.$$

Finally to obtain the assertion (b) of the theorem, we observe that for all $x \in X$ and $n \in \mathbb{N}$, like in the proof of Theorem 3.2, we have $v_n^* \circ j = \int_T v_n^*(t)\, d\mu(t)$. So according to (b$'$) we conclude that

$$\overline{x}^* = \lim_{n\to\infty} \left[\int_T u_n^*(t)\, d\mu(t) + s_n^* \circ j\right] \quad \text{in} \quad (X^*, \|\cdot\|). \qquad \square$$

When the separable Banach space X is *nonreflexive*, arguing as in section 4, we see that Theorem 5.2 still holds with nets $(u_i)_{i\in I}$, $(u_i^*)_{i\in I}$, $(v_i^*)_{i\in I}$, and $(s_i^*)_{i\in I}$ in place of sequences and the $w(X^*, X)$-topology in place of the $\|\ \|$-topology in (b). Indeed, the representation of the topological dual of $L^\infty(T, X)$ is given (see e.g. [2, Theorem VII.5]) by (5.1) with $L^1_{w^*}(T, X^*)$ in place of $L^1(T, X^*)$. Therefore, it is enough to use Theorem 4.1 in place of Theorem 3.1 and the result corresponding to Theorem 5.1 with $L^1_{w^*}(T, X^*)$ in place of $L^1(T, X^*)$ obtained by following the proof of Theorem 10 in [14].

Our aim now is to show how classical formulas under qualification conditions can be derived from the above sequential formulas for integral sums of convex functions. We begin with the following result of Ioffe and Levin [7, Theorem 3, p. 23]. The theorem is proved below with the assumption that the separable Banach space X is reflexive. The case of a *general separable Banach space* (not necessarily reflexive) is easily deduced via arguments similar to those above just past the proof of Theorem 5.2.

Theorem 5.3 (Ioffe & Levin [7])**.** *Let X be a separable reflexive Banach space and $f : T \times X \to \mathbb{R} \cup \{+\infty\}$ be a normal convex integrand. Assume that the hypotheses of Theorem 5.2 hold and assume that the following qualification condition*

(QC) *there exists some $x_0 \in X$ such that $\mathfrak{I}_{f,\infty}$ is bounded from above on some neighborhood of x_0 in $(L^\infty(T,X), \|\cdot\|_{L^\infty(X)})$*

also holds.

Then for any $\overline{x} \in \operatorname{dom} I_f$ one has

$$\partial I_f(\overline{x}) = \int_T \partial f_t(\overline{x})\, d\mu(t) + N(\operatorname{dom} I_f, \overline{x}),$$

where $\int_T \partial f_t(\overline{x})\, d\mu(t) := \{\int_T y^*(t)\, d\mu(t) : y^* \in L^1(T, X^*) \text{ and } y^*(t) \in \partial f_t(\overline{x}) \text{ a.e.}\}$.

Proof. The inclusion of the second member into the first one is not difficult to be verified. To prove the reverse inclusion, let us fix $\overline{x}$ and $\overline{x}^*$ with $\overline{x}^* \in \partial I_f(\overline{x})$. Theorem 5.2 yields the existence of sequences $(u_n^*)_n$ in $(L^\infty(T,X))^*$ and $(u_n)_n$ in $(L^\infty(T,X)$ such that

(a) $u_n^* \in \partial \mathfrak{I}_{f,\infty}(u_n)$;
(b) $u_n^* \circ j \to \overline{x}^*$ in $(X^*, \|\cdot\|)$ where j is the canonical embedding of X into $L^\infty(T,X)$;
(c) $\lim\limits_{n\to\infty} \|u_n(\cdot) - \overline{x}\|_{L^\infty(X)} = 0$ and $\lim\limits_{n\to\infty} \mathfrak{I}_{f,\infty}(u_n) = \mathfrak{I}_{f,\infty}(j\overline{x})$;
(d) $< u_n^*, u_n - j\overline{x} >_{(L^\infty(X))^*, L^\infty(X)} \to 0$.

By the qualification condition (QC) there are some $c > 0$ and $\varepsilon > 0$ such that for all $u \in L^\infty(T,X)$ with $\|u\|_{L^\infty(X)} \le \epsilon$ the inequality

$$|\mathfrak{I}_{f,\infty}(jx_0 + u)| \le c$$

holds. Then by (a), for any $u \in L^\infty(T,X)$ with $\|u\|_{L^\infty(X)} \le \varepsilon$ and all $n \in \mathbb{N}$ we have

$$\langle u_n^*, u\rangle_{(L^\infty)^*, L^\infty} \le \mathfrak{I}_{f,\infty}(jx_0 + u) - \mathfrak{I}_{f,\infty}(u_n) - \langle u_n^*, jx_0 - u_n\rangle_{(L^\infty)^*, L^\infty}$$

and consequently

$$\langle u_n^*, u\rangle_{(L^\infty)^*, L^\infty} \le c - \mathfrak{I}_{f,\infty}(u_n) - \langle u_n^*, jx_0 - u_n\rangle_{(L^\infty)^*, L^\infty}.$$

Write this inequality in the form $\langle u_n^*, u\rangle_{(L^\infty)^*, L^\infty} \le c + \theta_n$ where $\theta_n := -\mathfrak{I}_{f,\infty}(u_n) - \langle u_n^*, jx_0 - u_n\rangle_{(L^\infty)^*, L^\infty}$. The assertions (b) , (c) , (d) imply that the sequence $(\theta_n)_n$ converges in $\mathbb{R}$ and hence the sequence $(c + \theta_n)_{n \in \mathbb{N}}$ is bounded from above by some real number $M \ge 0$. This yields for any $n \in \mathbb{N}$ that

$$|\langle u_n^*, u\rangle_{(L^\infty)^*, L^\infty}| \le \frac{M}{\epsilon}\|u\| \quad \text{for all } u \in L^\infty(T,X)$$

and hence $\|u_n^*\| \leq \frac{M}{\epsilon}$. The Banach-Alaoglu-Bourbaki theorem allows us to take some subnet $(u_{k(i)}^*)_{i\in I}$ converging in the $w((L^\infty(X))^*, L^\infty(X))$-topology to some $u^* = v^* + s^* \in L^1(T, X^*) \oplus L^{sing}(T, X)$. Using this convergence and (b) we obtain

$$u_{k(i)}^* \circ j \to v^* \circ j + s \circ j = \overline{x}^*. \tag{5.6}$$

Further, writing

$$|\langle u_{k(i)}^*, u - u_{k(i)}\rangle - \langle u^*, u - j\overline{x}\rangle| \leq |\langle u_{k(i)}^* - u^*, u - j\overline{x}\rangle| + |\langle u_{k(i)}^*, j\overline{x} - u_{k(i)}\rangle|,$$

we see by (d) and by the $w((L^\infty(X))^*, L^\infty(X))$-convergence of $(u_{k(i)}^*)$ to u^* that

$$\lim_{i\in I}\langle u_{k(i)}^*, u - u_{k(i)}\rangle_{(L^\infty)^*,L^\infty} = \langle u^*, u - j\overline{x}\rangle_{(L^\infty)^*,L^\infty}. \tag{5.7}$$

In other respects , we know by the definition of subdifferential that for all $i \in I$ and all $u \in L^\infty(T, X)$

$$\langle u_{k(i)}^*, u - u_{k(i)}\rangle_{(L^\infty)^*,L^\infty} \leq \mathfrak{I}_{f,\infty}(u) - \mathfrak{I}_{f,\infty}(u_{k(i)})$$

and then by (c) and (5.7)

$$\langle u^*, u - j\overline{x}\rangle_{(L^\infty)^*,L^\infty} \leq \mathfrak{I}_{f,\infty}(u) - \mathfrak{I}_{f,\infty}(j\overline{x}),$$

that is $v^* + s^* = u^* \in \partial\mathfrak{I}_{f,\infty}(j\overline{x})$. Proposition 5.1 then says that $v^*(t) \in \partial f_t(\overline{x})$ a.e. and $s^* \in N(\operatorname{dom}\mathfrak{I}_{f,\infty}, j\overline{x})$. On the one hand, since $v^* \circ j = \int_T v^*(t)\,d\mu(t)$ like in the proof of Theorem 3.2, we have $v^* \circ j \in \int_T \partial f_t(j\overline{x})\,d\mu(t)$. On the other hand, observing that, for every $x \in \operatorname{dom} I_f$, we have $jx \in \operatorname{dom}\mathfrak{I}_{f,\infty}$, we see through the inclusion $s^* \in N(\operatorname{dom}\mathfrak{I}_{f,\infty}, j\overline{x})$ that

$$0 \geq \langle s^*, jx - j\overline{x}\rangle_{(L^\infty)^*,L^\infty} = \langle s^* \circ j, x - \overline{x}\rangle,$$

that is, $s^* \circ j \in N(\operatorname{dom} I_f, \overline{x})$. Taking the equality in the second member of (5.6) into account, we see that the proof is complete. □

In the case where X is a finite dimensional space the qualification condition (QC) can be rewritten to yield the following result of Ioffe and Tikhomirov [8].

Corollary 5.1. *Let $f : T \times X \to \mathbb{R}\cup\{+\infty\}$ be a normal convex integrand. In addition to the hypotheses of Theorem 5.2, assume that the space X is finite dimensional and that the following qualification condition*

(QC_0) *the interior in X of $\operatorname{dom} I_f$ is non empty*

also holds.

Then for any $\overline{x} \in \operatorname{dom} I_f$ one has

$$\partial I_f(\overline{x}) = \int_T \partial f_t(\overline{x})\,d\mu(t) + N(\operatorname{dom} I_f, \overline{x}).$$

Proof. We may suppose that $X = \mathbb{R}^m$ and that $\|\cdot\|$ is the supremum (box) norm. The assumptions and the qualification condition (QC_0) ensure that the convex function I_f is finite over a neighborhood in X of some point x_0, that is, there exist some positive number r such that $I_f(x_0 + x)$ is finite for all $x \in X$ with $\|x\| \leq r$. Fix a finite set of points $x_1, \cdots, x_N$ of X with $\|x_k\| \leq r$ for $k = 1, \cdots, N$ and whose convex hull contains the closed ball of X centered at the origin and with radius r.

Fix also any $u(\cdot) \in L^\infty(T,X)$ with $\|u\|_{L^\infty(X)} < r$. Then for a.e. $t \in T$ we have $\|u(t)\| \le r$ and hence the convexity of $f(t,\cdot)$ allows us to write

$$\int_T f(t, x_0 + u(t))\, d\mu(t) \le \int_T \max_{1\le k\le N} f(t, x_0 + x_k)\, d\mu(t).$$

Denoting by j the embedding mapping of X into $L^\infty(T,X)$ we see that $\mathfrak{I}_{f,\infty}$ is bounded from above on some neighborhood of jx_0 in $L^\infty(T,X)$. Thus the corollary follows from Theorem 5.3. □

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