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VECTOR-VALUED WEAKLY ALMOST PERIODIC FUNCTIONS AND MEAN ERGODIC THEOREMS IN BANACH SPACES

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Abstract. We prove weak and strong mean ergodic theorems for weakly almost periodic functions (in the sense of Eberlein) which are defined on an abstract semigroup and take values in a Banach space. Using these results, we obtain weak and strong mean ergodic theorems for noncommutative semigroups of nonexpansive mappings, affine nonexpansive mappings and linear bounded operators in Banach spaces. These results are also used to obtain well-known mean ergodic theorems in cases of discrete and one-parameter semigroups of linear and nonlinear mappings in Banach spaces.

1. Introduction

Let C be a closed and convex subset of a real Banach space. Then a mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

In 1975, Baillon [3] originally proved the first nonlinear ergodic theorem in the framework of Hilbert spaces: Let C be a closed and convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means

$$S_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

converge weakly to some $y \in F(T)$. In this case, putting $y = Px$ for each $x \in C$, we have that P is a nonexpansive retraction of C onto $F(T)$ such that $PT = TP = P$ and Px is contained in the closure of convex hull of $\{T^n x : n = 1, 2, \dots\}$ for each $x \in C$. We call such a retraction *"an ergodic retraction"*. In 1981, Takahashi [29, 31] proved the existence of ergodic retractions for amenable semigroups of nonexpansive mappings on Hilbert spaces. Rodé [24] also found a sequence of means on a semigroup, generalizing the Cesàro means, and extended Baillon's theorem. These results were extended to a uniformly convex Banach space whose norm is Fréchet differentiable in the case of commutative semigroups of nonexpansive mappings by Hirano, Kido and Takahashi [12]. In 1999, Lau, Shioji and Takahashi [17] extended Takahashi's result and Rodé's result to amenable semigroups of nonexpansive mappings in the Banach space.

By using Rodé's method, Kido and Takahashi [14] also proved a mean ergodic theorem for noncommutative semigroups of linear bounded operators in Banach spaces.

On the other hand, Edelstein [10] studied a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset in a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space, let T be a nonexpansive mapping of C into itself and let $\xi \in C$. Then, for each point x of the closure of convex hull of the ω-limit set $\omega(\xi)$ of ξ, the Cesàro means $1/n\sum_{k=0}^{n-1}T^kx$ converge to a fixed point of T, where the ω-limit set $\omega(\xi)$ of ξ is the set of cluster points of the sequence $\{T^n\xi : n = 1, 2, \dots\}$. By using results of Bruck [4], Atsushiba and Takahashi [1] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a strictly convex Banach space: Let C be a compact and convex subset of a strictly convex Banach space and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1}T^kx$ converge to a fixed point of T. This result was extended to commutative semigroups of nonexpansive mappings by Atsushiba, Lau and Takahashi [2]. Suzuki and Takahashi [28] constructed a nonexpansive mapping of a compact and convex subset C of a Banach space into itself such that for some $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1}T^kx$ converge to a point of C, but the limit point is not a fixed point of T. Motivated by the example of Suzuki and Takahashi, Miyake and Takahashi [21] proved a nonlinear ergodic theorem for nonexpansive mappings on a compact and convex subset of a general Banach space: Let C be a compact and convex subset of a Banach space and let T be a nonexpansive mapping of C into itself. Then, for each $x \in C$, the Cesàro means $1/n\sum_{k=0}^{n-1}T^kx$ converge. They also proved a nonlinear ergodic theorem for semigroups of nonexpansive mappings on a compact and convex subset of a general Banach space.

In this paper, motivated Kido and Takahashi [14], Hirano, Kido and Takahashi [12], Lau, Shioji and Takahashi [17], Atsushiba, Lau and Takahashi [2] and Miyake and Takahashi [21], we first prove weak and strong mean ergodic theorems for weakly almost periodic functions (in the sense of Eberlein) which are defined on an abstract semigroup and take values in a Banach space. Using these results, we obtain weak and strong mean ergodic theorems for noncommutative semigroups of nonexpansive mappings, affine nonexpansive mappings and linear bounded operators in Banach spaces. These results are used to obtain new and well-known mean ergodic theorems in cases of discrete and one-parameter semigroups of linear and nonlinear mappings in Banach spaces.

2. Preliminaries

Throughout this paper, we denote by S a semigroup with identity and by E a real Banach space. Let $\langle E, F\rangle$ be the duality between vector spaces E and F. For each $y \in F$, we define a linear functional f_y on E by $f_y(x) = \langle x, y\rangle$. We denote by $\sigma(E, F)$ the weak topology on E generated by $\{f_y : y \in F\}$. If X is a Banach space, we denote by X^* the topological dual of X. We also denote by $\langle\cdot,\cdot\rangle$ the canonical bilinear form between E and E^*, that is, for $x \in E$ and $x^* \in E^*$, $\langle x, x^*\rangle$ is the value of x^* at x. If A is a subset of E, then the closure of convex hull of A is denoted by $\overline{\mathrm{co}}A$.

We denote by $l^\infty(S)$ the Banach space of bounded real-valued functions on S with supremum norm. For each $s \in S$, we define operators $l(s)$ and $r(s)$ on $l^\infty(S)$ by

$$(l(s)f)(t) = f(st) \quad \text{and} \quad (r(s)f)(t) = f(ts)$$

for each $t \in S$ and $f \in l^\infty(S)$, respectively. Let X be a subspace of $l^\infty(S)$ which contains constants. Then, X is said to be *translation invariant* if $l(s)f \in X$ and

$r(s)f \in X$ for each $s \in S$ and $f \in X$. A linear functional μ on X is said to be a *mean* on X if $\|\mu\| = \mu(e) = 1$, where $e(s) = 1$ for each $s \in S$. We often write $\mu_s f(s)$ instead of $\mu(f)$ for each $f \in X$. For $s \in S$, we can define a *point evaluation* δ_s by $\delta_s(f) = f(s)$ for each $f \in X$. A convex combination of point evaluations is called a *finite mean* on S. As is well known, μ is a mean on X if and only if

$$\inf_{s \in S} f(s) \leq \mu(f) \leq \sup_{s \in S} f(s)$$

for each $f \in X$; see [32] for more details. If X is translation invariant, then a mean μ on X is said to be *left invariant* (resp. *right invariant*) if $\mu(l(s)f) = \mu(f)$ (resp. $\mu(r(s)f) = \mu(f)$) for each $s \in S$ and $f \in X$. A mean μ on X is said to be *invariant* if μ is both left and right invariant. If there exists an invariant mean on X, then X is said to be *amenable*. We know from [6] that if S is commutative, then X is amenable. Let $\{\mu_\alpha\}$ be a net of means on X. Then $\{\mu_\alpha\}$ is said to be *(strongly) asymptotically invariant* if for each $s \in S$, both $l(s)^*\mu_\alpha - \mu_\alpha$ and $r(s)^*\mu_\alpha - \mu_\alpha$ converge to 0 in the weak topology $\sigma(X^*, X)$ (the norm topology), where $l(s)^*$ and $r(s)^*$ are the adjoint operators of $l(s)$ and $r(s)$, respectively. Such nets were first studied by Day [6].

We denote by $l^\infty(S, E)$ the Banach space of vector-valued functions on S that take values in a Banach space E such that for each $f \in l^\infty(S, E)$, $f(S) \subset E$ is bounded. We also denote by $l_c^\infty(S, E)$ the subspace of those elements $f \in l^\infty(S, E)$ such that $f(S) = \{f(s) : s \in S\}$ is a relatively weakly compact subset of E. Let X be a subspace of $l^\infty(S)$ containing constants such that for each $f \in l_c^\infty(S, E)$ and $x^* \in E^*$, the function $s \mapsto \langle f(s), x^* \rangle$ is contained in X. Then, for each $\mu \in X^*$ and $f \in l_c^\infty(S, E)$, we define a bounded linear functional $\tau(\mu)f$ on E^* by

$$\tau(\mu)f : x^* \mapsto \mu\langle f(\cdot), x^* \rangle.$$

It follows from the bipolar theorem that $\tau(\mu)f$ is contained in E. We know that if μ is a mean on X, then $\tau(\mu)f$ is contained in the closure of convex hull of $\{f(s) : s \in S\}$. We also know that for each $\mu \in X^*$, $\tau(\mu)$ is a bounded linear mapping of $l_c^\infty(S, E)$ into E such that for each $f \in l_c^\infty(S, E)$, $\|\tau(\mu)f\| \leq \|\mu\|\|f\|$; see [13].

Let C be a closed and convex subset of E and let T be a mapping of C into itself. Then, T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for each $x, y \in C$. Let $L(E)$, $A(C)$ and $N(C)$ be the semigroups of linear bounded operators on E, affine nonexpansive mappings and nonexpansive mapings of C into itself under operator multiplication, respectively. If $\mathcal{S}$ is a semigroup homomorphism of S into $L(E)$ ($A(C)$ or $N(C)$), then $\mathcal{S} = \{T(s) : s \in S\}$ is said to be a *representation* of S as linear bounded operators on E (as affine nonexpansive mappings on C or as nonexpansive mappings on C). A subspace X of $l^\infty(S)$ is said to be *admissible* if for each $x \in E$ (or C) and $x^* \in E^*$, the function $s \mapsto \langle T(s)x, x^* \rangle$ is contained in X. We denote by $F(\mathcal{S})$ the set of common fixed points of $\mathcal{S}$, that is, $F(\mathcal{S}) = \cap_{s \in S}\{x \in C : T(s)x = x\}$.

Let C be a closed and convex subset of a Banach space E and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as linear bounded operators on E (as affine nonexpansive mappings on C or as nonexpansive mappings on C) such that $T(\cdot)x \in l_c^\infty(S, E)$ for some $x \in E$ (or C), let X be an admissible subspace of $l^\infty(S)$ which contains constants and let μ be a mean on X. Then, there exists a unique point x_0 of E such that $\mu\langle T(\cdot)x, x^*\rangle = \langle x_0, x^*\rangle$ for each $x^* \in E^*$. We denote such a point x_0 by $T(\mu)x$. Note that $T(\mu)x$ is contained in the closure of convex hull of $\{T(s)x : s \in S\}$ for each $x \in C$; see [29] and [12] for more details.

For each $s \in S$, we define the operators $R(s)$ and $L(s)$ on $l^\infty(S, E)$ by

$$(R(s)f)(t) = f(ts) \quad \text{and} \quad (L(s)f)(t) = f(st)$$

for each $t \in S$ and $f \in l^\infty(S, E)$, respectively. We denote by $\mathcal{LO}(f)$ (resp. $\mathcal{RO}(f)$) the set $\{L(s)f \in l^\infty(S, E) : s \in S\}$ of left translates of f (resp. the set $\{R(s)f \in l^\infty(S, E) : s \in S\}$ of right translates of f). A function $f \in l^\infty(S, E)$ is said to be *left (resp. right) almost periodic* if $\mathcal{LO}(f)$ (resp. $\mathcal{RO}(f)$) is relatively compact in $l^\infty(S, E)$; the notion of almost periodicity for real-valued functions on an abstract group is due to von Neumann [22]. A function $f \in l^\infty(S, E)$ is also said to be *left (resp. right) weakly almost periodic* if $\mathcal{LO}(f)$ (resp. $\mathcal{RO}(f)$) is relatively weakly compact in $l^\infty(S, E)$; the notion of weakly almost periodicity was introduced by Eberlein [9]. See also [8]. Note that every weakly almost periodic function $f \in l^\infty(S, E)$ is contained in $l_c^\infty(S, E)$.

3. Vector-valued weakly almost periodic functions

In this section, we prove weak and strong mean ergodic theorems for weakly almost periodic functions (in the sense of Eberlein) which are defined on an abstract semigroup and take values in a Banach space. Before proving the theorems, we need some lemmas.

Lemma 3.1. *Let X be a subspace of $l^\infty(S)$ which contains constants, let $\mathcal{M}$ be the set of means on X and let Λ be the set of finite means on S. Then, $\mathcal{M}$ is a compact convex subset of X^* and is the closure of Λ in the weak topology $\sigma(X^*, X)$.*

For the proof, see Day [6] and Takahashi [32].

Lemma 3.2. *Let $f \in l^\infty(S, E)$ be a right weakly almost periodic function and let K be the closure of convex hull of $\mathcal{RO}(f)$. Then, the weak topology on K is identical with the topology of pointwise weak convergence, that is, the topology on K induced by the product topology on the Cartesian product E^S of $(E, \sigma(E, E^*))$.*

Proof. Since f is right weakly almost periodic, K is weakly compact in $l^\infty(S, E)$. The weak topology on K is finer than the topology of pointwise weak convergence, that is, the topology on K induced by the product topology $\prod_{s\in S} \sigma(E, E^*)$ on the Cartesian product E^S of $(E, \sigma(E, E^*))$. Since the latter topology is Hausdorff, it follows that on K, the weak topology is identical with the topology of pointwise weak convergence. This completes the proof. □

Lemma 3.3. *Let $f \in l^\infty(S, E)$ be a right weakly almost periodic function, let X be a closed and translation invariant subspace of $l^\infty(S)$ containing constants such that for each $x^* \in E^*$, the function $s \mapsto \langle f(s), x^*\rangle$ is contained in X and let μ be a mean on X. Then, the function $s \mapsto \tau(l(s)^*\mu)f$ is contained in the closure K of convex hull of $\mathcal{RO}(f)$ in $l^\infty(S, E)$.*

Proof. Let $\mu = \delta(s)$ be a point evaluation. Then, we have, for each $t \in S$ and $x^* \in E^*$,

$$\begin{aligned}\langle \tau(l(t)^*\mu)f, x^*\rangle &= \langle \tau(l(t)^*\delta(s))f, x^*\rangle \\ &= l(t)^*\delta(s)\langle f(\cdot), x^*\rangle \\ &= \langle f(ts), x^*\rangle \\ &= \langle (R(s)f)(t), x^*\rangle,\end{aligned}$$

and hence $\tau(l(\cdot)^*\mu)f = R(s)f$.

Next, let $\mu = \sum_{i=1}^n \lambda_i\delta(s_i)$ be a finite mean on S with $\lambda_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n \lambda_i = 1$. Then, we have, for each $t \in S$ and $x^* \in E^*$,

$$\begin{aligned}\langle \tau(l(t)^*\mu)f, x^*\rangle = l(t)^*\mu\langle f(\cdot), x^*\rangle &= \mu\langle (L(t)f)(\cdot), x^*\rangle \\ &= (\sum_{i=1}^n \lambda_i\delta(s_i))\langle (L(t)f)(\cdot), x^*\rangle \\ &= \sum_{i=1}^n \lambda_i\delta(s_i)\langle (L(t)f)(\cdot), x^*\rangle \\ &= \sum_{i=1}^n \lambda_i\langle (R(s_i)f)(t), x^*\rangle \\ &= \langle (\sum_{i=1}^n \lambda_i R(s_i)f)(t), x^*\rangle\end{aligned}$$

and hence $\tau(l(\cdot)^*\mu)f = \sum_{i=1}^n \lambda_i R(s_i)f$.

Finally, let μ be a mean on X. It follows from Lemma 3.1 that there exists a net $\{\lambda_\alpha\}$ of finite means on S such that $\{\lambda_\alpha\}$ converges to μ in the weak topology $\sigma(X^*, X)$. Then, we have, for each $t \in S$ and $x^* \in E^*$,

$$\begin{aligned}\langle \tau(l(t)^*\lambda_\alpha)f, x^*\rangle &= l(t)^*\lambda_\alpha\langle f(\cdot), x^*\rangle \\ &= \lambda_\alpha\langle (L(t)f)(\cdot), x^*\rangle \\ &\to \mu\langle (L(t)f)(\cdot), x^*\rangle \\ &= l(t)^*\mu\langle f(\cdot), x^*\rangle \\ &= \langle \tau(l(t)^*\mu)f, x^*\rangle.\end{aligned}$$

So, $\{\tau(l(t)^*\lambda_\alpha)f\}$ converges weakly to $\tau(l(t)^*\mu)f$ in E for each $t \in S$. Since f is right weakly almost periodic in $l^\infty(S, E)$, it follows from Lemma 3.2 that $\{\tau(l(\cdot)^*\lambda_\alpha)f\}$ in K converges weakly to $\tau(l(\cdot)^*\mu)f$. Hence, by weak compactness of K, $\tau(l(\cdot)^*\mu)f$ is contained in K. This completes the proof. □

Lemma 3.4. *Let $f \in l^\infty(S, E)$ be a right weakly almost periodic function, let X be a closed and translation invariant subspace of $l^\infty(S)$ containing constants such that for each $x^* \in E^*$, the function $s \mapsto \langle f(s), x^*\rangle$ is contained in X, let μ be a right invariant mean on X and let λ be a finite mean on S. Then, $\tau(\mu)\tau(l(\cdot)^*\lambda)f = \tau(\mu)f$.*

Proof. Let $\lambda = \sum_{i=1}^n \alpha_i \delta(s_i)$ with $\alpha_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n \alpha_i = 1$. Then, we have, for each $x^* \in E^*$,

$$\begin{aligned}\langle \tau(\mu)\tau(l(\cdot)^*\lambda)f, x^*\rangle &= \mu\langle \tau(l(\cdot)^*\lambda)f, x^*\rangle = \mu\langle \tau(\lambda)(L(\cdot)f), x^*\rangle \\ &= \mu\langle \sum_{i=1}^n \alpha_i f(\cdot s_i), x^*\rangle = \sum_{i=1}^n \alpha_i \mu\langle f(\cdot s_i), x^*\rangle \\ &= \sum_{i=1}^n \alpha_i r(s_i)^* \mu\langle f(\cdot), x^*\rangle = \sum_{i=1}^n \alpha_i \mu\langle f(\cdot), x^*\rangle \\ &= \sum_{i=1}^n \alpha_i \langle \tau(\mu)f, x^*\rangle = \langle \tau(\mu)f, x^*\rangle\end{aligned}$$

and hence $\tau(\mu)\tau(l(\cdot)^*\lambda)f = \tau(\mu)f$. This completes the proof. □

Lemma 3.5. *Let $f \in l^\infty(S, E)$ be a right weakly almost periodic function and let X be a closed and translation invariant subspace of $l^\infty(S)$ containing constants such that for each $x^* \in E^*$, the function $s \mapsto \langle f(s), x^*\rangle$ is contained in X. If X has a left invariant mean, then there exists a unique constant function in the closure K of convex hull of $\mathcal{RO}(f)$. In this case, the constant function is $\tau(l(\cdot)^*\mu)f = \tau(\mu)f$ for each left invariant mean μ on X. In particular, if μ and ν are left invariant means on X, then $\tau(\mu)f = \tau(\nu)f$.*

Proof. Let μ be a left invariant mean on X. Then, we have, for each $t \in S$ and $x^* \in E^*$,

$$\langle \tau(l(t)^*\mu)f, x^*\rangle = l(t)^*\mu\langle f(\cdot), x^*\rangle = \mu\langle f(\cdot), x^*\rangle = \langle \tau(\mu)f, x^*\rangle$$

and hence $\tau(l(t)^*\mu)f = \tau(\mu)f$ for each $t \in S$. So, we have from Lemma 3.3 that $\tau(l(\cdot)^*\mu)f = \tau(\mu)f$ is a constant function in K.

Next, let $g = \sum_{i=1}^n \lambda_i R(s_i)f$ with $\lambda_i \geq 0$ $(i = 1, \ldots, n)$ and $\sum_{i=1}^n \lambda_i = 1$. Then, as in the same argument of Lemma 3.3, we have, for each $s \in S$,

$$g(s) = \sum_{i=1}^n \lambda_i R(s_i)f(s) = \tau(l(s)^*\lambda)f$$

where $\lambda = \sum_{i=1}^n \lambda_i \delta(s_i)$ is a finite mean on S. So, since $\tau(\mu)$ is a continuous linear mapping of $l^\infty(S, E)$ into E, it follows from Lemma 3.4 that $\tau(\mu)f = \tau(\mu)g$ for each g in K. If a constant function c is in K, then $c = \tau(\mu)c = \tau(\mu)f$. In particular, if μ and ν are left invariant means on X, then we have

$$\tau(\mu)f = \tau(\mu)(\tau(\nu)f) = \tau(\nu)f.$$

This completes the proof. □

Remark 3.6. In the above lemma, let us consider that $E = \mathbb{R}$. This implies that the Banach space $WAP(S)$ of real-valued weakly almost periodic functions defined on a semigroup S has at most one left (or right) invariant mean. See also [8].

Theorem 3.7. *Let $f \in l^\infty(S, E)$ be a right weakly almost periodic function in the sense of Eberlein, let X be a closed and translation invariant subspace of $l^\infty(S)$ containing constants such that for each $x^* \in E^*$, the function $s \mapsto \langle f(s), x^*\rangle$ is* *contained in X and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, $\{\tau(l(\cdot)^*\mu_\alpha)f\}$ converges weakly to a constant function p in the closure K of convex hull of $\mathcal{RO}(f)$. In this case, $p(\cdot) = \tau(\mu)f$ in E for each invariant mean μ on X.*

Proof. For each α, we define a function $f_\alpha \in l^\infty(S,E)$ by $f_\alpha(s) = \tau(l(s)^*\mu_\alpha)f$ for each $s \in S$. Then, since f is right weakly almost periodic, it follows from Lemma 3.3 that $\{f_\alpha\}$ is contained in a weakly compact subset K of $l^\infty(S,E)$. So, $\{f_\alpha\}$ has a cluster point g in K. Then, there exists a subnet $\{f_{\alpha_\beta}\}_{\beta\in\Gamma}$ of $\{f_\alpha\}$ such that $\{f_{\alpha_\beta}\}$ converges weakly to g in K. It also follows from Lemma 3.1 that there exists a cluster point μ of $\{\mu_{\alpha_\beta}\}$ in the weak topology $\sigma(X^*,X)$. We show that μ is an invariant mean on X. Without loss of generality, we can assume that $\{\mu_{\alpha_\beta}\}$ converges to μ in the weak topology $\sigma(X^*,X)$. Let $\epsilon > 0$, $s \in S$ and $h \in X$. Since $\{\mu_\alpha\}$ is asymptotically invariant, for each $h \in X$, there exists a $\beta_0 \in \Gamma$ such that for each $\beta \geq \beta_0$,

$$|\mu_{\alpha_\beta}(h) - \mu_{\alpha_\beta}(l(s)h)| \leq \epsilon/3.$$

Since $\{\mu_{\alpha_\beta}\}$ converges to μ in the weak topology $\sigma(X^*,X)$, we can choose a $\beta_1 \geq \beta_0$ such that

$$|\mu_{\alpha_{\beta_1}}(h) - \mu(h)| \leq \epsilon/3$$

and

$$|\mu_{\alpha_{\beta_1}}(l(s)h) - \mu(l(s)h)| \leq \epsilon/3.$$

Hence, we have

$$\begin{aligned}|\mu(h) - \mu(l(s)h)| &\leq |\mu(h) - \mu_{\alpha_{\beta_1}}(h)| + |\mu_{\alpha_{\beta_1}}(h) - \mu_{\alpha_{\beta_1}}(l(s)h)| \\ &\quad + |\mu_{\alpha_{\beta_1}}(l(s)h) - \mu(l(s)h)| \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $\mu(h) = \mu(l(s)h)$ for each $s \in S$ and $h \in X$. Similarly, we also have $\mu(h) = \mu(r(s)h)$ for each $s \in S$ and $h \in X$. This implies that μ is an invariant mean on X.

The above argument shows that there exists a subnet of $\{\mu_{\alpha_\beta}\}$ of $\{\mu_\alpha\}$ such that $\{\mu_{\alpha_\beta}\}$ converges to μ in the weak topology $\sigma(X^*,X)$ and $\{f_{\alpha_\beta}\}$ converges weakly to g in K. Since, for each $t \in S$ and $x^* \in E^*$,

$$\begin{aligned}\langle f_{\alpha_\beta}(t), x^*\rangle &= \langle \tau(l(t)^*\mu_{\alpha_\beta})f, x^*\rangle = (l(t)^*\mu_{\alpha_\beta})\langle f(\cdot), x^*\rangle \\ &= \mu_{\alpha_\beta}\langle (L(t)f)(\cdot), x^*\rangle \\ &\to \mu\langle (L(t)f)(\cdot), x^*\rangle \\ &= (l(t)^*\mu)\langle f(\cdot), x^*\rangle = \mu\langle f(\cdot), x^*\rangle \\ &= \langle \tau(\mu)f, x^*\rangle,\end{aligned}$$

$\{f_{\alpha_\beta}(t)\}$ converges weakly to $\tau(\mu)f$ in C for each $t \in S$. Then, by Lemma 3.2, $\{f_{\alpha_\beta}\}$ converges weakly to $\tau(\mu)f$ in $l^\infty(S,E)$. So, we have $g(\cdot) = \tau(\mu)f$. Hence, it follows from Lemma 3.5 that $\{\tau(l(\cdot)^*\mu_\alpha)f\}$ converges weakly to a constant function p in K and $p(\cdot) = \tau(\mu)f$ in E for each invariant mean μ on X. Moreover, since μ is an invariant mean on X, we have $\tau(\mu)f = \tau(r(s)^*\mu)f = \tau(\mu)(R(s)f)$ for each

$s \in S$ and hence $\tau(\mu)g = \tau(\mu)f$ for each $g \in K$. So, $q(s) = \tau(\mu)q = \tau(\mu)f$ for each constant function $q \in K$ and $s \in S$. This completes the proof. □

Theorem 3.8. *Let $f \in l^\infty(S,E)$ be a right weakly almost periodic function in the sense of Eberlein, let X be a closed and translation invariant subspace of $l^\infty(S)$ containing constants such that for each $x^* \in E^*$, the function $s \mapsto \langle f(s), x^*\rangle$ is contained in X and let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net of means on X. Then, $\{\tau(l(\cdot)^*\mu_\alpha)f\}$ converges strongly to a constant function p in the closure K of convex hull of $\mathcal{RO}(f)$. In this case, $p(\cdot) = \tau(\mu)f$ for each invariant mean μ on X.*

Proof. We have, for each $x^* \in E^*$ with $\|x^*\| = 1$ and $s, t \in S$,

$$\begin{aligned}
&|\langle \tau(l(t)^*\mu_\alpha)(R(s)f) - \tau(l(t)^*\mu_\alpha)f, x^*\rangle| \\
&= |(l(t)^*\mu_\alpha)\langle (R(s)f)(\cdot) - f(\cdot), x^*\rangle| \\
&= |\mu_\alpha\langle (L(t)R(s)f)(\cdot) - (L(t)f)(\cdot), x^*\rangle| \\
&= |\mu_\alpha\langle (R(s)L(t)f)(\cdot) - (L(t)f)(\cdot), x^*\rangle| \\
&= |(r(s)^*\mu_\alpha - \mu_\alpha)\langle (L(t)f)(\cdot), x^*\rangle| \\
&\le \|r(s)^*\mu_\alpha - \mu_\alpha\|\|f\|\|x^*\| \\
&= \|r(s)^*\mu_\alpha - \mu_\alpha\|\|f\|
\end{aligned}$$

and hence, for each $s \in S$,

$$\|\tau(l(\cdot)^*\mu_\alpha)(R(s)f) - \tau(l(\cdot)^*\mu_\alpha)f\| \le \|r(s)^*\mu_\alpha - \mu_\alpha\|\|f\|.$$

Let $g \in K$ and let $\epsilon > 0$. Then, there exists a $h = \sum_{i=1}^n \alpha_i R(s_i)f$ with $s_i \in S$ $(i = 1, \dots, n)$ and $\sum_{i=1}^n \alpha_i = 1$ such that $\|g - h\| \le \epsilon/2$. We can choose an α_0 such that for each $\alpha \ge \alpha_0$,

$$\sum_{i=1}^n \|r(s_i)^*\mu_\alpha - \mu_\alpha\|\|f\| \le \frac{\epsilon}{2}.$$

So, since, for each $x^* \in E^*$ and $t \in S$,

$$\begin{aligned}
&\|\tau(l(t)^*\mu_\alpha)g - \tau(l(t)^*\mu_\alpha)f\| \\
&\le \|\tau(l(t)^*\mu_\alpha)g - \tau(l(t)^*\mu_\alpha)h\| \\
&\qquad + \|\tau(l(t)^*\mu_\alpha)h - \tau(l(t)^*\mu_\alpha)f\| \\
&\le \|\tau(\mu_\alpha)\|\|L(t)g - L(t)h\| \\
&\qquad + \sum_{i=1}^n \alpha_i\|\tau(l(t)^*\mu_\alpha)(R(s_i)f) - \tau(l(t)^*\mu_\alpha)f\| \\
&\le \|g - h\| + \sum_{i=1}^n \alpha_i\|r(s_i)^*\mu_\alpha - \mu_\alpha\|\|f\| \\
&\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,
\end{aligned}$$

we have, for each $g \in K$,

$$\lim_\alpha \|\tau(l(\cdot)^*\mu_\alpha)g - \tau(l(\cdot)^*\mu_\alpha)f\| = 0.$$

It follows from Theorem 3.7 that $\{\tau(l(\cdot)^*\mu_\alpha)f\}$ converges weakly to a constant function p in K such that $p(\cdot) = \tau(\mu)f$ for each invariant mean μ on X. Hence, we have

$$\lim_\alpha \|\tau(l(\cdot)^*\mu_\alpha)f - p\| = \lim_\alpha \|\tau(l(\cdot)^*\mu_\alpha)f - \tau(l(\cdot)^*\mu_\alpha)p\| = 0.$$

This completes the proof. □

4. Mean ergodic theorems for semigroups of operators

In this section, using Theorems 3.7 and 3.8, we obtain weak and strong mean ergodic theorems for noncommutative semigroups of nonexpansive mappings, affine nonexpansive mappings and linear bounded operators in Banach spaces.

Theorem 4.1. *Let C be a compact and convex subset of a Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings on C, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a point p uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X.*

Proof. For each $x \in C$, we define a function $f_x \in l^\infty(S, E)$ by $f_x(s) = T(s)x$ for each $s \in S$. We show that for each $x \in C$, f_x is right almost periodic. In fact, we have, for each $s \in S$,

$$(R(s)f_x)(t) = T(ts)x = T(t)T(s)x = f_{T(s)x}(t)$$

for each $t \in S$. Hence, $\mathcal{RO}(f_x)$ is contained in $\{f_y : y \in C\}$. We define a mapping Φ of C into $l^\infty(S, E)$ by $\Phi(x) = f_x$ for each $x \in C$. Then, we have, for each $x, y \in C$,

$$\begin{aligned}\|\Phi(x) - \Phi(y)\| &= \|f_x - f_y\| \\ &= \sup_{t\in S} \|f_x(t) - f_y(t)\| \\ &= \sup_{t\in S} \|T(t)x - T(t)y\| \\ &\le \|x - y\|\end{aligned}$$

and hence Φ is norm-to-norm continuous. Since C is compact, $\mathcal{RO}(f_x)$ is contained in a compact subset $\Phi(C)$ of $l^\infty(S, E)$. So, for each $x \in C$, $f_x \in l^\infty(S, E)$ is right almost periodic.

It follows from Theorem 3.7 that $\{T(l(\cdot)^*\mu_\alpha)x\}$ converges strongly to a constant function q in $l^\infty(S, E)$. In this case, $q(\cdot) = T(\mu)x$ for each invariant mean μ on X. Hence, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a point $T(\mu)x$ uniformly in $h \in S$ where μ is an invariant mean on X. This completes the proof. □

Remark 4.2. In [28], Suzuki and Takahashi constructed a nonexpansive mapping T of a compact convex subset C of a Banach space E without strict convexity into itself such that for some $x \in C$, the Cesàro means $1/n \sum_{k=0}^{n-1} T^k x$ converge to a point which is not a fixed point of T.

Lemma 4.3. *Let C be a compact and convex subset of a Banach space E, let S be a commutative semigroup with identity and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation* *of S as nonexpansive mappings on C. Then, $\mathcal{S}$ is asymptotically isometric on C, that is, for each $x, y \in C$,*

$$\lim_{s\in S} \|T(s+h)x - T(s+k)y\|$$

exists uniformly in $h, k \in S$.

Proof. For each $s \in S$, we define a real-valuded function f_s on a compact metric space $C \times C$ by $f_s(x, y) = \|T(s)x - T(s)y\|$ for each $x, y \in C$. Then, by nonexpansiveness of $T(s)$ $(s \in S)$, $\{f_s : s \in S\}$ is equicontinuous. Since, for each $x, y \in C$ and $s, t \in S$,

$$\|T(s+t)x - T(s+t)y\| \le \|T(s)x - T(s)y\|,$$

$\lim_{s\in S} \|T(s)x - T(s)y\| = \lim_{s\in S} f_s(x, y)$ exists for each $x, y \in C$. So, putting $f(x, y) = \lim_{s\in S} \|T(s)x - T(s)y\|$ for each $x, y \in C$, we have that $\{f_s\}$ converges to f uniformly on $C \times C$. This completes the proof. □

Lemma 4.4. *Let C be a compact and convex subset of a strictly convex Banach space E, let S be a commutative semigroup with identity and let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings on C. Then, for each $x \in C$,*

$$\lim_{s\in S} \|T(h)T(r(s)^*\lambda)x - T(r(h+s)^*\lambda)x\| = 0$$

uniformly in $h \in S$ and $\lambda \in \Lambda$, where Λ is the set of finite means on S.

For the proof, see Lemma 4.3 and [13, Proposition 3.8].

Theorem 4.5. *Let C be a compact and convex subset of a strictly convex Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as nonexpansive mappings on C, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on X. Then, for each $x \in C$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a common fixed point p of $\mathcal{S}$ uniformly in $h \in S$. In this case, $p = T(\mu)x$ for each invariant mean μ on X. Moreover, if S is commutative, then*

$$\{T(\mu)x\} = \cap_{s\in S}\overline{co}\{T(s+t)x : t \in S\} \cap F(\mathcal{S})$$

for each invariant mean μ on X.

Proof. We know from Theorem 3.7 and [21, Theorem 2] that for each $x \in C$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a common fixed point p uniformly in $h \in S$ and, in this case, $p = T(\mu)x$ for each invariant mean μ on X. Since, for each $s \in S$,

$$T(\mu)x = T(r(s)^*\mu)x = T(\mu)T(s)x \in \overline{\text{co}}\{T(s+t)x : t \in S\},$$

we have

$$T(\mu)x \in \cap_{s\in S}\overline{\text{co}}\{T(s+t)x : t \in S\} \cap F(\mathcal{S}).$$

So, it suffices to show that if $p \in \cap_{s\in S}\overline{\text{co}}\{T(s+t)x : t \in S\} \cap F(\mathcal{S})$, then $p = T(\mu)x$.

Let $x \in C$, let $\epsilon > 0$ and let $p \in \cap_{s\in S}\overline{\text{co}}\{T(s+t)x : t \in S\} \cap F(\mathcal{S})$. Then, by Lemma 4.4, there exists a $s_0 \in S$ such that

$$\|T(h)T(r(s_0)^*\lambda)x - T(r(s_0+h)^*\lambda)x\| < \frac{\epsilon}{2}$$

for each $h \in S$ and $\lambda \in \Lambda$, where Λ is the set of finite means on S. Then, by $p \in \overline{\text{co}}\{T(s_0 + t)x : t \in S\}$, there exists a finite mean λ_0 on S such that

$$\|p - T(r(s_0)^*\lambda_0)x\| = \|p - T(\lambda_0)T(s_0)x\| < \frac{\epsilon}{2}.$$

So, from Lemma 3.4 we have

$$\begin{aligned}
\|p - T(\mu)x\| &\leq \|p - T(\mu)T(r(s_0)^*\lambda_0)x\| \\
&\quad + \|T(\mu)T(r(s_0)^*\lambda_0)x - \tau(\mu)T(r(\cdot + s_0)^*\lambda)x\| \\
&\quad + \|\tau(\mu)T(r(\cdot + s_0)^*\lambda)x - T(\mu)x\| \\
&\leq \|p - T(r(s_0)^*\lambda_0)x\| \\
&\quad + \sup_{s \in S} \|T(s)T(r(s_0)^*\lambda_0)x - T(r(s + s_0)^*\lambda)x\| \\
&\quad + \|\tau(\mu)T(r(\cdot)^*\lambda)T(s_0)x - T(\mu)x\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} + \|T(\mu)T(s_0)x - T(\mu)x\| \\
&= \epsilon.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $p = T(\mu)x$. This completes the proof. □

Theorem 4.6. *Let $\mathcal{S} = \{T(s) : s \in S\}$ of S be a representation of S as linear bounded operators on a Banach space E such that for $s \in S$, $\|T(s)\| \leq M$ and for each $x \in E$, $\{T(s)x : s \in S\}$ is relatively weakly compact, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net of means on X. Then, for each $x \in E$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a common fixed point p of $\mathcal{S}$ uniformly in $h \in S$. In this case, $p = T(\mu)x$ and*

$$\{T(\mu)x\} = \overline{co}\{T(s)x : s \in S\} \cap F(\mathcal{S})$$

for each invariant mean μ on X.

Proof. For each $x \in E$, we define a function $f_x \in l^\infty(S, E)$ by $f_x(s) = T(s)x$ for each $s \in S$. We show that for each $x \in E$, f_x is right weakly almost periodic. In fact, we have, for each $s \in S$,

$$(R(s)f_x)(t) = T(ts)x = T(t)T(s)x = f_{T(s)x}(t)$$

for each $t \in S$. Hence, $\mathcal{RO}(f_x)$ is contained in $\{f_y : y \in C\}$, where $C = \bar{co}\{T(s)x : s \in S\}$. We define a mapping Φ of E into $l^\infty(S, E)$ by $\Phi(x) = f_x$ for each $x \in E$. Then, Φ is a bounded linear mapping and hence is weak-to-weak continuous. Since C is weakly compact, $\mathcal{RO}(f_x)$ is contained in a weakly compact subset $\Phi(C)$ of $l^\infty(S, E)$. So, for each $x \in E$, $f_x \in l^\infty(S, E)$ is right weakly almost periodic.

It follows from Theorem 3.8 that $\{T(l(\cdot)^*\mu_\alpha)x\}$ converges strongly to a constant function q in $l^\infty(S, E)$. In this case, $q(\cdot) = T(\mu)x$ for each invariant mean μ on X. Hence, for each $x \in E$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a point $T(\mu)x$ in C uniformly in $h \in S$ where μ is an invariant mean on X. Since, for each $s \in S$ and

$x^* \in E^*$,

$$\begin{aligned}\langle T(s)T(\mu)x, x^*\rangle &= \langle T(\mu)x, T(s)^*x^*\rangle = \mu\langle T(\cdot)x, T(s)^*x^*\rangle \\ &= \mu\langle T(s)T(\cdot)x, x^*\rangle = \mu\langle T(s\cdot)x, x^*\rangle \\ &= l(s)^*\mu\langle T(\cdot), x^*\rangle = \mu\langle T(\cdot), x^*\rangle \\ &= \langle T(\mu)x, x^*\rangle\end{aligned}$$

where $T(s)^*$ is the adjoint operator of $T(s)$, we have $T(s)T(\mu)x = T(\mu)x$ for each $s \in S$.

It remains to show that $\{T(\mu)x\} = \overline{\text{co}}\{T(s)x : s \in S\} \cap F(\mathcal{S})$ for each $x \in C$. Since μ is an invariant mean on X, we have $T(\mu)x = T(r(s)^*\mu)x = T(\mu)T(s)x$ for each $s \in S$ and hence $T(\mu)x = T(\mu)y$ for each $y \in \overline{\text{co}}\{T(s)x : s \in S\}$. This completes the proof. □

Lemma 4.7. *Let C be a bounded closed and convex subset of a Banach space E, let F be a Banach space and let T be an affine continuous mapping of C into F. Then, T is weak-to-weak continuous.*

Proof. It suffices to show that for each $x^* \in F^*$, the function $x \mapsto \langle Tx, x^*\rangle$ is weakly continuous on C. Let $x^* \in F^*$ and let $\{x_\alpha\}$ be a net in C such that $\{x_\alpha\}$ converges weakly to x. Then since $\{\langle Tx_\alpha, x^*\rangle\}$ has a cluster point w in the real numbers, there exists a subnet $\{x_{\alpha_\beta}\}$ of $\{x_\alpha\}$ such that $\{\langle Tx_{\alpha_\beta}, x^*\rangle\}$ converges to w. Let $\epsilon > 0$. Then, there exists a β_0 such that for each $\beta \geq \beta_0$,

$$|\langle Tx_{\alpha_\beta}, x^*\rangle - w| \leq \epsilon.$$

Since x is contained in the closure of convex hull of $\{x_{\alpha_\beta}\}$ and T is continuous, there exists a $y = \sum_{i=1}^n \lambda_i x_{\alpha_{\beta_i}}$ with $\beta_i \geq \beta_0$, $\lambda_i \geq 0$ $(i = 1, \dots, n)$ and $\sum_{i=1}^n \lambda_i = 1$ such that $\|Tx - Ty\|\|x^*\| \leq \epsilon$. Then, we have

$$\begin{aligned}|w - \langle Tx, x^*\rangle| &\leq |w - \langle Ty, x^*\rangle| + |\langle Ty - Tx, x^*\rangle| \\ &\leq \left| w - \langle T(\sum_{i=1}^n \lambda_i x_{\alpha_{\beta_i}}), x^*\rangle \right| + \|Ty - Tx\|\|x^*\| \\ &\leq \left| w - \langle \sum_{i=1}^n \lambda_i Tx_{\alpha_{\beta_i}}, x^*\rangle \right| + \epsilon \\ &\leq \sum_{i=1}^n \lambda_i |w - \langle Tx_{\alpha_{\beta_i}}, x^*\rangle| + \epsilon \\ &\leq \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, we have $w = \langle Tx, x^*\rangle$. It follows that the function $x \mapsto \langle Tx, x^*\rangle$ is weakly continuous on C. This completes the proof. □

From Day [7, Lemma 2] and Lemma 4.7, we can prove the following lemma.

Lemma 4.8. *Let C be a weakly compact and convex subset of a Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as affine nonexpansive mappings on C, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let μ be a mean on X. Then, for each $x \in C$ and $s \in S$,*

$T(l(s)^*\mu)x = T(s)T(\mu)x$. In particular, if μ is an invariant mean on X, then $T(\mu)x$ is a common fixed point of $\mathcal{S}$.

Theorem 4.9. *Let C be a weakly compact and convex subset of a Banach space E, let $\mathcal{S} = \{T(s) : s \in S\}$ be a representation of S as affine nonexpansive mappings on C, let X be a closed, translation invariant and admissible subspace of $l^\infty(S)$ containing constants and let $\{\mu_\alpha\}$ be a strongly asymptotically invariant net of means on X. Then, for each $x \in C$, $\{T(r(h)^*\mu_\alpha)x\}$ converges strongly to a common fixed point p of $\mathcal{S}$ uniformly in $h \in S$. In this case, $p = T(\mu)x$ and*

$$\{T(\mu)x\} = \overline{co}\{T(s)x : s \in S\} \cap F(\mathcal{S})$$

for each invariant mean μ on X.

Proof. For each $x \in C$, we define a function $f_x \in l^\infty(S, E)$ by $f_x(s) = T(s)x$ for each $s \in S$. We show that for each $x \in C$, f_x is right weakly almost periodic. In fact, we have, for each $s \in S$,

$$(R(s)f_x)(t) = T(ts)x = T(t)T(s)x = f_{T(s)x}(t)$$

for each $t \in S$. Hence, $\mathcal{RO}(f_x)$ is contained in $\{f_y : y \in C\}$. We define a mapping Φ of C into $l^\infty(S, E)$ by $\Phi(x) = f_x$ for each $x \in C$. Since Φ is an affine continuous mapping of C into $l^\infty(S, E)$, we have from Lemma 4.7 that Φ is weak-to-weak continuous. Then, by weak compactness of C, $\mathcal{RO}(f_x)$ is contained in a weakly compact subset $\Phi(C)$ of $l^\infty(S, E)$. So, for each $x \in C$, $f_x \in l^\infty(S, E)$ is right weakly almost periodic.

It follows from Theorem 3.8 that for each $x \in C$, $\{T(l(\cdot)^*\mu_\alpha)x\}$ converges strongly to a constant function q in $l^\infty(S, E)$. In this case, $q(\cdot) = T(\mu)x$ in C for each invariant mean μ on X. Hence, for each $x \in C$, $\{T(l(h)^*\mu_\alpha)x\}$ converges strongly to a point $T(\mu)x$ uniformly in $h \in S$ where μ is an invariant mean on X. By Lemma 4.8, $T(\mu)x$ is a common fixed point of $\mathcal{S}$.

It remains to show that $\{T(\mu)x\} = \overline{\text{co}}\{T(s)x : s \in S\} \cap F(\mathcal{S})$ for each $x \in C$. Since μ is an invariant mean on X and $T(\mu)$ is an affine nonexpansive mapping, we have $T(\mu)x = T(r(s)^*\mu)x = T(\mu)T(s)x$ for each $s \in S$ and hence $T(\mu)x = T(\mu)y$ for each $y \in \overline{\text{co}}\{T(s)x : s \in S\}$. This completes the proof. □

5. Some nonlinear ergodic theorems

In this section, using the generalized nonlinear ergodic theorems for nonexpansive semigroups in Section 4, we obtain some nonlinear ergodic theorems in cases of discrete and one-parameter semigroups of nonexpansive mappings. We denote by $\mathbb{N}$, $\mathbb{N}_+$ and $\mathbb{R}_+$ the set of positive integers, the set of non-negative integers and the set of non-negative real numbers, respectively.

Let C be a compact and convex subset of a Banach space E. In the case of a single nonexpansive mapping T of C into itself, choose $S = \mathbb{N}_+$. For each $n \in \mathbb{N}$, define

$$\mu_n(f) = \frac{1}{n}\sum_{i=0}^{n-1} f(i)$$

for each $f \in l^\infty(\mathbb{N}_+)$. Then, $\{\mu_n\}$ is an asymptotically invariant sequence of means on $l^\infty(\mathbb{N}_+)$; for more details, see [32]. We have, for each $x^* \in E^*$,

$$(\mu_n)_i\langle T^i x, x^*\rangle = \frac{1}{n}\sum_{i=0}^{n-1}\langle T^i x, x^*\rangle = \left\langle \frac{1}{n}\sum_{i=0}^{n-1} T^i x, x^* \right\rangle$$

and hence

$$T_{\mu_n}x = \frac{1}{n}\sum_{i=0}^{n-1} T^i x.$$

Therefore, it follows from Theorem 4.1 that $1/n\sum_{i=0}^{n-1} T^{i+h}x$ converges uniformly in $h \in \mathbb{N}_+$. Further, if E is a strictly convex Banach space, then we have from Theorem 4.5 that, for each $x \in C$,

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$$

converges to a fixed point of T uniformly in $h \in \mathbb{N}_+$.

In the case of a finite family of nonexpansive mappings $T_1, \dots, T_k$ of C into itself such that $T_iT_j = T_jT_i$ for each $i, j = 1, \dots, k$, consider $S = {\mathbb{N}_+}^k$. For each $n \in \mathbb{N}$, define

$$\mu_n(f) = \frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} f(i_1, \dots, i_k)$$

for each $f \in l^\infty({\mathbb{N}_+}^k)$. Then, $\{\mu_n\}$ is an asymptotically invariant sequence of means on $l^\infty({\mathbb{N}_+}^k)$; for more details, see [32]. As in the above argument, we have

$$T_{\mu_n}x = \frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} {T_1}^{i_1}\dots{T_k}^{i_k}x.$$

Therefore, it follows from Theorem 4.1 that

$$\frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} {T_1}^{i_1+h_1}\dots{T_k}^{i_k+h_k}x$$

converges uniformly in $h_1, \dots, h_k \in \mathbb{N}_+$. If E is a strictly convex Banach space, then we have from Theorem 4.5 that, for each $x \in C$,

$$\frac{1}{n^k}\sum_{i_1=0}^{n-1}\cdots\sum_{i_k=0}^{n-1} {T_1}^{i_1+h_1}\dots{T_k}^{i_k+h_k}x$$

converges to a common fixed point of $T_1, \dots, T_k$ uniformly in $h_1, \dots, h_k \in \mathbb{N}_+$.

Let $Q = \{q_{n,m}\}_{n,m\in\mathbb{N}_+}$ be a matrix satisfying the following conditions:

(a) $\sup_{n\geq 0}\sum_{m=0}^{\infty}|q_{n,m}| < \infty$;
(b) $\sum_{m=0}^{\infty} q_{n,m} = 1$ for each $n \in \mathbb{N}$;
(c) $\lim_{n\to\infty}\sum_{m=0}^{\infty}|q_{n,m+1} - q_{n,m}| = 0$.

Such a matrix Q is called *strongly regular* in the sense of Lorentz [18]. If Q is a strongly regular matrix, then for each $m \in \mathbb{N}$, we have $|q_{n,m}| \to 0$ as $n \to \infty$; see [12]. For each $n \in \mathbb{N}$, define

$$\mu_n(f) = \sum_{m=0}^{\infty} q_{n,m} f(m)$$

for each $f \in l^\infty(\mathbb{N}_+)$. Then, $\{\mu_n\}$ is an asymptotically invariant sequence of means; for more details, see [32]. We have, for each $x^* \in E^*$,

$$\begin{aligned}(\mu_n)_m \langle T^m x, y\rangle &= \sum_{m=0}^{\infty} q_{n,m} \langle T^m x, y\rangle \\ &= \left\langle \sum_{m=0}^{\infty} q_{n,m} T^m x, y \right\rangle\end{aligned}$$

and hence

$$T_{\mu_n} x = \sum_{m=0}^{\infty} q_{n,m} T^m x.$$

Therefore, it follows from Theorem 4.1 that $\sum_{m=0}^{\infty} q_{n,m} T^{m+h} x$ converges uniformly in $h \in \mathbb{N}_+$. Further, if E is a strictly convex Banach space, then we also have from Theorem 4.5 that, for each $x \in C$,

$$\sum_{m=0}^{\infty} q_{n,m} T^{m+h} x$$

converges to a fixed point of T uniformly in $h \in \mathbb{N}_+$.

In the case of a strongly continuous one-parameter semigroup of nonexpansive mappings of C into itself, consider $S = \mathbb{R}_+$. For $t > 0$, define

$$\mu_t(f) = \frac{1}{t} \int_0^t f(s)\, ds$$

for each $f \in C(\mathbb{R}_+)$, where $C(\mathbb{R}_+)$ is the space of real-valued, bounded and continuous functions on $\mathbb{R}_+$ with supremum norm. Then, $\{\mu_t\}$ is an asymptotically invariant net of means on $C(\mathbb{R}_+)$; for more details, see [32]. We have, for each $x^* \in E^*$,

$$\begin{aligned}\mu_t \langle T(\cdot)x, x^*\rangle &= \frac{1}{t} \int_0^t \langle T(s)x, x^*\rangle\, ds \\ &= \left\langle \frac{1}{t} \int_0^t T(s)x\, ds, x^* \right\rangle\end{aligned}$$

and hence

$$T(\mu_t)x = \frac{1}{t} \int_0^t T(s)x\, ds.$$

Therefore, it follows from Theorem 4.1 that $1/t \int_0^t T(s+h)x\, ds$ converges uniformly in $h \in \mathbb{R}_+$. Further if E is a strictly convex Banach space, then we also have from

Theorem 4.5 that, for each $x \in C$,

$$\frac{1}{t}\int_0^t T(s+h)x\ ds$$

converges to a common fixed point of $\mathcal{S}$ uniformly in $h \in \mathbb{R}_+$.

Finally, for $r > 0$, define

$$\mu_r(f) = r\int_0^\infty \exp(-rs)f(s)\ ds$$

for each $f \in C(\mathbb{R}_+)$. Then, $\{\mu_r\}$ is an asymptotically invariant net of means on $C(\mathbb{R}_+)$; for more details, see [32]. We have, for each $x^* \in E^*$,

$$\begin{aligned}\mu_r\langle T(\cdot)x, x^*\rangle &= r\int_0^\infty \exp(-rs)\langle T(s)x, x^*\rangle\ ds\\ &= \left\langle r\int_0^\infty \exp(-rs)T(s)x\ ds, x^*\right\rangle\end{aligned}$$

and hence

$$T(\mu_r)x = r\int_0^\infty \exp(-rs)T(s)x\ ds.$$

Therefore, it follows from Theorem 4.1 that $r\int_0^\infty \exp(-rs)T(s+h)x\ ds$ converges uniformly in $h \in \mathbb{R}_+$. In the case of strict convexity of E, we have from Theorem 4.5 that, for each $x \in C$,

$$r\int_0^\infty \exp(-rs)T(s+h)x\ ds$$

converges to a common fixed poit of $\mathcal{S}$ uniformly in $h \in \mathbb{R}_+$.

Similarly, we can prove such mean ergodic theorems for linear bounded operator T of a Banach space E into itself such that $\|T^n\| \le M$ for all $n \in \mathbb{N}$ or a commutative family $\{T(s) : s \in S\}$ of linear bounded operators such that $\|T(s)\| \le M$ for all $s \in S$. We also obtain mean ergodic theorems for an affine nonexpansive mapping of a closed convex subset of a Banach space into into itself, or strongly continuous one-parameter semigroups of affine nonexpansive mappings. As in the above argument, the following corollaries are obtained by using Theorem 4.9:

Corollary 5.1. *Let C be a weakly compact and convex subset of a Banach space E and let T be an affine nonexpansive mapping of C into itself. Then, for each $x \in C$,*

$$\frac{1}{n}\sum_{i=0}^{n-1} T^{i+h}x$$

converges to a fixed point of T uniformly in $h \in \mathbb{N}_+$.

Corollary 5.2. *Let C be a weakly compact and convex subset of a Banach space E, let $Q = \{q_{n,m}\}_{n,m\in\mathbb{N}_+}$ be a strongly regular matrix and let T be an affine nonexpansive mapping of C into itself. Then, for each $x \in C$,*

$$\sum_{m=0}^{\infty} q_{n,m}T^{m+h}x$$

converges to a fixed point of T uniformly in $h \in \mathbb{N}_+$.

Corollary 5.3. *Let C be a weakly compact and convex subset of a Banach space E and let $\mathcal{S} = \{T(s) : s \in \mathbb{R}_+\}$ be a strongly continuous one-parameter semigroup of affine nonexpansive mappings of C into itself. Then, for each $x \in C$,*

$$\frac{1}{t}\int_0^t T(s+h)x\, ds$$

converges to a common fixed point of $\mathcal{S}$ uniformly in $h \in \mathbb{R}_+$.

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