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COUPLING VISCOSITY METHODS WITH THE EXTRAGRADIENT ALGORITHM FOR SOLVING EQUILIBRIUM PROBLEMS

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Abstract. We make use of the viscosity technique to develop a convergent method for approximating a common element of the set of fixed points of a demicontractive operator and the set of solutions of a monotone equilibrium problem. The proposed algorithm is obtained by coupling a modified hybrid steepest descent method with the extragradient algorithm. Under mild conditions, the strong convergence of the sequences generated by the algorithm is obtained. Using this result we obtain two corollaries which improve or develop several corresponding results in this field.

1. Introduction

Throughout this paper, $\mathcal{H}$ is a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $| \,.\, |$. Let C be a nonempty closed convex subset of $\mathcal{H}$ and denote by S_F the set of solutions of the following equilibrium problem:

$$\text{find} \quad u \in C \quad \text{such that} \quad F(u,y) \geq 0, \quad \forall y \in C, \tag{1.1}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction.

Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minmax problems, Nash equilibrium problem in noncooperative games and others (see, for instance [2], [27] and the references quoted therein). In recent years, methods for solving equilibrium problems have been studied extensively. In [18], Moudafi extended the proximal method to monotone equilibrium problems and in [9] Konnov used the proximal method to solve problem (1.1) with weakly monotone bifunctions. Recently, Mastroeni in [16] extended the so-called auxiliary problem principle to (1.1) involving strong monotone equilibrium problems. Other solution methods such as bundle methods and extragradient methods are extended to (1.1) in [27] and [28].

Now, consider a (possibly) nonlinear mapping $T : \mathcal{H} \to \mathcal{H}$ with a fixed point set denoted by $Fix(T) := \{x \in \mathcal{H}; \quad Tx = x\}$ and satisfying $Fix(T) \cap S_F \neq \emptyset$. In this paper, we are interested in approximating a solution of the mixed problem:

$$\text{find} \quad u \in S_F \cap Fix(T). \tag{1.2}$$

It is worth noting that numerous algorithms were proposed for solving fixed point problems for nonexpansive and even more general mappings [1, 7, 11, 13, 19, 22, 31, 32]. Other numerical methods were proposed for solving (1.2) in the special case

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when $F(x,y) = \langle Ax, y-x\rangle$, where $A : C \to \mathcal{H}$ is a monotone and Lipschitz continuous mapping and T is nonexpansive. In this latter case, the proposed methods can be regarded as a suitable combination of the extra-gradient method initiated in [10] and either a Mann's type iteration [14, 20], an Halpern's type process [6, 33] or the hybrid steepest descent method [21, 29]. Very recently, a numerical approach was considered in [24] for solving the more general problem (1.2) where the bifunction F verifies the following usual conditions:

(A1) $F(x,x) = 0 \quad$ for all $\quad x, y \in C$;
(A2) F is monotone, i.e., $\quad F(x,y) + F(y,x) \leq 0 \quad$ for all $\quad x, y \in C$;
(A3) $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$for any $x, y, z \in C$;
(A4) for each $x \in C$, $y \to F(x,y)$ is convex and lower-semicontinuous.

The main result of [24] can be summarized as follows.

Theorem 1.1. ([24], Theorem 3.2) *Let C be a nonempty closed convex subset of $\mathcal{H}$. Let F be a bi-function from $C \times C$ to $\mathbb{R}$ satisfying [*(A1)-(A4)*] and let T be a nonexpansive mapping of C into $\mathcal{H}$ such that $S_F \cap Fix(T) \neq \emptyset$. Let f be a contraction of $\mathcal{H}$ into itself and let (x_n) and (u_n) be sequences generated by $x_0 \in \mathcal{H}$ and*

$$\left[\begin{array}{l} F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) T u_n, \end{array}\right. \tag{1.3}$$

for all $n \in \mathbb{N}$, where $(\alpha_n) \subset (0,1]$ and $(r_n) \subset (0,\infty)$ satisfy:

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_n \alpha_n = \infty, \quad \sum_n |\alpha_{n+1} - \alpha_n| < \infty,$$
$$\liminf_{n\to\infty} r_n > 0, \quad \textit{and} \quad \sum_n |r_{n+1} - r_n| < \infty.$$

Then, (x_n) and (y_n) converge strongly to z in $Fix(T) \cap S_F$, where $z = P_{Fix(T)\cap S_F} f(z)$.

Motivated by the above work and based upon the extragradient method [10, 20, 33], we propose an alternative method for solving (1.2) in the more general case when T is demicontractive and demi-closed. Then, we prove a strong convergence theorem which improves or develops several corresponding results in this field.

First of all we recall that T is demicontractive means that there exists a constant $\beta \in [0,1)$ such that $|Tx - q|^2 \leq |x-q|^2 + \beta|x - Tx|^2$, for all $(x,q) \in \mathcal{H} \times Fix(T)$, which is equivalent to (see [17])

$$\langle x - Tx, x - q\rangle \geq \frac{1-\beta}{2}|x - Tx|^2, \quad \forall (x,q) \in \mathcal{H} \times Fix(T). \tag{1.4}$$

Let us also recall that T is called demi-closed (see [5]) if for any sequence $(z_k) \subset \mathcal{H}$ and $z \in \mathcal{H}$, we have:

$$z_k \to z \text{ weakly}, \quad (I - T)(z_k) \to 0 \text{ strongly} \quad \Rightarrow \quad z \in Fix(T).$$

An operator satisfying (1.4) will be refered to as a β-demicontractive mapping. It is worth noting that the class of demicontractive maps contains important operators such as the quasi-nonexpansive maps and the strictly pseudocontractive maps with fixed points (see [7, 15, 17]). Finally, let us recall that a mapping $T : \mathcal{H} \to \mathcal{H}$ is called:

i) *nonexpansive* if $|Tx - Ty| \leq |x - y|$ for all $(x, y) \in \mathcal{H} \times \mathcal{H}$;

ii) *quasi-nonexpansive* if $|Tx - q| \leq |x - q|$ for all $(x, q) \in \mathcal{H} \times Fix(T)$;

iii) *strictly pseudocontractive* if $|Tx - Ty|^2 \leq |x - y|^2 + \rho|x - y - (Tx - Ty)|^2$ for all $(x, y) \in \mathcal{H} \times \mathcal{H}$ (for some $\rho \in [0, 1)$).

Observe also that the nonexpansive operators are both quasi-nonexpansive and strictly pseudocontractive maps and are well-known for being demi-closed.

In view of selecting a particular solution of (1.2), we consider an operator $\mathcal{F} : C \to \mathcal{H}$ satisfying the following two conditions:

(LC): $\mathcal{F}$ is L-Lipschitz continuous (for some $L > 0$), i.e. $|\mathcal{F}(x) - \mathcal{F}(y)| \leq L|x - y|$ for all $x, y \in C$;

(SM): $\mathcal{F}$ is η-strongly monotone (for some $\eta > 0$), i.e. $\langle \mathcal{F}(x) - \mathcal{F}(y), x - y\rangle \geq \eta|x - y|^2$ for all $x, y \in C$,

and we investigate the asymptotic behavior of the sequence (x_n) generated, from an arbitrary x_0 in $\mathcal{H}$, by the following algorithm:

$$(1.5)\qquad \left[\begin{array}{l} \bullet\quad x_0 \in \mathcal{H}; \\ \\ \bullet\quad \text{compute } u_n \text{ such that:} \\ \\ \qquad\qquad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ \\ \bullet\quad x_{n+1} := [(1 - w)I + wT]v_n, \quad v_n := u_n - \alpha_n \mathcal{F}(u_n), \end{array}\right.$$

where T is assumed to be demicontractive, $I : \mathcal{H} \to \mathcal{H}$ stands for the identity mapping and the parameters are such that: $(\alpha_n) \subset [0, 1)$, $(r_n) \subset (0, \infty)$ and $w \in (0, 1)$.

More precisely, we will prove that the limit is the solution to the following well-posed variational inequality problem $VIP(\mathcal{F}, S_F \cap Fix(T))$:

$$(1.6)\qquad \text{find } x_* \in Fix(T) \cap S_F \text{ such that } \langle v - x_*, \mathcal{F}(x_*)\rangle \geq 0, \quad \forall v \in Fix(T) \cap S_F.$$

It is worth mentioning that the existence and the uniqueness of the solution of (1.6) are ensured by the conditions (LC), (SM) and by the fact that $S_F \cap Fix(T)$ is a nonempty closed and convex set.

We would like to emphasize that when $F \equiv 0$ and $C = \mathcal{H}$, (1.5) reduces to a modified version of the hybrid steepest descent method recently investigated in [13] as an algorithmic solution for solving $VIP(\mathcal{F}, Fix(T))$. The convergence in norm of the iterates generated by this scheme is obtained in the more general case when T is demicontractive. On the other hand, we would like to emphasize that the relaxation process induced by the mapping $(1 - w)I + wT$ in (1.5) was mainly suggested by the work of Suzuki [22] (see also [7, 14, 11]) and permits to relax substantially the conditions on parameters α_n and r_n. Finally, let us notice that when $C = \mathcal{H}$ and $F(x, y) = \max_{u \in Ax}\langle u, y - x\rangle$, where A is a maximal monotone operator, (1.1) amounts to finding zeroes of the operator A and the sequence u_n given by (1.5) is nothing but the resolvent operator associated to A at x_n, namely

$u_n = J^A_{r_n} x_n = (I + r_n A)^{-1} x_n$ so that the algorithm (1.5) reduces to $x_{n+1} := [(1-w)I + wT]v_n$ with $v_n := (I - \alpha_n \mathcal{F})(J^A_{r_n} x_n)$.

Under classical assumptions on the operators and the parameters, we will prove that the sequences (x_n) and (u_n) generated by the scheme (1.5) converge strongly to the unique solution of (1.6). Thus by algorithm (1.5), we provide an efficient selecting method for solving the initial mixed problem (1.2) for a new broad class of maps. Moreover, the techniques of proofs are simple and different from the usual ones.

2. Preliminaries

We begin with the following preliminary results (see [2] and [8]).

Lemma 2.1. *Let C be a nonempty closed convex subset of $\mathcal{H}$ and let F be a bi-function from $C \times C$ into $\mathbb{R}$ satisfying [(A1)-(A4)].*
i) *Let $r > 0$ and $x \in \mathcal{H}$. Then there exists $z \in C$ such that:*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

ii) *Let $T_r : \mathcal{H} \to C$ be the mapping defined by*

$$T_r(x) = \{z \in C : \quad F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C\}.$$

Then the following hold:

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, i.e.,*
$|T_r x_1 - T_r x_2|^2 \leq \langle T_r x_1 - T_r x_2, x_1 - x_2\rangle$;
(3) *$Fix(T_r) = S_F$ (S_F being the set of solutions of (1.2));*
(4) *S_F is closed and convex.*

The following lemmas, which appear implicitly in [24], will be needed in the sequel.

Lemma 2.2. *Assume that (x_n) and (u_n) are two sequences in $\mathcal{H}$ verifying $r_n > 0$ and $u_n = T_{r_n} x_n$ for all $n \geq 0$. Then:*

$$|u_n - u|^2 \leq |x_n - u|^2 - |x_n - u_n|^2, \quad \forall n \geq 0, \tag{2.1}$$

where u is any element in S_F.

Proof. Indeed, for any u in S_F, we successively have

$$\begin{aligned} |u_n - u|^2 &= |T_{r_n} x_n - T_{r_n} u|^2 \\ &\leq \langle T_{r_n} x_n - T_{r_n} u, x_n - u\rangle \\ &= \langle u_n - u, x_n - u\rangle \\ &= \frac{1}{2}\left(|u_n - u|^2 + |x_n - u|^2 - |x_n - u_n|^2\right), \end{aligned}$$

which clearly leads to (2.1). □

Lemma 2.3. *Let (x_n) and (u_n) are two sequences in $\mathcal{H}$ verifying $u_n = T_{r_n}x_n$ for all $n \geq 0$ and assume that $r_n \in [\delta, \infty)$ for some $\delta > 0$. If, in addition, there exists a subsequence (u_{n_k}) of (u_n) such that:*
i) *(u_{n_k}) converges weakly to some u in $\mathcal{H}$;*
ii) *$|u_{n_k} - x_{n_k}| \to 0$,*
then u belongs in S_F.

Proof. Since $u_n = T_{r_n}x_n$, for any $y \in C$, we can write

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

Monotonicity of F together with property (A2) yields

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq -F(u_n, y) \geq F(y, u_n),$$

which, by replacing n by n_k, implies

$$\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\rangle \geq F(y, u_{n_k}).$$

As (u_{n_k}) is a bounded sequence (since it is weakly convergent), by passing to the limit in the previous inequality and by taking into account that $\frac{u_{n_k} - x_{n_k}}{r_{n_k}} \to 0$ and $u_{n_k} \rightharpoonup u$ weakly, we deduce that

$$0 \geq F(y, u).$$

Now, by setting $y_t = ty + (1-t)u$ (for $t \in (0, 1]$) and thanks to the fact that $y \in C$ and $u \in C$, we have $y_t \in C$ so that $F(y_t, u) \leq 0$.
Hence by virtue of (A1) and (A4), we get

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, u) \leq tF(y_t, y).$$

Consequently, we deduce

$$F(y_t, y) \geq 0,$$

which in the light of (A3) entails

$$F(u, y) \geq 0, \quad \forall y \in C,$$

thus $u \in S_F$. □

Now, let us state a key property of the relaxed operator $T_w := (1-w)I + wT$.

Remark 2.4. (See also [13]) Let T be a β-demicontractive self-mapping on $\mathcal{H}$ with $Fix(T) \neq \emptyset$ and set $T_w := (1-w)I + wT$ for $w \in (0, 1]$. Then T_w is quasi-nonexpansive if $w \in [0, 1-\beta]$.
Indeed, for any arbitrary element $(x, q) \in \mathcal{H} \times Fix(T)$, we have

$$|T_w x - q|^2 = |x - q|^2 - 2w\langle x - q, x - Tx\rangle + w^2|Tx - x|^2,$$

which according to (1.4) yields

$$|T_w x - q|^2 \leq |x - q|^2 - w(1 - \beta - w)|Tx - x|^2. \tag{2.2}$$

Furthermore, we clearly have $Fix(T) = Fix(T_w)$ if $w \neq 0$. As a consequence, the operator T_w is quasi-nonexpansive for $w \in [0, 1-\beta]$ and $Fix(T)$ is then a closed convex subset of $\mathcal{H}$ (see [32], Proposition 1).

The following lemma shows that the sequences (x_n) and (u_n) generated by (1.5) are bounded.

Lemma 2.5. *Suppose $T : \mathcal{H} \to \mathcal{H}$ is β-demicontractive with $S_F \cap Fix(T) \neq \emptyset$ and let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be an operator satisfying* (LC) *and* (SM). *Assume, in addition, that $w \in (0, 1-\beta]$, $(r_n) \subset (0,\infty)$ and $(\alpha_n) \subset [0, \delta)$ (for some small enough $\delta > 0$). Then the sequences (x_n) and (u_n) generated by (1.5) are bounded.*

Proof. Without loss of generality, we may assume $0 < \eta < L$. Given $\mu \in (0,\infty)$ and $x, y \in \mathcal{H}$, by using properties (SM) and (LC), we can write

$$\begin{aligned} &|(\mu\mathcal{F} - I)(x) - (\mu\mathcal{F} - I)(y)|^2 \\ &\qquad = \mu^2 |\mathcal{F}(x) - \mathcal{F}(y)|^2 - 2\mu\langle x - y, \mathcal{F}(x) - \mathcal{F}(y)\rangle + |x - y|^2 \\ &\qquad \leq \mu^2 L^2 |x - y|^2 - 2\mu\eta |x - y|^2 + |x - y|^2, \end{aligned}$$

so that

$$|(\mu\mathcal{F} - I)(x) - (\mu\mathcal{F} - I)(y)| \leq (\sqrt{1 - 2\mu\eta + \mu^2 L^2})|x - y|. \tag{2.3}$$

Furthermore, taking $q \in Fix(T) \cap S_F$ and recalling that $v_n = u_n - \alpha_n \mathcal{F}(u_n)$, we have

$$\begin{aligned} |v_{n+1} - (q - \alpha_{n+1}\mathcal{F}(q))| &= |(u_{n+1} - \alpha_{n+1}\mathcal{F}(u_{n+1})) - (q - \alpha_{n+1}\mathcal{F}(q))| \\ &= \left|(1 - \frac{\alpha_{n+1}}{\mu})(u_{n+1} - q) - \frac{\alpha_{n+1}}{\mu}((\mu\mathcal{F} - I)(u_{n+1}) - (\mu\mathcal{F} - I)(q)\right| \\ &\leq (1 - \frac{\alpha_{n+1}}{\mu})|u_{n+1} - q| + \frac{\alpha_{n+1}}{\mu}|(\mu\mathcal{F} - I)(u_{n+1}) - (\mu\mathcal{F} - I)(q)|, \end{aligned}$$

provided that $(\alpha_n) \subset [0, \mu)$, which by (2.3) yields

$$|v_{n+1} - (q - \alpha_{n+1}\mathcal{F}(q))| \leq (1 - \frac{\alpha_{n+1}}{\mu}\nu)|u_{n+1} - q|, \tag{2.4}$$

where $\nu := 1 - \sqrt{1 - 2\mu\eta + \mu^2 L^2}$. Clearly, we have that $\nu \in (0, 1)$ when $\mu \in (0, \mu_0)$ for some small enough μ_0. Using (2.1) and observing that $T_w := (1 - w)I + wT$ is quasi-nonexpansive for $w \in (0, 1 - \beta]$ (see remark 2.4), by (2.1) we additionally have

$$|u_{n+1} - q| \leq |x_{n+1} - q| = |T_w v_n - q| \leq |v_n - q|. \tag{2.5}$$

Combining (2.4) and (2.5), we then get

$$|v_{n+1} - (q - \alpha_{n+1}\mathcal{F}(q))| \leq (1 - \frac{\alpha_{n+1}}{\mu}\nu)|v_n - q|.$$

As a consequence, we deduce

$$\begin{aligned} |v_{n+1} - q| &\leq |v_{n+1} - (q - \alpha_{n+1}\mathcal{F}(q))| + |(q - \alpha_{n+1}\mathcal{F}(q)) - q| \\ &\leq (1 - \frac{\alpha_{n+1}\nu}{\mu})|v_n - q| + \alpha_{n+1}|\mathcal{F}(q)| \\ &= (1 - \frac{\alpha_{n+1}\nu}{\mu})|v_n - q| + (\frac{\alpha_{n+1}\nu}{\mu})(\frac{\mu|\mathcal{F}(q)|}{\nu}), \end{aligned}$$

and hence

$$\max\left\{|v_{n+1} - q|, \frac{\mu|\mathcal{F}(q)|}{\nu}\right\} \leq \max\left\{|v_n - q|, \frac{\mu|\mathcal{F}(q)|}{\nu}\right\},$$

so that for all $n \geq 0$,

$$|v_n - q| \leq \max\left\{|v_0 - q|, \frac{\mu|\mathcal{F}(q)|}{\nu}\right\}. \tag{2.6}$$

Thus (v_n) is bounded, which by (2.5) leads to the boundedness of (x_n) and (u_n). □

3. Main convergence results

In order to prove our main convergence theorem, we start with some key preliminary results.

Lemma 3.1. *Suppose $T : \mathcal{H} \to \mathcal{H}$ be a β-demicontractive mapping with $Fix(T) \cap S_F \neq \emptyset$ and let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be a given operator. Assume in addition that the following condition holds:*

(H1) : $w \in (0, \frac{1-\beta}{2}]$.

Then, for all $n \geq 0$ the sequences (x_n) and (u_n), given by (1.5), satisfy the following inequality:

$$\begin{aligned} &|x_{n+1} - q|^2 - |x_n - q|^2 \\ &\quad +|x_{n+1} - u_n|^2 + |x_n - u_n|^2 \leq -2\alpha_n\langle x_{n+1} - q, \mathcal{F}(u_n)\rangle, \end{aligned} \tag{3.1}$$

where q is any element in $Fix(T) \cap S_F$.

Proof. Let $q \in Fix(T) \cap S_F$. From (1.5) and (2.2) we obtain

$$|x_{n+1} - q|^2 \leq |v_n - q|^2 - w(1 - \beta - w)|v_n - Tv_n|^2 \tag{3.2}$$

and by virtue of (1.5) we also have $Tv_n - v_n = \frac{1}{w}(x_{n+1} - v_n)$. Consequently, setting $\rho := \frac{1}{w}(1 - \beta - w)$, we obtain

$$|x_{n+1} - q|^2 \leq |v_n - q|^2 - \rho|x_{n+1} - v_n|^2, \tag{3.3}$$

hence if $w \in (0, \frac{1-\beta}{2}]$ (so that $\rho \geq 1$) we get

$$\begin{aligned} |x_{n+1} - q|^2 &\leq |v_n - q|^2 - |x_{n+1} - v_n|^2 \\ &= |(u_n - q) - \alpha_n\mathcal{F}(u_n)|^2 - |(u_n - x_{n+1}) - \alpha_n\mathcal{F}(u_n)|^2 \\ &= |u_n - q|^2 - 2\alpha_n\langle x_{n+1} - q, \mathcal{F}(u_n)\rangle - |x_{n+1} - u_n|^2. \end{aligned} \tag{3.4}$$

Furthermore, thanks to (2.1) we have

$$|u_n - q|^2 \leq |x_n - q|^2 - |x_n - u_n|^2, \tag{3.5}$$

which, combined with (3.4), entails the desired result. □

Lemma 3.2. *Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be any operator satisfying* (LC). *Suppose that $T : \mathcal{H} \to \mathcal{H}$ is demi-closed with $Fix(T) \cap S_F \neq \emptyset$ and that the following conditions on the parameters hold:*

(C1): $(r_n) \subset [\delta, \infty)$ *(for some $\delta > 0$);*

(H2): $(\alpha_n) \subset [0, 1)$, $\alpha_n \to 0$.

Let (x_n), (u_n) be the sequences generated by (1.5) and assume further the existence of a subsequence (u_{n_k}) of (u_n) such that:

i) $|u_{n_k} - x_{n_k+1}| \to 0$; ii) $|u_{n_k} - x_{n_k}| \to 0$.

Then any weak-cluster point of (u_{n_k}) belongs to $Fix(T) \cap S_F$. Moreover, if in addition (u_{n_k}) is bounded, then

$$\liminf_{k\to\infty} \langle u_{n_k} - u_*, \mathcal{F}(u_*)\rangle \geq 0, \tag{3.6}$$

where u_* is any solution of (1.6).

Proof. Let $u \in \mathcal{H}$ be a weak-cluster point of (u_{n_k}). Then there exists a bounded subsequence of (u_{n_k}) (labeled (u_{m_k})) which weakly converges to u. By i) and ii), we also have $|x_{m_k+1} - u_{m_k}| \to 0$ and $|x_{m_k} - u_{m_k}| \to 0$. If in addition $\alpha_n \to 0$, we easily deduce that $v_{m_k} := u_{m_k} - \alpha_{m_k}\mathcal{F}(u_{m_k})$ weakly converges to u (because $\mathcal{F}(u_{m_k})$ is bounded thanks to (LC)), hence $\alpha_{m_k}|\mathcal{F}(u_{m_k})| \to 0$), which by (1.5) entails

$$|Tv_{m_k} - v_{m_k}| = \frac{1}{w}|x_{m_k+1} - v_{m_k}| = \frac{1}{w}|(x_{m_k+1} - u_{m_k}) + \alpha_{m_k}\mathcal{F}(u_{m_k})| \to 0.$$

Now, as T is assumed to be demi-closed, we then obtain $u \in Fix(T)$. Furthermore, recalling that $|x_{m_k} - u_{m_k}| \to 0$ and assuming $r_n \geq \delta > 0$, by Lemma 2.3 we get $u \in S_F$. Consequently, the set of weak cluster points of (u_{n_k}) is included in $S_F \cap Fix(T)$. If (u_{n_k}) is also a bounded sequence, so is the quantity $\langle u_{n_k} - u_*, \mathcal{F}(u_*)\rangle$. It is then immediate that there exists a subsequence of (u_{n_k}) (denoted (u_{m_k})) which converges weakly to some element v in $\mathcal{H}$ (hence $v \in Fix(T) \cap S_F$) and such that $\liminf_{k\to\infty}\langle u_{n_k} - u_*, \mathcal{F}(u_*)\rangle = \lim_{k\to\infty}\langle u_{m_k} - u_*, \mathcal{F}(u_*)\rangle$. Thus, by the weak convergence of (u_{m_k}) and by reminding that u_* is the solution of (1.6), we easily deduce $\liminf_{k\to\infty}\langle u_{n_k} - u_*, \mathcal{F}(u_*)\rangle = \langle v - u_*, \mathcal{F}(u_*)\rangle \geq 0$. This ends the proof. □

Lemma 3.3. *Assume that $T : \mathcal{H} \to \mathcal{H}$ is β-demicontractive, demi-closed and such that $Fix(T) \cap S_F \neq \emptyset$; $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ satisfies* (LC) *and* (SM) *and suppose in addition that:*

(C1): $(r_n) \subset [\delta, \infty)$ *(for some $\delta > 0$);*
(H2): $(\alpha_n) \subset [0, 1)$, $\alpha_n \to 0$.

Let (x_n), (u_n) be the sequences generated by (1.5) and assume furthermore the existence of a bounded subsequence (u_{n_k}) of (u_n) such that :

i) $|u_{n_k} - x_{n_k+1}| \to 0$*;* ii) $|x_{n_k} - u_{n_k}| \to 0$*;*
iii) $\langle x_{n_k+1} - x_*, \mathcal{F}(u_{n_k})\rangle \leq 0$*, where x_* is the solution of (1.6).*

Then (x_{n_k}) and (u_{n_k}) converge strongly to x_*.

Proof. From the boundedness of (u_{n_k}), we can extract a subsequence (again labeled (u_{n_k})) which converges weakly to some q in $\mathcal{H}$ and such that i), ii) and iii) still hold. Thanks to Lemma 3.2 we infer that $q \in Fix(T) \cap S_F$. Furthermore, by (SM) we observe that

$$\begin{aligned}\eta|u_{n_k} - x_*|^2 &\leq \langle u_{n_k} - x_*, \mathcal{F}(u_{n_k}) - \mathcal{F}(x_*)\rangle \\ &= \langle x_{n_k+1} - x_*, \mathcal{F}(u_{n_k})\rangle + \langle u_{n_k} - x_{n_k+1}, \mathcal{F}(u_{n_k})\rangle - \langle u_{n_k} - x_*, \mathcal{F}(x_*)\rangle,\end{aligned}$$

which in the light of iii) entails

$$\eta|u_{n_k} - x_*|^2 \leq \langle u_{n_k} - x_{n_k+1}, \mathcal{F}(u_{n_k})\rangle - \langle u_{n_k} - x_*, \mathcal{F}(x_*)\rangle. \tag{3.7}$$

Hence by (3.7), ii) and (1.6) we obviously have

$$\limsup_{k\to+\infty} |u_{n_k} - x_*|^2 \leq -(1/\eta)\langle q - x_*, \mathcal{F}(x_*)\rangle \leq 0.$$

Therefore, we obtain $\lim_{k\to+\infty} |u_{n_k} - x_*| = 0$ and by virtue of ii) and the uniqueness of x_* we deduce that (x_{n_k}) also converge strongly to x_*, which completes the proof. □

Lemma 3.4. *Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be an operator satisfying* (LC) *and* (SM)*, suppose that $T : \mathcal{H} \to \mathcal{H}$ is β-demicontractive, demi-closed with $Fix(T) \cap S_F \neq \emptyset$ and assume that the following conditions on the parameters hold:*

(H1): $w \in (0, \frac{1-\beta}{2}]$;
(H2): $(\alpha_n) \subset [0, 1)$, $\alpha_n \to 0$;
(C2): $(r_n) \subset [\delta, \infty)$ *(for $\delta > 0$)*;
(SP): $\sum_{n\geq 0} \alpha_n = \infty$.

Assume furthermore that the sequences (x_n) and (u_n) given by (1.5) satisfy :

i) $|u_n - x_{n+1}| \to 0$; ii) $|x_n - u_n| \to 0$;
iii) $\lim_{n\to\infty} |x_n - x_*|$ *exists, x_* being the solution of (1.6).*

Then (x_n) and (u_n) converge strongly to x_*.

Proof. Using condition (SM), we obviously obtain

$$\langle x_{n+1} - x_*, \mathcal{F}(u_n)\rangle \geq \eta|u_n - x_*|^2 + \langle u_n - x_*, \mathcal{F}(x_*)\rangle + \langle x_{n+1} - u_n, \mathcal{F}(u_n)\rangle. \tag{3.8}$$

Set $\lim_{n\to\infty} |x_n - x_*| = \lambda \geq 0$. (x_n) is thus bounded and, thanks to ii), so is (u_n). Since Lemma 3.2 is applicable, we also have

$$\liminf_{n\to\infty} \langle u_n - x_*, \mathcal{F}(x_*)\rangle \geq 0. \tag{3.9}$$

Therefore, by (3.8), (3.9) and i) we get

$$\liminf_{n\to\infty} \langle x_{n+1} - x_*, \mathcal{F}(u_n)\rangle \geq \eta\lambda^2.$$

Hence, for $\varepsilon > 0$, from Lemma 3.1, we deduce that for $n \geq n_0$ (for some n_0 large enough),

$$|x_{n+1} - x_*|^2 - |x_n - x_*|^2 \leq -2\alpha_n(\eta\lambda^2 - \varepsilon).$$

This easily leads to

$$|x_{n+1} - x_*|^2 - |x_{n_0} - x_*|^2 \leq -2(\lambda^2\eta - \varepsilon) \sum_{k=n_0}^{n} \alpha_k.$$

Assuming $\sum \alpha_k = \infty$, we observe that this last inequality is absurd for $\lambda > 0$, since (x_n) is bounded. As a straightforward consequence, we obtain $\lambda = 0$, namely (x_n) converges strongly to x_* and according to ii) so is (u_n). This completes the proof. □

We are now in a position to give the following main convergence theorem.

Theorem 3.5. *Suppose $T : \mathcal{H} \to \mathcal{H}$ is β-demicontractive, demi-closed with $Fix(T) \cap S_F \neq \emptyset$. Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be satisfying* (LC) *and* (SM) *and assume the following conditions hold:*

(H1): $w \in (0, \frac{1-\beta}{2}]$;
(H2): $(\alpha_n) \subset [0, 1)$, $\alpha_n \to 0$;
(H3): $(r_n) \subset [\delta, \infty)$ *(where $\delta > 0$)*;
(SP): $\sum_{n\geq 0} \alpha_n = \infty$.

Then the sequences (x_n) and (u_n) generated by (1.5) converge strongly to x_*, the unique solution of the variational inequality (1.6).

Proof. The boundedness of (x_n) and (u_n) is deduced from Lemma 2.5, so that there exists a constant $C \geq 0$ such that $|\langle x_{n+1} - x_*, \mathcal{F}(u_n)\rangle| \leq C$ for all $n \geq 0$. Consequently, by Lemma 3.1 we get

$$|x_{n+1} - x_*|^2 - |x_n - x_*|^2 + |x_{n+1} - u_n|^2 + |x_n - u_n|^2 \leq 2C\alpha_n. \tag{3.10}$$

The rest of the proof can be divided into two cases:

Case 1) Assume $(|x_n - x_*|)$ is a monotone sequence. In other words, for n_0 large enough, $(|x_n - x_*|)_{n \geq n_0}$ is either non-decreasing or non-increasing and being also bounded, $(|x_n - x_*|)$ is thus convergent. Clearly, we then have $|x_{n+1} - x_*|^2 - |x_n - x_*|^2 \to 0$, which by (3.10) yields $|x_{n+1} - u_n| \to 0$ and $|x_n - u_n| \to 0$. Consequently, by Lemma 3.4 we deduce that (x_n), (u_n) converge strongly to x_*.

Case 2) Assume $(|x_n - x_*|)$ is not a monotone sequence, set $\Gamma_n := |x_n - x_*|^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be the map defined for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N}; \quad k \leq n, \quad \Gamma_k \leq \Gamma_{k+1}\}. \tag{3.11}$$

Clearly, τ is a non-decreasing sequence such that $\tau(n) \to +\infty$ (as $n \to +\infty$) and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ (for $n \geq n_0$), which by (3.10) entails

$$|x_{\tau(n)+1} - u_{\tau(n)}|^2 + |x_{\tau(n)} - u_{\tau(n)}|^2 \leq 2C\alpha_{\tau(n)} \to 0,$$

thus $|x_{\tau(n)+1} - u_{\tau(n)}| \to 0$ and $|x_{\tau(n)} - u_{\tau(n)}| \to 0$, so that $|x_{\tau(n)+1} - x_{\tau(n)}| \to 0$. Furthermore, by Lemma 3.1 we have

$$\text{for any } j \geq 0 \quad \langle x_{j+1} - x_*, \mathcal{F}(u_j)\rangle > 0 \quad \Rightarrow \quad \Gamma_{j+1} < \Gamma_j.$$

As a consequence, since $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$, we get

$$\langle x_{\tau(n)+1} - x_*, \mathcal{F}(u_{\tau(n)})\rangle \leq 0 \quad \text{for all } n \geq n_0.$$

Applying Lemma 3.3, we deduce that $|x_{\tau(n)} - x_*| \to 0$ and it is then immediate that $\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0$, since $|x_{\tau(n)+1} - x_{\tau(n)}| \to 0$. Furthermore, for $n \geq n_0$, it is easily observed that $\Gamma_n \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, if $\tau(n) < n$), because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \quad \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence $\lim_{n\to\infty} \Gamma_n = 0$, that is (x_n) converges strongly to x_*. In view of (3.10), we also obtain the strong converge of (u_n) to x_*, which completes the proof. □

We end this section with two important particular cases. First, when $F \equiv 0$, we have $u_n = P_C x_n$. So as a direct consequence of Theorem 3.5, we obtain that the sequence generated from x_0 by

$$x_{n+1} := [(1-w)I + wT](I - \alpha_n \mathcal{F})(P_C x_n)$$

converges strongly to $x^* \in FixT$ which solves the variational inequality

$$\langle v - x_*, \mathcal{F}(x_*)\rangle \geq 0, \quad \forall v \in Fix(T).$$

In the case when $T = I$, algorithm (5.1) generates from an arbitrary $x_0 \in \mathcal{H}$ two sequences (u_n) and (x_n) by the following rule

$$(3.12) \qquad \left[\begin{array}{l} \bullet \text{ compute } u_n \text{ such that:} \\ \qquad F(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C; \\ \bullet \; x_{n+1} = u_n - \alpha_n \mathcal{F}(u_n). \end{array}\right.$$

As a direct consequence of Theorem 3.5, we obtain that the sequences (x_n) and (u_n) strongly converge to $x^* \in S_F$ which solves the variational inequality

$$\langle v - x_*, \mathcal{F}(x_*)\rangle \geq 0, \quad \forall v \in S_F.$$

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