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MINIMAL ELEMENT THEOREM WITH SET-RELATIONS

AKIRA SHIMIZU AND TAMAKI TANAKA

Abstract. Hamel and Löhne proved the existence results for minimal points of subsets of the product space $X \times 2^Y$, where X and Y are a separated uniform space and a topological vector space, respectively. In this paper, we present a similar minimal element theorem with set-relations to those introduced in [3].

1. Introduction

In [3], Hamel and Löhne proved the existence results for minimal points of subsets of the product space $X \times 2^Y$, where X is a separated uniform space and Y a topological vector space. These principles are based on the following order relations in 2^Y:

$$V_1 \leq_C^l V_2 \Leftrightarrow V_2 \subseteq V_1 + C \text{ and } V_1 \leq_C^u V_2 \Leftrightarrow V_1 \subseteq V_2 - C$$

where $C \subseteq Y$ is a convex cone and $V_1, V_2 \in 2^Y$. Moreover, they derived from them new versions of Ekeland's principle for set-valued maps.

In this paper, we introduce a different scalarization function to present a minimal element theorem with set-relations which is similar to those introduced in [3].

This paper is organized as follows. In Section 2, we define two ordering relations and boundedness on 2^Y, where Y is a topological vector space. In Section 3, we introduce two types of nonlinear scalarization functions defined on 2^Y. Those functions are used for the proofs of minimal element theorems introduced in Section 4. In Section 4, we give minimal element theorems with set-relation based on the Brézis-Browder principle.

2. Relationships between two sets and boundedness concepts for subsets of 2^Y

Throughout this section, let Y be a real ordered topological vector space with the vector ordering $\leq_C$ induced by a nonempty convex cone C: for $x, y \in Y$,

$$x \leq_C y \text{ if } y - x \in C.$$

We introduce two relations $\leq_C^u, \leq_C^l$ on 2^Y and boundedness concepts for subsets of 2^Y.

Definition 2.1 (Set-relationships in [4]). Given nonempty sets $A, B \subset Y$, we define two types of relationships between A and B as follows:

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$$A \leq_C^l B \Longleftrightarrow B \subseteq A + C,$$
$$A \leq_C^u B \Longleftrightarrow A \subseteq B - C.$$

Definition 2.2 (Boundedness in [3])**.** A subset $\mathbb{V} \subseteq 2^Y$ is said to be $\leq_C^l$-*bounded below* if there exists some topologically bounded subset $\tilde{V} \subseteq Y$ such that $\tilde{V} \leq_C^l V$ holds for all $V \in \mathbb{V}$. The set $\tilde{V}$ is called a *lower* $\leq_C^l$-*bound* of $\mathbb{V}$. A subset $\mathbb{V} \subseteq 2^Y$ is said to be $\leq_C^u$-*bounded above* and $\tilde{V}$ is called an *upper* $\leq_C^u$-*bound* of $\mathbb{V}$ if $-\mathbb{V} := \{-V : V \in \mathbb{V}\}$ is $\leq_C^l$-bounded below with the lower $\leq_C^l$-bound $-\tilde{V}$.

Definition 2.3 (Boundedness in [3])**.** A subset $\mathbb{V} \subseteq 2^Y$ is said to be $\leq_C^u$-*bounded below* if there exists some topologically bounded subset $\tilde{V} \subseteq Y$ such that $\tilde{V} \leq_C^u V$ holds for all $V \in \mathbb{V}$. The set $\tilde{V}$ is called a *lower* $\leq_C^u$-*bound* of $\mathbb{V}$. A subset $\mathbb{V} \subseteq 2^Y$ is said to be $\leq_C^l$-*bounded above* and $\tilde{V}$ is called an *upper* $\leq_C^l$-*bound* of $\mathbb{V}$ if $-\mathbb{V} := \{-V : V \in \mathbb{V}\}$ is $\leq_C^u$-bounded below with the lower $\leq_C^u$-bound $-\tilde{V}$.

3. Nonlinear salarization methods on 2^Y

In this section, we introduce two types of scalarization functionals defined on 2^Y, which have the monotonicity with respect to the set-relation $\leq_C^l$; $z : 2^Y \to \mathbb{R}\cup\{\pm\infty\}$ is said to be monotone with respect to $\leq_C^l$ if $V_1 \leq_C^l V_2$ implies $z(V_1) \leq z(V_2)$.

Theorem 3.1 ([3])**.** *Let Y be a topological vector space, $C \subseteq Y$ a convex cone and $k^0 \in C\backslash(-\mathrm{cl}C)$. Let $\mathbb{V} \subseteq 2^Y$ be nonempty and $\leq_C^l$-bounded, i.e., there is a topological bounded set $V' \subseteq Y$ and a nonempty set $V'' \subseteq Y$ such that*

$$\forall V \in \mathbb{V},\ V' \leq_C^l V \leq_C^l V''.$$

Then, the function $z_{k^0}^l : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$z_{k^0}^l(V) := \inf\{t \in \mathbb{R} : tk^0 + V'' \subseteq V + \mathrm{cl}C\},$$

has the following properties:

(i) $z_{k^0}^l$ *is bounded on* $\mathbb{V}$*;*
(ii) $V \in \mathbb{V}, \alpha \in \mathbb{R}$ *implies* $z_{k^0}^l(V + \alpha k^0) = z_{k^0}^l(V) + \alpha$*;*
(iii) $z_{k^0}^l$ *is monotone with respect to* $\leq_C^l$*.*

When we assume that $\mathbb{V}$ is "$\leq_C^l$-bounded below" instead of a stronger condition "$\leq_C^l$-bounded," we propose another scalarization function which has similar properties to those of $z_{k^0}^l$ above.

Theorem 3.2. *Let Y be a topological vector space, $C \subseteq Y$ a convex cone and $k^0 \in C\backslash(-\mathrm{cl}C)$. Let $\{\emptyset\} \neq \mathbb{V} \subseteq 2^Y$ be nonempty and $\leq_C^l$-bounded below, i.e., there is a topological bounded set $V' \subseteq Y$ such that*

$$\forall V \in \mathbb{V},\ V' \leq_C^l V.$$

Then, the function $g_{k^0}^l : 2^Y \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$g_{k^0}^l(V) := \inf\{t \in \mathbb{R} : tk^0 + V' \subseteq V + \mathrm{cl}C\},$$

has the following properties:

(i) $g_{k^0}^l$ *is bounded below on* $\mathbb{V}$*;*
(ii) $V \in \mathbb{V}, \alpha \in \mathbb{R}$ *implies* $g_{k^0}^l(V + \alpha k^0) = g_{k^0}^l(V) + \alpha$*;*
(iii) $g_{k^0}^l$ *is monotone with respect to* $\leq_C^l$*.*

Proof. The statements (ii) and (iii) are obvious. We shall show the statement (i). Since $\mathbb{V} \neq \{\emptyset\}$ and $\mathbb{V}$ is $\leq_C^l$-bounded below, we have $V' \neq \emptyset$. Assume that $g_{k^0}^l$ is not bounded below. Then for each $n \in \mathbb{N}$, there exist $t_n < -n$ and $V_n \in \mathbb{V}$ such that $t_n k^0 + V' \subseteq V_n + \text{cl}C$. Then

$$-nk^0 + V' = (-n - t_n)k^0 + t_n k^0 + V' \subseteq (C \backslash (-\text{cl}C)) + V_n + \text{cl}C \subseteq V_n + \text{cl}C,$$

and $V_n \subseteq V' + C$ since $\mathbb{V}$ is $\leq_C^l$-bounded below lower $\leq_C^l$-bound V'. Consequently, we have $-nk^0 + V' \subseteq V' + \text{cl}C$ for all $n \in \mathbb{N}$. For arbitrary fixed $v_0 \in V'$ and $n \in \mathbb{N}$, $-nk^0 + v_0 \in V' + \text{cl}C$ and hence

$$-k^0 + \frac{1}{n}v_0 \in \frac{1}{n}V' + \text{cl}C. \tag{3.1}$$

Since V' is nonempty and topological bounded, $\frac{1}{n}V' \to \{0\}$. By (3.1) with $\frac{1}{n}v_0 \to 0$ we have $-k^0 \in \text{cl}C$, which is a contradiction to $k^0 \in C \backslash (-\text{cl}C)$. □

Remark 3.3. Under the conditions of Theorem 3.2, functional $g_{k^0}^l$ takes $+\infty$ possibly.

4. Minimal element theorems in 2^Y

In this section, we present minimal element theorems with set-relations. The proofs are indebted to the following existence principle for minimal elements in quasi-ordered sets due to Brézis and Browder [2], 1976.

Theorem 4.1 (Brézis-Browder principle in [2])**.** *Let $(W, \preceq)$ be a quasi-ordered set (i.e., $\preceq$ is a reflexive and transitive relation on W) and let $\phi : W \to \mathbb{R}$ be a function satisfying*

(A1) *ϕ is bounded below;*
(A2) *$w_1 \preceq w_2$ implies $\phi(w_1) \leq \phi(w_2)$;*
(A3) *For every $\preceq$-decreasing sequence $\{w_n\}_{n \in \mathbb{N}} \subseteq W$ there exists some $w \in W$ such that $w \preceq w_n$ for all $n \in \mathbb{N}$.*

Then, for every $w_0 \in W$ there exists some $\bar{w} \in W$ such that

(i) *$\bar{w} \preceq w_0$;*
(ii) *$\hat{w} \preceq \bar{w}$ implies $\phi(\hat{w}) = \phi(\bar{w})$.*

Using Theorem 4.1, we get several types of minimal element theorems with set-relations. Let $\mathbb{A}$ be a subset of $X \times 2^Y$, where X is a separated uniform space equipped with a families of quasi-metrics $\{q_\lambda\}_{\lambda \in \Lambda}$ (see [3] for the detail) and Y is a topological vector space. We define the following notation:

$$\Phi(\mathbb{A}) := \{V \in 2^Y : \exists x \in X, (x, V) \in \mathbb{A}\}.$$

Using the relation $\leq_C^l$ we introduce the following ordering relation on $X \times 2^Y$:

$$(x_1, V_1) \leq_{k^0}^l (x_2, V_2) \Longleftrightarrow \forall \lambda \in \Lambda, V_1 + k^0 q_\lambda(x_1, x_2) \leq_C^l V_2.$$

This ordering has reflexivity and transitivity on $X \times 2^Y$.

Theorem 4.2 ([3])**.** *Let X be a separated uniform space and Y a topological vector space, $C \subseteq Y$ a convex cone and $k^0 \in C \backslash (-\text{cl}C)$. Let $\mathbb{A}$ be a nonempty subset of $X \times 2^Y$ such that for some $(x_0, V_0) \in \mathbb{A}$ and for $\mathbb{A}_0 := \{(x, V) \in \mathbb{A} : (x, V) \leq_{k^0}^l (x_0, V_0)\}$ the following conditions are satisfied:*

(M1) $\Phi(\mathbb{A}_0)$ *is* $\leq_C^l$*-bounded above, i.e.,* V_0 *is nonempty;*
(M2) $\Phi(\mathbb{A}_0)$ *is* $\leq_C^l$*-bounded below;*
(M3) *For every* $\leq_{k^0}^l$*-decreasing sequence* $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{A}_0$ *there exists some* $(x, V) \in \mathbb{A}_0$ *such that* $(x, V) \leq_{k^0}^l (x_n, V_n)$ *for all* $n \in \mathbb{N}$.

Then, there exists $(\bar{x}, \bar{V}) \in \mathbb{A}$ *such that*
(i) $(\bar{x}, \bar{V}) \leq_{k^0}^l (x_0, V_0)$*;*
(ii) $(\hat{x}, \hat{V}) \in \mathbb{A}$ *and* $(\hat{x}, \hat{V}) \leq_{k^0}^l (\bar{x}, \bar{V}) \Longrightarrow \hat{x} = \bar{x}$.

The theorem above was proved in [3] by applying theorem 4.1 with the result of Theorem 3.1. In the same manner, we use Theorem 3.2 to present another minimal element theorem with the same set-relation.

Theorem 4.3. *Let* X *be a separated uniform space and* Y *a topological vector space.* $C \subseteq Y$ *is a convex cone and* $k^0 \in C\backslash(-\mathrm{cl}C)$*. Let* $\mathbb{A}$ *be a nonempty subset of* $X \times 2^Y$ *such that for some* $(x_0, V_0) \in \mathbb{A}$ *with* $V_0 \neq \emptyset$ *and for* $\mathbb{A}_0 := \{(x, V) \in \mathbb{A} : (x, V) \leq_{k^0}^l (x_0, V_0)\}$ *the following conditions are satisfied:*
(M1) $\Phi(\mathbb{A}_0)$ *is* $\leq_C^l$*-bounded below with lower* $\leq_C^l$*-bound* V' *of* $\Phi(\mathbb{A}_0)$*;*
(M2) $V_0 \in \mathbb{V}(k^0, V') := \{V \in 2^Y : tk^0 + V' \subseteq V + \mathrm{cl}C \text{ for some } t \in \mathbb{R}\}$*;*
(M3) *For every* $\leq_{k^0}^l$*-decreasing sequence* $\{(x_n, V_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{A}_0$ *there exists some* $(x, V) \in \mathbb{A}_0$ *such that* $(x, V) \leq_{k^0}^l (x_n, V_n)$ *for all* $n \in \mathbb{N}$.

Then, there exists $(\bar{x}, \bar{V}) \in \mathbb{A}$ *such that*
(i) $(\bar{x}, \bar{V}) \leq_{k^0}^l (x_0, V_0)$*;*
(ii) $(\hat{x}, \hat{V}) \in \mathbb{A}$ *and* $(\hat{x}, \hat{V}) \leq_{k^0}^l (\bar{x}, \bar{V}) \Longrightarrow \hat{x} = \bar{x}$.

Proof. We define $\phi : \mathbb{A}_0 \to \mathbb{R}$ by $\phi(x, V) := g_{k^0}^l(V)$ where $g_{k^0}^l : \Phi(\mathbb{A}_0) \to \mathbb{R}$ is the scalarization function defined in Theorem 3.2.

We shall check the assumptions of Theorem 4.1. By (ii) and (iii) of Theorem 3.2,

$$(x_1, V_1) \leq_{k^0}^l (x_2, V_2) \Longrightarrow \phi(x_1, V_1) \leq \phi(x_2, V_2).$$

Hence ϕ is $\leq_{k^0}^l$-monotone on $\mathbb{A}_0$ (i.e., (A2) holds). By (M1), (M2) and (i) of Theorem 3.2, ϕ is well-defined and bounded below (i.e., (A1) holds). Also (M3) implies (A3) of Theorem 4.1.

Therefore, it follows from Theorem 4.1 that the existence of an element $(\bar{x}, \bar{V}) \in \mathbb{A}_0$ such that
(I) $(\bar{x}, \bar{V}) \leq_{k^0}^l (x, V)$ for any $(x, V) \in \mathbb{A}_0$;
(II) $(\hat{x}, \hat{V}) \in \mathbb{A}_0$ and $(\hat{x}, \hat{V}) \leq_{k^0}^l (\bar{x}, \bar{V}) \Rightarrow \phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V})$.

The property (I) implies the statement (i) of the theorem. To show the statement (ii), let $(\hat{x}, \hat{V}) \in \mathbb{A}$ such that $(\hat{x}, \hat{V}) \leq_{k^0}^l (\bar{x}, \bar{V})$. Since $(\bar{x}, \bar{V}) \in \mathbb{A}_0$, the transitivity of $\leq_{k^0}^l$ yields $(\hat{x}, \hat{V}) \in \mathbb{A}_0$. Hence using the property (II), we have

$$\phi(\hat{x}, \hat{V}) = \phi(\bar{x}, \bar{V}). \tag{4.1}$$

On the other hand, by the definition of $\leq_{k^0}^l$ we have $\hat{V} + k^0 q_\lambda(\hat{x}, \bar{x}) \leq \bar{V}$ for all $\lambda \in \Lambda$. Using the properties (ii) and (iii) of Theorem 3.2, we get for all $\lambda \in \Lambda$,

$$\phi(\hat{x}, \hat{V}) + q_\lambda(\hat{x}, \bar{x}) \leq \phi(\bar{x}, \bar{V}). \tag{4.2}$$

By (4.1) and (4.2), $q_\lambda(\hat{x}, \bar{x}) = 0$ for all $\lambda \in \Lambda$. Since X is separated, we have $\hat{x} = \bar{x}$ by Theorem 4.3 in [3]. □

References

[1] J. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhauser, Boston, 1990.
[2] H. Brézis and F. Browder, *A General Principle on Ordered Sets in Nonlinear Functional Analysis*, Advances in Mathematics **21** (1976), 355–364.
[3] A. Hamel and A. Löhne, *Minimal element theorems and Ekeland's principle with set relations*, J. of Nonlinear and Convex Anal. 7 (2006), 19–37.
[4] D. Kuroiwa, T. Tanaka and T. X. D. Ha, *On cone convexity of set-valued maps*, Nonlinear Anal. **30** (1997), 1487–1496.
[5] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, 319, Springer, Berlin, 1989.
[6] S. Nishizawa, M. Onodsuka and T. Tanaka, *Alternative Theorems for Set-Valued Maps based on a Nonlinear Scalarization*, Pacific J. Optim. **1** (2005), 147–159.

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Akira Shimizu
Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan
E-mail address: akira@m.sc.niigata-u.ac.jp

Tamaki Tanaka
Department of Mathematical Science, Graduate School of Science and Technology, Niigata University, Niigata 950-2181, Japan
E-mail address: tamaki@math.sc.niigata-u.ac.jp