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CONVERGENCE THEOREMS FOR FIXED POINT PROBLEMS AND VARIATIONAL INEQUALITY PROBLEMS

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ABSTRACT. In this paper, we introduce an iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an α-inverse-strongly monotone mapping in a Hilbert space. We show that the sequence converges strongly to a common element of two sets under the some mild conditions on parameters.

1. Introduction

Let C be a closed convex subset of a real Hilbert space H. Recall that a mapping S of C into itself is called *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. A mapping A of C into H is called *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0 \tag{1.1}$$

for all $u, v \in C$ and A is called *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2$$

for all $u, v \in C$. It is well known that the variational inequality problem VI(C, A) is to find $x^* \in C$ such that

$$\langle Ax^*, v - x^*\rangle \geq 0$$

for all $v \in C$ (see [1], [3], [7]). The variational inequality has been extensively studied in the literature. See, e.g., [10], [11], [12] and the references therein.

For finding an element of $F(S) \cap VI(C, A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive and a mapping A of C into H is α-inverse-strongly monotone, Takahashi and Toyoda [8] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n) \tag{1.2}$$

for every $n = 0, 1, 2, \cdots$, where P_C is the metric projection of H onto C, $x_0 = x \in C$, $\{\alpha_n\}$ is a sequence in $(0, 1)$, and $\{\lambda_n\}$ is a sequence in $(0, 2\alpha)$. They showed that, if $F(S) \cap VI(C, A)$ is nonempty, then the sequence $\{x_n\}$ generated by (1.2) converges weakly to some $z \in F(S) \cap VI(C, A)$. In 2005, Iiduka and Takahashi [2] further

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considered a new iterative scheme for a nonexpansive mapping and an α-inverse-strongly monotone mapping and obtained the following strong convergence theorem.

Theorem 1.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose $x_1 = x \in C$ and $\{x_n\}$ is given by*

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n) \tag{1.3}$$

for every $n = 1, 2, \cdots$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ defined by (1.3) converges strongly to $P_{F(S)\cap VI(C,A)}x$.

In this paper, motivated by the iterative schemes (1.2) and (1.3), we introduce a new iterative scheme for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse-strongly monotone mapping. We obtain a strong convergence theorem under the some mild conditions on parameters.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and let C be a closed convex subset of H. It is well known that, for any $u \in H$, there exists a unique $u_0 \in C$ such that

$$\|u - u_0\| = \inf\{\|u - x\| : x \in C\}.$$

We denote u_0 by $P_C u$, where P_C is called the *metric projection* of H onto C. The metric projection P_C of H onto C has the following basic properties:

Property (i): $\|P_C x - P_C y\| \leq \|x - y\|$ for all $x, y \in H$;
Property (ii): $\langle x - y, P_C x - P_C y \rangle \geq \|P_C x - P_C y\|^2$ for every $x, y \in H$;
Property (iii): $\langle x - P_C x, y - P_C x \rangle \leq 0$ for all $x \in H$ and $y \in C$;
Property (iv): $\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2$ for all $x \in H$ and $y \in C$.

Such properties of P_C will be crucial in the proofs of our main results. Let A be a monotone mapping of C into H. In the context of the variational inequality problem, it is easy to see from Property (iv) that

$$u \in VI(C, A) \Leftrightarrow u = P_C(u - \lambda A u), \quad \forall \lambda > 0. \tag{2.1}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$; i.e.,

$$N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}.$$

Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$ (see [2],[5]).

Now, we introduce several lemmas for our main results in this paper.

Lemma 2.1 ([6]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty} \|y_n - x_n\| = 0$.*

Lemma 2.2 ([4]). *Let H be a real Hilbert space. Then the following inequality holds: for each $x, y \in H$, we have*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle.$$

Lemma 2.3 ([9]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

Now we state and prove our main results in this section.

Theorem 3.1. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Suppose fixed $u \in C$ and given $x_0 \in C$ arbitrarily. Let $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta x_n + (1 - \beta)S[\alpha_n u + (1 - \alpha_n)P_C(x_n - \lambda_n Ax_n)], \quad \forall n \geq 0, \tag{3.1}$$

where $\beta \in (0, 1)$ is a constant, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$ and

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\quad \sum_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then $\{x_n\}$ defined by (3.1) converges strongly to $P_{F(S)\cap VI(C,A)}u$.

Proof. Since $\lambda_n \in [0, 2\alpha]$ and A is an α-inverse-strongly monotone mapping, we have, for all $x, y \in C$,

$$\begin{aligned} \|(I - \lambda_n A)x - (I - \lambda_n A)y\|^2 &= \|(x - y) - \lambda_n(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda_n\langle x - y, Ax - Ay\rangle \end{aligned}$$

$$+\lambda_n^2\|Ax-Ay\|^2 \tag{3.2}$$
$$\leq \|x-y\|^2+\lambda_n(\lambda_n-2\alpha)\|Ax-Ay\|^2,$$

which implies that $I-\lambda_n A$ is nonexpansive.

Let $x^* \in F(S)\cap VI(C,A)$. Then $x^*=P_C(x^*-\lambda_n Ax^*)$. Setting $y_n=\alpha_n u+(1-\alpha_n)P_C(x_n-\lambda_n Ax_n)$ for all $n\geq 0$, we have from Property (i) and (3.2) that

$$\begin{aligned}
\|y_n-x^*\| &= \|\alpha_n(u-x^*)+(1-\alpha_n)[P_C(x_n-\lambda_n Ax_n)-x^*]\| \\
&= \|\alpha_n(u-x^*)+(1-\alpha_n)[P_C(x_n-\lambda_n Ax_n)-P_C(x^*-\lambda_n Ax^*)]\| \\
&\leq \alpha_n\|u-x^*\|+(1-\alpha_n)\|P_C(x_n-\lambda_n Ax_n)-P_C(x^*-\lambda_n Ax^*)\| \qquad (3.3)\\
&\leq \alpha_n\|u-x^*\|+(1-\alpha_n)\|(x_n-\lambda_n Ax_n)-(x^*-\lambda_n Ax^*)\| \\
&\leq \alpha_n\|u-x^*\|+(1-\alpha_n)\|x_n-x^*\|.
\end{aligned}$$

By (3.1) and (3.3), we get

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\beta(x_n-x^*)+(1-\beta)(Sy_n-x^*)\| \\
&\leq \beta\|x_n-x^*\|+(1-\beta)\|y_n-x^*\| \\
&\leq \beta\|x_n-x^*\|+(1-\beta)\alpha_n\|u-x^*\| \\
&\quad +(1-\beta)(1-\alpha_n)\|x_n-x^*\| \\
&= [1-(1-\beta)\alpha_n]\|x_n-x^*\|+(1-\beta)\alpha_n\|u-x^*\| \\
&\leq \max\{\|u-x^*\|,\|x_0-x^*\|\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Hence $\{y_n\}$, $\{Sy_n\}$ and $\{Ax_n\}$ are also bounded.

Note that

$$\begin{aligned}
y_{n+1}-y_n &= (\alpha_{n+1}-\alpha_n)u+(1-\alpha_{n+1})P_C(x_{n+1}-\lambda_{n+1}Ax_{n+1}) \\
&\quad -(1-\alpha_n)P_C(x_n-\lambda_n Ax_n) \\
&= (\alpha_{n+1}-\alpha_n)u+(1-\alpha_{n+1})[P_C(x_{n+1}-\lambda_{n+1}Ax_{n+1}) \\
&\quad -P_C(x_n-\lambda_n Ax_n)]+(\alpha_n-\alpha_{n+1})P_C(x_n-\lambda_n Ax_n).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{n+1}-y_n\| &\leq |\alpha_{n+1}-\alpha_n|(\|u\|+\|P_C(x_n-\lambda_n Ax_n)\|) \\
&\quad +(1-\alpha_{n+1})\|P_C(x_{n+1}-\lambda_{n+1}Ax_{n+1})-P_C(x_n-\lambda_n Ax_n)\| \\
&\leq |\alpha_{n+1}-\alpha_n|(\|u\|+\|P_C(x_n-\lambda_n Ax_n)\|) \\
&\quad +(1-\alpha_{n+1})\|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_n Ax_n)\| \\
&= |\alpha_{n+1}-\alpha_n|(\|u\|+\|P_C(x_n-\lambda_n Ax_n)\|) \qquad (3.4)\\
&\quad +(1-\alpha_{n+1})\|(x_{n+1}-\lambda_{n+1}Ax_{n+1})-(x_n-\lambda_{n+1}Ax_n) \\
&\quad +(\lambda_n-\lambda_{n+1})Ax_n\| \\
&\leq |\alpha_{n+1}-\alpha_n|(\|u\|+\|P_C(x_n-\lambda_n Ax_n)\|) \\
&\quad +\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|Sy_{n+1}-Sy_n\| &\leq \|y_{n+1}-y_n\| \\
&\leq |\alpha_{n+1}-\alpha_n|(\|u\|+\|P_C(x_n-\lambda_n Ax_n)\|) \\
&\quad +\|x_{n+1}-x_n\|+|\lambda_{n+1}-\lambda_n|\|Ax_n\|,
\end{aligned}$$

which implies that

$$\limsup_{n\to\infty}(\|Sy_{n+1} - Sy_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Hence, by Lemma 2.1, we obtain $\|Sy_n - x_n\| \to 0$ as $n \to \infty$. Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta)\|Sy_n - x_n\| = 0. \tag{3.5}$$

From (3.4) and (3.5), we also have $\|y_{n+1} - y_n\| \to 0$ as $n \to \infty$.

For $x^* \in F(S) \cap VI(C, A)$, from (3.2), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta(x_n - x^*) + (1-\beta)(Sy_n - x^*)\|^2 \\
&\le [\beta\|x_n - x^*\| + (1-\beta)\|Sy_n - x^*\|]^2 \\
&= \beta^2\|x_n - x^*\|^2 + (1-\beta)^2\|Sy_n - x^*\|^2 \\
&\quad + 2\beta(1-\beta)\|x_n - x^*\|\|Sy_n - x^*\| \\
&\le \beta^2\|x_n - x^*\|^2 + (1-\beta)^2\|Sy_n - x^*\|^2 \\
&\quad + \beta(1-\beta)(\|x_n - x^*\|^2 + \|Sy_n - x^*\|^2) \\
&= \beta\|x_n - x^*\|^2 + (1-\beta)\|Sy_n - x^*\|^2 \\
&\le \beta\|x_n - x^*\|^2 + (1-\beta)\|y_n - x^*\|^2 \\
&= \beta\|x_n - x^*\|^2 + (1-\beta)[\|\alpha_n(u - x^*) \\
&\quad + (1-\alpha_n)(P_C(x_n - \lambda_n Ax_n) - P_C(x^* - \lambda_n Ax^*))\|]^2 \\
&\le \beta\|x_n - x^*\|^2 + (1-\beta)[\alpha_n\|u - x^*\|^2 \\
&\quad + (1-\alpha_n)\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2] \\
&\le \beta\|x_n - x^*\|^2 + (1-\beta)\{\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|x_n - x^*\|^2 \\
&\quad + (1-\alpha_n)\lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ax^*\|^2\} \\
&\le (1-\beta)\alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad + (1-\beta)(1-\alpha_n)a(b - 2\alpha)\|Ax_n - Ax^*\|^2.
\end{aligned}$$

Then we have

$$\begin{aligned}
&-(1-\beta)(1-\alpha_n)a(b-2\alpha)\|Ax_n - Ax^*\|^2 \\
&\le (1-\beta)\alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= (1-\beta)\alpha_n\|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad \times (\|x_n - x^*\| - \|x_{n+1} - x^*\|) \\
&\le (1-\beta)\alpha_n\|u - x^*\|^2 + (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \times \|x_n - x_{n+1}\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$, we obtain $\|Ax_n - Ax^*\| \to 0$ as $n \to \infty$.

Setting $z_n = P_C(x_n - \lambda_n Ax_n)$ for all $n \geq 0$, from Property (ii), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|P_C(x_n - \lambda_n Ax_n) - P_C(x^* - \lambda_n Ax^*)\|^2 \\
&\leq \langle (x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*), z_n - x^* \rangle \\
&= \frac{1}{2}\{\|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*)\|^2 + \|z_n - x^*\|^2 \\
&\quad - \|(x_n - \lambda_n Ax_n) - (x^* - \lambda_n Ax^*) - (z_n - x^*)\|^2\} \\
&\leq \frac{1}{2}\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 \\
&\quad - \|(x_n - z_n) - \lambda_n(Ax_n - Ax^*)\|^2\} \\
&= \frac{1}{2}\{\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|x_n - z_n\|^2 \\
&\quad + 2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle - \lambda_n^2\|Ax_n - Ax^*\|^2\}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|x_n - z_n\|^2 \\
+ 2\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle - \lambda_n^2\|Ax_n - Ax^*\|^2,
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta(x_n - x^*) + (1-\beta)(Sy_n - x^*)\|^2 \\
&\leq \beta\|x_n - x^*\|^2 + (1-\beta)\|y_n - x^*\|^2 \\
&\leq \beta\|x_n - x^*\|^2 + (1-\beta)[\alpha_n\|u - x^*\|^2 + (1-\alpha_n)\|z_n - x^*\|^2] \\
&\leq \alpha_n\|u - x^*\|^2 + \beta\|x_n - x^*\|^2 + (1-\beta)\|z_n - x^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - (1-\beta)\|x_n - z_n\|^2 \\
&\quad + 2(1-\beta)\lambda_n\langle x_n - z_n, Ax_n - Ax^*\rangle - (1-\beta)\lambda_n^2\|Ax_n - Ax^*\|^2 \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - (1-\beta)\|x_n - z_n\|^2 \\
&\quad + 2(1-\beta)\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
(1-\beta)\|x_n - z_n\|^2 &\leq \alpha_n\|u - x^*\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2(1-\beta)\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\| \\
&\leq \alpha_n\|u - x^*\|^2 + \|x_n - x_{n+1}\| \times (\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\
&\quad + 2(1-\beta)\lambda_n\|x_n - z_n\|\|Ax_n - Ax^*\|.
\end{aligned}$$

Since $\alpha_n \to 0, \|x_n - x_{n+1}\| \to 0$ and $\|Ax_n - Ax^*\| \to 0$ as $n \to \infty$, we have $\|x_n - z_n\| \to 0$ as $n \to \infty$. At the same time, we note that

$$y_n - z_n = \alpha_n(u - z_n),$$

then we have

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.6}$$

Since

$$\begin{aligned}\|Sz_n - z_n\| &\le \|Sz_n - Sy_n\| + \|Sy_n - x_n\| + \|x_n - z_n\| \\ &\le \|z_n - y_n\| + \|Sy_n - x_n\| + \|x_n - z_n\|,\end{aligned}$$

we can conclude that $\|Sz_n - z_n\| \to 0$ as $n \to \infty$.

Next we show that

$$\limsup_{n\to\infty}\langle u - z_0, z_n - z_0\rangle \le 0,$$

where $z_0 = P_{F(S)\cap VI(C,A)}u$.

To show it, we choose a subsequence $\{z_{n_i}\}$ of $\{z_n\}$ such that

$$\limsup_{n\to\infty}\langle u - z_0, Sz_n - z_0\rangle = \lim_{i\to\infty}\langle u - z_0, Sz_{n_i} - z_0\rangle.$$

As $\{z_{n_i}\}$ is bounded, we have that a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ converges weakly to z. We may assume without loss of generality that $z_{n_i} \rightharpoonup z$. Since $\|Sz_n - z_n\| \to 0$, we obtain $Sz_{n_i} \rightharpoonup z$ as $i \to \infty$. Then we can obtain $z \in F(S) \cap VI(C, A)$. In fact, let us first show that $z \in VI(C, A)$.

Let

$$Tv = \begin{cases} Av + N_C v, v \in C, \\ \qquad\quad \emptyset, v \notin C. \end{cases}$$

Then T is maximal monotone. Let $(v, w) \in G(T)$. Since $w - Av \in N_C v$ and $z_n \in C$, we have $\langle v - z_n, w - Av\rangle \ge 0$. On the other hand, from $z_n = P_C(x_n - \lambda_n Ax_n)$, we have $\langle v - z_n, z_n - (x_n - \lambda_n Ax_n)\rangle \ge 0$, that is,

$$\langle v - z_n, \frac{z_n - x_n}{\lambda_n} + Ax_n\rangle \ge 0.$$

Therefore, we have

$$\begin{aligned}\langle v - z_{n_i}, w\rangle &\ge \langle v - z_{n_i}, Av\rangle \\ &\ge \langle v - z_{n_i}, Av\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} + Ax_{n_i}\rangle \\ &= \langle v - z_{n_i}, Av - Ax_{n_i} - \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &= \langle v - z_{n_i}, Av - Az_{n_i}\rangle + \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle \\ &\quad - \langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle \\ &\ge \langle v - z_{n_i}, Az_{n_i} - Ax_{n_i}\rangle - \langle v - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}}\rangle.\end{aligned}$$

Hence we obtain $\langle v - z, w\rangle \ge 0$ as $i \to \infty$. Since T is maximal monotone, we have $z \in T^{-1}0$ and hence $z \in VI(C, A)$. Let us show that $z \in F(S)$. Assume $z \notin F(S)$. From Opial's condition, we have

$$\begin{aligned}\liminf_{i\to\infty}\|z_{n_i} - z\| &< \liminf_{i\to\infty}\|z_{n_i} - Sz\| \\ &= \liminf_{i\to\infty}\|z_{n_i} - Sz_{n_i} + Sz_{n_i} - Sz\| \\ &\le \liminf_{i\to\infty}\|Sz_{n_i} - Sz\| \\ &\le \liminf_{i\to\infty}\|z_{n_i} - z\|.\end{aligned}$$

This is a contradiction. Thus, we obtain $z \in F(S)$. Then we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u-z_0, z_n-z_0\rangle &= \limsup_{n\to\infty}\langle u-z_0, Sz_n-z_0\rangle \\ &= \lim_{i\to\infty}\langle u-z_0, Sz_{n_i}-z_0\rangle \\ &= \langle u-z_0, z-z_0\rangle \\ &\le 0. \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\limsup_{n\to\infty}\langle u-z_0, y_n-z_0\rangle \le 0. \tag{3.8}$$

Therefore, form Lemma 2.2 and (3.1), we have

$$\begin{aligned}\|x_{n+1}-z_0\|^2 &\le \beta\|x_n-z_0\|^2+(1-\beta)\|y_n-z_0\|^2 \\ &\le \beta\|x_n-z_0\|^2+(1-\beta)[(1-\alpha_n)\|z_n-z_0\|^2 \\ &\quad +2\alpha_n\langle u-z_0, y_n-z_0\rangle] \\ &\le \beta\|x_n-z_0\|^2+(1-\beta)[(1-\alpha_n)\|x_n-z_0\|^2 \\ &\quad +2\alpha_n\langle u-z_0, y_n-z_0\rangle] \\ &= [1-(1-\beta)\alpha_n]\|x_n-z_0\|^2+(1-\beta)\alpha_n\{2\langle u-z_0, y_n-z_0\rangle\} \\ &= (1-\gamma_n)\|x_n-z_0\|^2+\delta_n, \end{aligned} \tag{3.9}$$

where $\gamma_n = 1-(1-\beta)\alpha_n$ and $\delta_n = (1-\beta)\alpha_n\{2\langle u-z_0, y_n-z_0\rangle\}$. It is easily seen that $\sum_{n=0}^{\infty}\gamma_n = \infty$ and

$$\limsup_{n\to\infty}\delta_n/\gamma_n = \limsup_{n\to\infty}\{2\langle u-z_0, y_n-z_0\rangle\} \le 0.$$

Thus by Lemma 2.3 and (3.9), we can obtain the desired conclusion. This completes the proof. □

Corollary 3.2. *Let C be a closed convex subset of a real Hilbert space H. Let A be an α-inverse-strongly monotone mapping of C into H such that $VI(C,A) \neq \emptyset$. Suppose fixed $u \in C$ and given $x_0 \in C$ arbitrarily. Let $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta x_n + (1-\beta)[\alpha_n u + (1-\alpha_n)P_C(x_n - \lambda_n A x_n)], \quad \forall n \ge 0, \tag{3.10}$$

where $\beta \in (0,1)$ is constant, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a,b with $0<a<b<2\alpha$ and

(i) $\lim_{n\to\infty}\alpha_n = 0, \quad \sum_{n=0}^{\infty}\alpha_n = \infty,$
(ii) $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0,$

then $\{x_n\}$ defined by (3.10) converges strongly to $P_{VI(C,A)}u$.

A mapping $T: C \to C$ is called strictly pseudocontractive if there exists k with $0 \le k < 1$ such that

$$\|Tx-Ty\|^2 \le \|x-y\|^2 + k\|(I-T)x-(I-T)y\|^2$$

for all $x, y \in C$. Put $A = I - T$. Then we have

$$\|(I-A)x-(I-A)y\|^2 \le \|x-y\|^2 + k\|Ax-Ay\|^2.$$

On the other hand,

$$\|(I-A)x-(I-A)y\|^2 = \|x-y\|^2 + \|Ax-Ay\|^2 - 2\langle x-y, Ax-Ay\rangle.$$

Hence we have

$$\langle x-y, Ax-Ay\rangle \geq \frac{1-k}{2}\|Ax-Ay\|^2.$$

Now we can get the following result.

Theorem 3.3. *Let C be a closed convex subset of a real Hilbert space H. Let T be a k-strictly pseudocontractive mapping of C into itself and let S be a nonexpansive mapping of C into itself such that $F(T)\cap F(S) \neq \emptyset$. Suppose fixed $u \in C$ and given $x_0 \in C$ arbitrarily. Let $\{x_n\}$ be generated iteratively by*

$$x_{n+1} = \beta x_n + (1-\beta)S\{\alpha_n u + (1-\alpha_n)[(1-\lambda_n)x_n + \lambda_n T x_n]\}, \quad \forall n \geq 0, \tag{3.11}$$

where $\beta \in (0,1)$ is constant, $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence in $[0,2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < 2\alpha$ and

(i) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=0}^{\infty} \alpha_n = \infty,$

(ii) $\lim_{n\to\infty}(\lambda_{n+1}-\lambda_n) = 0,$

then $\{x_n\}$ defined by (3.11) converges strongly to $P_{F(T)\cap F(S)}u$.

Proof. Put $A = I - T$. Then A is $(1-k)/2$-inverse-strongly monotone. We have $F(T) = VI(C,A)$ and $P_C(x_n - \lambda_n A x_n) = (1-\lambda_n)x_n + \lambda_n T x_n$. So, by Theorem 3.1, we can obtain the desired result. This completes the proof. □

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