Journal of Nonlinear and Convex Analysis
Volume 9, Number 2, 2008, 215–238

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

APPROXIMATION PROCESSES OF CONVOLUTION TYPE OPERATORS IN BANACH SPACES

TOSHIHIKO NISHISHIRAHO

Abstract. We consider the convergence of approximation processes of convolution type operators in Banach spaces and give quantitative estimates of the rate of their convergence in terms of the modulus of continuity and higher order abosolute moments of approximate kernels. Furthermore, applications are discussed for various summation processes and multiplier operators in connection with Fourier series expansions corresponding to a total, fundamental sequence of mutually orthogonal projections as well as for homogeneous Banach spaces which include the certain classical function spaces, as particular cases. We also give several concrete examples of approximating operators from a probabilistic point of view. These can be induced by various probability density functuions, together with various positive summability kernels.

1. Introduction

Let X be a Banach space with norm $\|\cdot\|_X$. Let $\mathfrak{L} = \{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of operators from X to itself, where D is a directed set and Λ is an index set. Then the family $\mathfrak{L}$ is called an approximation process on X if for every $f \in X$,

$$\lim_{\alpha} \|L_{\alpha,\lambda}(f) - f\|_X = 0 \qquad \text{uniformly in } \lambda \in \Lambda. \tag{1.1}$$

Here we consider a family of convolutuion type operators on X defined as follows:

Let $\mathfrak{T} = \{T_\alpha(t) : \alpha \in D, t \in \mathbb{R}\}$ be a family of operators from X to itself, where $\mathbb{R}$ denotes the real line, such that for each $\alpha \in D, f \in X$, the mapping $t \mapsto T_\alpha(t)(f)$ is bounded and strongly continuous on $\mathbb{R}$. Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1(\mathbb{R})$, which denotes the Banach space of all Lebesgue integrable functions g on $\mathbb{R}$ with the norm

$$\|g\|_1 = \int_{\mathbb{R}} |g(t)|\, dt.$$

Then we define a convolution type operator by the form

$$L_{\alpha,\lambda}(f) = \int_{\mathbb{R}} k_{\alpha,\lambda}(t) T_\alpha(t)(f)\, dt \qquad (f \in X), \tag{1.2}$$

which exists as a Bochner integral.

2000 *Mathematics Subject Classification.* Primary 41A35, 41A25, 41A65, 44A35.

Key words and phrases. Approximation process, approximate kernel, absolute moment, convolution type operator, modulus of continuity, stronggly continuous group of operators, summation process, probability density function, summability kernel, Fourier series expansion, multiplier operator, homogeneous Banach space.

The purpose of this paper is to consider the convergence behavior (1.1) for the family $\mathfrak{L}$ of convolution type operators defined by (1.2) and give quantitative estimates of the rate of its convergence under certain appropriate conditions. Furthermore, applications are discussed for various summation processes (cf. [1], [12]) and multiplier operators on X (cf. [4], [11], [12], [16]) as well as for the homogeneous Banach spaces on $\mathbb{R}$ (cf. [9], [11], [15], [17]), which include the Banach space $BUC(\mathbb{R})$ of all bounded uniformly continuous functions f on $\mathbb{R}$ with the norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in \mathbb{R}\}$$

and the Banach space $L^p(\mathbb{R})$ of all the pth power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \Big(\int_{\mathbb{R}} |f(t)|^p\,dt\Big)^{1/p} \qquad (1 \le p < \infty),$$

as special cases.

Certain portions of the method treated below can be carried out more generally for spaces of functions defined on the r-dimensional Euclidean space $\mathbb{R}^r$. However, here we shall consider as carrier space only $\mathbb{R}^1 = \mathbb{R}$ in view of the great precision of the results achievable, and the importance of this case in applications.

2. Convergence theorems

A family $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ of functions in $L^1(\mathbb{R})$ is called an approximate kernel if

$$\limsup_{\alpha}(\sup\{\|k_{\alpha,\lambda}\|_1 : \lambda \in \Lambda\}) < \infty, \tag{2.1}$$

$$\lim_{\alpha} \int_{\mathbb{R}} k_{\alpha,\lambda}(t)\,dt = 1 \qquad \text{uniformly in } \lambda \in \Lambda \tag{2.2}$$

and for each fixed $\delta > 0$,

$$\lim_{\alpha} \int_{|t|\ge\delta} |k_{\alpha,\lambda}(t)|\,dt = 0 \qquad \text{uniformly in } \lambda \in \Lambda. \tag{2.3}$$

If $k_{\alpha,\lambda}(t) \ge 0$ (a.e.t) for all $\alpha \in D$ and all $\lambda \in \Lambda$, then $\mathfrak{K}$ is said to be positive. If

$$\int_{\mathbb{R}} k_{\alpha,\lambda}(t)\,dt = 1$$

for all $\alpha \in D$ and all $\lambda \in \Lambda$, then $\mathfrak{K}$ is said to be normal.

Remark 1. Let $\mathfrak{G} = \{g_\alpha\}_{\alpha\in D}$ be a net of functions in $L^1(\mathbb{R})$. We define $k_{\alpha,\lambda} = g_\alpha$ for all $\alpha \in D$ and all $\lambda \in \Lambda$. Then $\mathfrak{G}$ is called an approximate identity (or summability kernel) if $\mathfrak{K}$ is an approximate kernel. Thererfore, (2.1), (2.2) and (2.3) reduce to the following conditions, respectively (cf. [3], [9]):

$$\limsup_{\alpha} \|g_\alpha\|_1 < \infty, \tag{2.4}$$

$$\lim_{\alpha} \int_{\mathbb{R}} g_\alpha(u)\,du = 1 \tag{2.5}$$

and for each fixed $\delta > 0$,

$$\lim_{\alpha} \int_{|u|\geq\delta} |g_\alpha(u)|\,du = 0. \tag{2.6}$$

Let $\mathbb{I}$ be an interval of $\mathbb{R}$ and $0 \in \mathbb{I}$. Let $\{f_{(\alpha,t)}\}_{(\alpha,t)\in D\times\mathbb{I}}$ be a net of elements in X and $f \in X$. Then $\lim_{(\alpha,t)_0} f_{(\alpha,t)} = f$ implies that for any $\epsilon > 0$, there exist an $\alpha_0 \in D$ and a $\delta > 0$ such that if $\alpha \geq \alpha_0, \alpha \in D$ and $0 < |t| < \delta, t \in \mathbb{I}$, then $\|f_{(\alpha,t)} - f\|_X < \epsilon$.

Lemma 2.1. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be an approximate kernel. Let $\{\varphi_\alpha\}_{\alpha\in D}$ be a net of bounded continuous functions from $\mathbb{R}$ to X which satisfy*

$$\limsup_{\alpha}(\sup\{\|\varphi_\alpha(t)\|_X : t \in \mathbb{R}\}) < \infty \tag{2.7}$$

and

$$\lim_{(\alpha,t)_0} \varphi_\alpha(t) = f. \tag{2.8}$$

Then we have

$$\lim_{\alpha}\Big\|\int_{\mathbb{R}} k_{\alpha,\lambda}(t)\varphi_\alpha(t)\,dt - f\Big\|_X = 0 \qquad \textit{uniformly in } \lambda \in \Lambda. \tag{2.9}$$

Proof. Let $\epsilon > 0$ be given. Then by (2.8), there exist an $\alpha_1 \in D$ and a $\delta > 0$ such that $\|\varphi_\alpha(t) - f\|_X < \epsilon$ for all $\alpha \in D$ with $\alpha \geq \alpha_1$ and all $t \in \mathbb{R}$ with $0 < |t| < \delta$. Also, by (2.1) and (2.7), there exist an $\alpha_2 \in D$ and a constant $C > 0$ such that $\|k_{\alpha,\lambda}\|_1 \leq C$ and $\|\varphi_\alpha(t)\|_X \leq C$ for all $\alpha \geq \alpha_2, \alpha \in D, \lambda \in \Lambda$ and all $t \in \mathbb{R}$. Now, we have

$$\begin{aligned}\Big\|\int_{\mathbb{R}} k_{\alpha,\lambda}(t)\varphi_\alpha(t)\,dt - f\|_X &\leq \int_{\mathbb{R}} |k_{\alpha,\lambda}(t)|\|\varphi_\alpha(t) - f\|_X\,dt + \Big|\int_{\mathbb{R}} k_{\alpha,\lambda}(t)\,dt - 1\Big|\|f\|_X \\ &= I_{\alpha,\lambda} + J_{\alpha,\lambda},\end{aligned}$$

say. Therefore, setting $\alpha_3 = \sup\{\alpha_1, \alpha_2\}$ we obtain

$$I_{\alpha,\lambda} \leq C\epsilon + (C + \|f\|_X)\int_{|t|\geq\delta} |k_{\alpha,\lambda}(t)|\,dt \qquad (\alpha \geq \alpha_3, \lambda \in \Lambda),$$

which together with (2.3) implies $\lim_\alpha I_{\alpha,\lambda} = 0$ uniformly in $\lambda \in \Lambda$. Also, by (2.2) we have $\lim_\alpha J_{\alpha,\lambda} = 0$ uniformly in $\lambda \in \Lambda$. Thus (2.9) holds. □

For each $k \in L^1(\mathbb{R})$ and $q > 0$, we define

$$\mu(k; q) = \int_{\mathbb{R}} |t|^q |k(t)|\,dt,$$

which is called the qth absolute moment of k.

From now on we suppose that for each $f \in X$,

$$\limsup_{\alpha}(\sup\{\|T_\alpha(t)(f)\|_X : t \in \mathbb{R}\}) < \infty \tag{2.10}$$

and

$$\lim_{(\alpha,t)_0} T_\alpha(t)(f) = f. \tag{2.11}$$

Theorem 2.2. *If $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an approximate kernel, then $\mathfrak{L}$ is an approximation process on X.*

Proof. Let $f \in X$ be fixed. Then we define $\varphi_\alpha(t) = T_\alpha(t)(f)$ for all $\alpha \in D$ and all $t \in \mathbb{R}$. Then (2.10) and (2.11) imply (2.7) and (2.8), respectively. Thus the desired result follows from Lemma 2.1. □

Theorem 2.3. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1(\mathbb{R})$ satisfying (2.1) and (2.2). If*

$$\lim_{\alpha} \mu(k_{\alpha,\lambda}; q) = 0 \qquad \text{uniformly in } \lambda \in \Lambda \tag{2.12}$$

for some $q > 0$, then $\mathfrak{L}$ is an approximation process on X.

Proof. We have

$$\int_{|t|\geq\delta} |k_{\alpha,\lambda}(t)|\, dt \leq \frac{\mu(k_{\alpha,\lambda}; q)}{\delta^q} \qquad (\delta > 0)$$

for all $\alpha \in D$ and all $\lambda \in \Lambda$, and so (2.12) implies (2.3). Therefore, the desired result follows from Theorem 2.2. □

In the rest of this section, let $\{T(t) : t \in \mathbb{R}\}$ be a family of operators from X to itself such that $T(0) = I$ (the identity operator) and for each $f \in X$, the mapping $t \mapsto T(t)(f)$ is bounded and strongly continuous on $\mathbb{R}$, and let $\{\epsilon_\alpha\}_{\alpha\in D}$ be a net of positive real numbers.

Corollary 2.4. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be an approximate kernel. Then the following statements hold:*

(a) *Suppose that $\lim_\alpha \epsilon_\alpha = 0$, and define $T_\alpha(t) = T(t + \epsilon_\alpha)$ for all $\alpha \in D$ and all $t \in \mathbb{R}$. Then $\mathfrak{L}$ is an approximation process on X.*

(b) *Suppose that $\limsup_\alpha \epsilon_\alpha < \infty$, and define $T_\alpha(t) = T(t\epsilon_\alpha)$ for all $\alpha \in D$ and all $t \in \mathbb{R}$. Then $\mathfrak{L}$ is an approximation process on X.*

Corollary 2.5. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1(\mathbb{R})$ satisfying (2.1) and (2.2). If (2.12) holds for some $q > 0$, then statements (a) and (b) in Corollary 2.4 hold.*

Lemma 2.6. *Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be continuous and let $F : \mathbb{R} \to X$ be bounded and strongly continuous. Let $\mathfrak{G} = \{g_\alpha\}_{\alpha\in D}$ be an approximate identity. We define*

$$F_\alpha(t) = \int_{\mathbb{R}} g_\alpha(u) F(\varphi(t, u))\, du \qquad (\alpha \in D, t \in \mathbb{R}). \tag{2.13}$$

Then the following statements hold:

(a) *For each $\alpha \in D$, F_α is bounded and strongly continuous on $\mathbb{R}$.*

(b) *We have*

$$\lim_{(\alpha,t)_0} F_\alpha(t) = F(\varphi(0, 0)). \tag{2.14}$$

Proof. (a) For all $(t, u) \in \mathbb{R}^2$ we have $\|F(\varphi(t, u))\|_X \leq C$, where $C = \sup\{\|F(s)\|_X : s \in \mathbb{R}\} < \infty$, and so

$$\|F_\alpha(t)\|_X \leq C\|g_\alpha\|_1 \tag{2.15}$$

for all $\alpha \in D$ and all $t \in \mathbb{R}$. We have $\|g_\alpha(u)F(\varphi(t,u))\|_X \le C|g_\alpha(u)|$ for all $t, u \in \mathbb{R}$ and all $\alpha \in D$. Let $t_0 \in \mathbb{R}$. Then $\lim_{t\to t_0} \|g_\alpha(u)(F(\varphi(t,u)) - F(\varphi(t_0,u)))\|_X = 0$ for each $u \in \mathbb{R}$. Therefore, the dominated convergence theorem implies $\lim_{t\to t_0} \|F_\alpha(t) - F_\alpha(t_0)\|_X = 0$.

(b) Let $\epsilon > 0$ be given. Then there exist a $\delta > 0$ such that $\sqrt{t^2+u^2} < \delta$ implies $\|F(\varphi(t,u)) - F(\varphi(0,0))\|_X < \epsilon$. By (2.4), there exist a constant $K > 0$ and an $\alpha_0 \in D$ such that $\|g_\alpha\|_1 \le K$ for all $\alpha \in D$ with $\alpha \ge \alpha_0$. Now we have

$$\|F_\alpha(t) - F(\varphi(0,0))\|_X \le \int_{\mathbb{R}} |g_\alpha(u)| \|F(\varphi(t,u)) - F(\varphi(0,0))\|_X \, du$$
$$+ \left|\int_{\mathbb{R}} g_\alpha(u)\, du - 1\right| \|F(\varphi(0,0))\|_X = I_\alpha(t) + J_\alpha,$$

say. Let $\alpha \ge \alpha_0, \alpha \in D$ and $0 < |t| < \delta/2$. Then we have $\sqrt{t^2+u^2} \le |t| + |u| < \delta$ whenever $|u| < \delta/2$, and so

$$I_\alpha(t) \le K\epsilon + (C + \|F(\varphi(0,0))\|_X) \int_{|u|\ge\delta/2} |g_\alpha(u)|\, du,$$

which together with (2.6) implies $\lim_{(\alpha,t)_0} I_\alpha(t) = 0$. Also, by (2.5) we have $\lim_\alpha J_\alpha = 0$. Thus (2.14) holds. □

Theorem 2.7. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be an approximate kernel. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $\varphi(0,0) = 0$. Suppose that $\lim_\alpha \epsilon_\alpha = 0$, and define*

$$T_\alpha(t)(f) = \frac{1}{\epsilon_\alpha}\int_0^{\epsilon_\alpha} T(\varphi(t,u))(f)\, du \qquad (\alpha \in D, t \in \mathbb{R}, f \in X). \tag{2.16}$$

Then $\mathfrak{L}$ is an approximation process on X.

Proof. We define $g_\alpha = (1/\epsilon_\alpha)\chi_{[0,\epsilon_\alpha]}$ for all $\alpha \in D$, where χ_A denotes the characteristic function of the set A. Then $\mathfrak{G} = \{g_\alpha\}_{\alpha\in D}$ is an approximate identity. Let $f \in X$ and we define $F(s) = T(s)(f)$ for all $s \in \mathbb{R}$ in Lemma 2.6. Then (2.13) reduces to (2.16) and since $\|g_\alpha\|_1 = 1$ for all $\alpha \in D$, (2.10) hold because of (2.15). Also, (2.14) implies (2.11) since $\varphi(0,0) = 0$ and $T(0) = I$. Thus the desired result follows from Theorem 2.2. □

Corollary 2.8. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1(\mathbb{R})$ satisfying (2.1) and (2.2). Suppose that $\lim_\alpha \epsilon_\alpha = 0$, and let $\mathfrak{T} = \{T_\alpha(t) : \alpha \in D, t \in \mathbb{R}\}$ be the family of operators definded by (2.16). If (2.12) holds for some $q > 0$, then $\mathfrak{L}$ is an approximation process on X.*

Theorem 2.9. *Let $\mathfrak{K} = \{k_{\alpha,\lambda}; \alpha \in D, \lambda \in \Lambda\}$ be an approximate kernel. Suppose that $\lim_\alpha \epsilon_\alpha = 0$. Then the following statements hold:*

(a) *We define*

$$T_\alpha(t)(f) = \frac{1}{\epsilon_\alpha}\int_0^{\epsilon_\alpha} T(t+u)(f)\, du \qquad (\alpha \in D, t \in \mathbb{R}, f \in X).$$

Then $\mathfrak{L}$ is an approximation process on X.

(b) *We define*

$$T_\alpha(t)(f) = \frac{1}{\epsilon_\alpha}\int_0^{\epsilon_\alpha} T(tu)(f)\, du \qquad (\alpha \in D, t \in \mathbb{R}, f \in X).$$

Then $\mathfrak{L}$ is an approximation process on X.

Proof. (a) Take $\varphi(t,u) = t+u, (t,u) \in \mathbb{R}^2$ in Theorem 2.7.
(b) Take $\varphi(t,u) = tu, (t,u) \in \mathbb{R}^2$ in Theorem 2.7. □

Corollary 2.10. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1(\mathbb{R})$ satisfying (2.1) and (2.2). If $\lim_\alpha \epsilon_\alpha = 0$ and if (2.12) holds for some $q > 0$, then the statements (a) and (b) in Theorem 2.9 hold.*

Theorem 2.11. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be an approximate kernel. Let $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be continuous and $\varphi(0,0) = 0$. Suppose that $\lim_\alpha \epsilon_\alpha = +\infty$, and define*

$$T_\alpha(t)(f) = \epsilon_\alpha \int_0^\infty e^{-\epsilon_\alpha u} T(\varphi(t,u))(f)\, du \quad (\alpha \in D, t \in \mathbb{R}, f \in X). \tag{2.17}$$

Then $\mathfrak{L}$ is an approximation process on X.

Proof. We define $g_\alpha(u) = \epsilon_\alpha e^{-\epsilon_\alpha u} \chi_{[0,\infty)}(u)$ for all $\alpha \in D$ and all $u \in \mathbb{R}$. Then $\mathfrak{G} = \{g_\alpha\}_{\alpha \in D}$ is an approximate identity. Let $f \in X$ and define $F(s) = T(s)(f)$ for all $s \in \mathbb{R}$ in Lemma 2.6. Then (2.13) reduces to (2.17) and since $\|g_\alpha\|_1 = 1$ for all $\alpha \in D$, (2.10) holds because of (2.15). Also, (2.14) implies (2.11) since $\varphi(0,0) = 0$ and $T(0) = I$. Thus the desired result follows from Theorem 2.2. □

Corollary 2.12. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1(\mathbb{R})$ satisfying (2.1) and (2.2). Suppose that $\lim_\alpha \epsilon_\alpha = +\infty$, and let $\mathfrak{T} = \{T_\alpha(t) : \alpha \in D, t \in \mathbb{R}\}$ be a family of operators defined by (2.17). If (2.12) holds for some $q > 0$, then $\mathfrak{L}$ is an approximation process on X.*

Theorem 2.13. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be an approximate kernel. Suppose that $\lim_\alpha \epsilon_\alpha = +\infty$. Then the following statements hold:*
(a) *We define*

$$T_\alpha(t)(f) = \epsilon_\alpha \int_0^\infty e^{-\epsilon_\alpha u} T(t+u)(f)\, du \qquad (\alpha \in D, t \in \mathbb{R}, f \in X).$$

Then $\mathfrak{L}$ is an approximation process on X.
(b) *We define*

$$T_\alpha(t)(f) = \epsilon_\alpha \int_0^\infty e^{-\epsilon_\alpha u} T(tu)(f)\, du \qquad (\alpha \in D, t \in \mathbb{R}, f \in X).$$

Then $\mathfrak{L}$ is an approximation process on X.

Proof. (a) Take $\varphi(t,u) = t+u$ for all $(t,u) \in \mathbb{R}^2$ in Theorem 2.11.
(b) Take $\varphi(t,u) = tu$ for all $(t,u) \in \mathbb{R}^2$ in Theorem 2.11. □

Corollary 2.14. *Let $\mathfrak{K} = \{k_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1(\mathbb{R})$ satisfying (2.1) and (2.2). If $\lim_\alpha \epsilon_\alpha = +\infty$ and if (2.12) holds for some $q > 0$, then the statements (a) and (b) in Theorem 2.13 hold.*

3. Convergence rates

Let $f \in X$ and let $\delta > 0$. Then we define

$$\omega_\alpha(f,\delta) = \sup\{\|T_\alpha(t)(f) - f\|_X : 0 < |t| \le \delta\} \qquad (\alpha \in D), \tag{3.1}$$

which is called the modulus of continuity of f.

Lemma 3.1. (a) *For each* $\alpha \in D$ *and* $f \in X$, $\omega_\alpha(f, \cdot)$ *is a monotone increasing function on* $(0, \infty)$.

(b) *For every* $f \in X$, $\lim_{(\alpha,\delta)+0} \omega_\alpha(f, \delta) = 0$. *In particular, if* $\{\xi_\alpha\}_{\alpha \in D}$ *is a net of positive real numbers converging to zero, then* $\lim_\alpha \omega_\alpha(f, \xi_\alpha) = 0$ *for every* $f \in X$.

Proof. Part (a) immediately follows from the definition (3.1). Part (b) follows from the limit relation (2.11). □

In order to give quantitative estimate of the rate of convergence for approximation processes $\mathfrak{L}$ on X, we suppose that there exist constants $C \geq 1$ and $K > 0$ such that

$$\omega_\alpha(f, \xi\delta) \leq (C + K\xi)\omega_\alpha(f, \delta) \tag{3.2}$$

for all $\alpha \in D, f \in X$ and all $\xi, \delta > 0$.

Lemma 3.2. *Suppose that there exists a constant* $K \geq 1$ *such that*

$$\|T_\alpha(t)(f) - T_\alpha(s)(f)\|_X \leq K\|T_\alpha(t-s)(f) - f\|_X \tag{3.3}$$

for all $\alpha \in D, t, s \in \mathbb{R}$ *and all* $f \in X$. *Then (3.2) holds with* $C = 1$ *for all* $\alpha \in D, f \in X$ *and all* $\xi, \delta > 0$.

Proof. If $\delta_1, \delta_2 > 0$ and $0 < |t| \leq \delta_1 + \delta_2$, then there exist $t_1, t_2 \in \mathbb{R}$ such that $t = t_1 + t_2$ and $0 < |t_1| \leq \delta_1, 0 < |t_2| \leq \delta_2$. Therefore, by (3.3), we have

$$\begin{aligned}\|T_\alpha(t)(f) - f\|_X &\leq \|T_\alpha(t_1 + t_2)(f) - T_\alpha(t_2)\|_X + \|T_\alpha(t_2)(f) - f\|_X \\ &\leq K\|T_\alpha(t_1) - f\|_X + \|T_\alpha(t_2) - f\|_X \leq K\omega_\alpha(f, \delta_1) + \omega_\alpha(f, \delta_2),\end{aligned}$$

which implies

$$\omega_\alpha(f, \delta_1 + \delta_2) \leq K\omega_\alpha(f, \delta_1) + \omega_\alpha(f, \delta_2).$$

Thus it follows from induction on n that $\omega_\alpha(f, n\delta) \leq (1 + (n-1)K)\omega_\alpha(f, \delta)$ for all $n \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of all natural numbers. Therefore, if $\xi \geq 1$, then denoting the largest positive integer not exceeding ξ by m, we have

$$\omega_\alpha(f, \xi\delta) \leq \omega_\alpha(f, (m+1)\delta) \leq (1 + mK)\omega_\alpha(f, \delta) \leq (1 + \xi K)\omega_\alpha(f, \delta).$$

If $0 < \xi < 1$, then $\omega_\alpha(f, \xi\delta) \leq \omega_\alpha(f, \delta) \leq (1 + \xi K)\omega_\alpha(f, \delta)$. □

Let $B[X]$ denote the Banach algebra of all bounded linear operators from X to itself with the usual operator norm $\|\cdot\|_{B[X]}$.

Remark 2. If for each $\alpha \in D$, $\{T_\alpha(t) : t \in \mathbb{R}\}$ is a strongly continuous group of operators in $B[X]$ satisfying $K = \sup\{\|T_\alpha(t)\|_{B[X]} : \alpha \in D, t \in \mathbb{R}\} < \infty$, then (3.3) holds for all $\alpha \in D, t, s \in \mathbb{R}$ and all $f \in X$. If, in addition, each $T_\alpha(t)$ is isometric, then (3.3) reduces to

$$\|T_\alpha(t)(f) - T_\alpha(s)(f)\|_X = \|T_\alpha(t-s)(f) - f\|_X.$$

For the basic theory of semigroups of operators on Banach spaces, we refer to [2], [5], [6], [7], [8] and [13].

Lemma 3.3. *Let $k \in L^1(\mathbb{R})$ and $f \in X$. Then for all $\alpha \in D, \delta > 0$ and all $q \geq 1$,*

$$\Big\| \int_{\mathbb{R}} k(t)(T_\alpha(t)(f) - f)\, dt \Big\|_X \leq (C\|k\|_1 + Kc(k; q, \delta))\omega_\alpha(f, \delta), \tag{3.4}$$

where

$$c(k; q, \delta) = \min\{\delta^{-q}\mu(k; q),\ \delta^{-1}(\mu(k; q))^{1/q}\|k\|_1^{1-1/q}\}.$$

Proof. If $|t| > \delta$, then by (3.2) we have

$$\|T_\alpha(t)(f) - f\|_X \leq (C + K\delta^{-q}|t|^q)\omega_\alpha(f, \delta),$$

which always holds for $0 < |t| \leq \delta$ on account of (3.1) and $C \geq 1$. Thus we derive

$$\Big\| \int_{\mathbb{R}} k(t)(T_\alpha(t)(f) - f)\, dt \Big\|_X \leq (C\|k\|_1 + K\delta^{-q}\mu(k; q))\omega_\alpha(f, \delta). \tag{3.5}$$

On the other hand, since

$$\|T_\alpha(t)(f) - f\|_X \leq (C + K|t|/\delta)\omega_\alpha(f, \delta),$$

using Hölder inequality we obtain

$$\Big\| \int_{\mathbb{R}} k(t)(T_\alpha(t)(f) - f)\, dt \Big\|_X \leq (C\|k\|_1 + K\delta^{-1}(\mu(k; q))^{1/q}\|k\|_1^{1-1/q})\omega_\alpha(f, \delta),$$

which together with (3.5) establishes (3.4) for $q > 1$. If $q = 1$, then (3.5) is clearly identical with (3.4). □

We assume that for each $\alpha \in D$,

$$0 < x_\alpha := \sup\{\|k_{\alpha,\lambda}\|_1 : \lambda \in \Lambda\} < \infty$$

and

$$0 < y_{\alpha,q} := (\sup\{\mu(k_{\alpha,\lambda}; q) : \lambda \in \Lambda\})^{1/q} < \infty \qquad (q \geq 1).$$

Furthermore, for any $\alpha \in D, f \in X$ we define

$$z_\alpha = \sup\Big\{\Big| \int_{\mathbb{R}} k_{\alpha,\lambda}(t)\, dt - 1\Big| : \lambda \in \Lambda\Big\}$$

and

$$\|L_\alpha(f) - f\|_\Lambda = \sup\{\|L_{\alpha,\lambda}(f) - f\|_X : \lambda \in \Lambda\}.$$

Note that $\mathfrak{L} = \{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ is an approximation process on X if and only if $\lim_\alpha \|L_\alpha(f) - f\|_\Lambda = 0$ for all $f \in X$.

We are now position to recast Theorem 2.3 in a quantitative form with the rate of convergence. Let $\{\delta_\alpha\}_{\alpha\in D}$ be a net of positive real numbers.

Theorem 3.4. *For all $f \in X, \alpha \in D$ and all $q \geq 1$,*

$$\|L_\alpha(f) - f\|_\Lambda \leq \|f\|_X z_\alpha + \ (Cx_\alpha + K\min\{\delta_\alpha^{-q},\ \delta_\alpha^{-1}x_\alpha^{1-1/q}\})\omega_\alpha(f, \delta_\alpha y_{\alpha,q}). \tag{3.6}$$

In particular, if $\mathfrak{K}$ is positive and normal, then (3.6) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \leq (C + K\min\{\delta_\alpha^{-1},\ \delta_\alpha^{-q}\})\omega_\alpha(f, \delta_\alpha y_{\alpha,q}).$$

Proof. Taking $k = k_{\alpha,\lambda}$ in Lemma 3.3, we obtain

$$\Big\|L_{\alpha,\lambda}(f) - \int_{\mathbb{R}} k_{\alpha,\lambda}(t)\,dt f\Big\|_X \le (Cx_\alpha + K\min\{\delta^{-q}y_{\alpha,q}^q,\ \delta^{-1}y_{\alpha,q}x_\alpha^{1-1/q}\})\omega_\alpha(f,\delta).$$

Thus putting $\delta = \delta_\alpha y_{\alpha,q}$ in the above inequality, the desired estimate (3.6) follows from the inequality

$$\|L_{\alpha,\lambda}(f) - f\|_X \le \Big|\int_{\mathbb{R}} k_{\alpha,\lambda}(t)\,dt - 1\Big|\|f\|_X + \Big\|L_{\alpha,\lambda}(f) - \int_{\mathbb{R}} k_{\alpha,\lambda}(t)\,dt f\Big\|_X \qquad (\alpha \in D, \lambda \in \Lambda). \tag{3.7}$$

□

As an immediate consequence of Theorem 3.4, we have the following corollary which can be more convenient for later applications.

Corollary 3.5. *For all $f \in X$ and all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \le \|f\|_X z_\alpha + (Cx_\alpha + K\min\{\delta_\alpha^{-2},\ \delta_\alpha^{-1}x_\alpha^{1/2}\})\omega_\alpha(f,\delta_\alpha y_{\alpha,2}). \tag{3.8}$$

In particular, if $\mathfrak{K}$ is positive and normal, then (3.8) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \le (C + K\min\{\delta_\alpha^{-1},\ \delta_\alpha^{-2}\})\omega_\alpha(f,\delta_\alpha y_{\alpha,2}).$$

Let $f \in X$ and let $\delta > 0$. Then we define

$$\omega_\alpha^*(f,\delta) = \sup\{\|T_\alpha(t)(f) + T_\alpha(-t)(f) - 2f\|_X : 0 < t \le \delta\}, \tag{3.9}$$

which is called the generalized modulus of continuity of f.

Lemma 3.6. (a) *For each $\alpha \in D$ and $f \in X$, $\omega_\alpha^*(f,\cdot)$ is a monotone increasing function on $(0,\infty)$.*

(b) *$\omega_\alpha^*(f,\delta) \le 2\omega_\alpha(f,\delta)$ for all $\alpha \in D, f \in X$ and all $\delta > 0$.*

(c) *For every $f \in X$, $\lim_{(\alpha,\delta)+0}\omega_\alpha^*(f,\delta) = 0$. In particular, if $\{\xi_\alpha\}_{\alpha\in D}$ is a net of positive real numbers converging to zero, then $\lim_\alpha \omega_\alpha^*(f,\xi_\alpha) = 0$ for every $f \in X$.*

Proof. Parts (a) and (b) immediately follow from the definition (3.9). Part (c) follows from Part (b) and Lemma 3.1 (b). □

Here let us impose the following condition on the generalized modulus of continuity: There exist positive constants A and B such that

$$\omega_\alpha^*(f,\xi\delta) \le (A + B\xi)^2\omega_\alpha^*(f,\delta) \tag{3.10}$$

for all $\alpha \in D, f \in X$ and all $\xi, \delta > 0$.

Lemma 3.7. *Suppose that for all $\alpha \in D, s, t, u \in \mathbb{R}$ and all $f \in X$,*

$$\|T_\alpha(s)(f) + T_\alpha(t)(f) - 2T_\alpha(u)(f)\|_X = \|T_\alpha(s-u)(f) + T_\alpha(t-u)(f) - 2f\|_X \tag{3.11}$$

and $T_\alpha(0) = I$. Then (3.10) holds with $A = B = 1$ for all $\alpha \in D, f \in X$ and all $\xi, \delta > 0$.

Proof. Let $n \in \mathbb{N}, s \in \mathbb{R}$ and $f \in X$. Then in view of (3.11), we have

$$\|T_\alpha(ns)(f) + T_\alpha(-ns)(f) - 2f\|_X = \|T_\alpha(2ns)(f) - 2T_\alpha(ns)(f) + T_\alpha(0)(f)\|_X$$
$$= \Big\|\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}(T_\alpha((i+j+2)s)(f) - 2T_\alpha((i+j+1)s)(f) + T_\alpha((i+j)s)(f))\Big\|_X$$
$$\le n^2\|T_\alpha(s)(f) + T_\alpha(-s)(f) - 2f\|_X.$$

Therefore, if $\delta > 0$ and $0 < t \le n\delta$, then we have

$$\|T_\alpha(t) + T_\alpha(-t)(f) - 2f\|_X \le n^2\|T_\alpha(t/n)(f) + T_\alpha(-t/n)(f) - 2f\|_X,$$

which yields $\omega_\alpha^*(f, n\delta) \le n^2\omega_\alpha^*(f,\delta)$. Therefore, if $\xi \ge 1$, then denoting the largest positive integer not exceeding ξ by m, we obtain

$$\omega_\alpha^*(f,\xi\delta) \le \omega_\alpha^*(f,(m+1)\delta) \le (m+1)^2\omega_\alpha^*(f,\delta) \le (\xi+1)^2\omega_\alpha^*(f,\delta).$$

If $0 < \xi < 1$, then $\omega_\alpha^*(f,\xi\delta) \le \omega_\alpha^*(f,\delta) \le (1+\xi)^2\omega_\alpha^*(f,\delta)$. □

Remark 3. If $T_\alpha(0) = I$, then (3.11) implies that

$$\|T_\alpha(t)(f) - T_\alpha(s)(f)\|_X = \|T_\alpha(t-s)(f) - f\|_X \qquad (\alpha \in D, f \in X, t, s \in \mathbb{R}),$$

and so (3.3) holds with $K = 1$. If for each $\alpha \in D$, $\{T_\alpha(t) : t \in \mathbb{R}\}$ is a strongly continuous group of isometric operators in $B[X]$, then (3.11) always holds.

Lemma 3.8. *Let k be an even function in $L^1(\mathbb{R})$ and $f \in X$. Then for all $\alpha \in D$ and all $\delta > 0$,*

$$\Big\|\int_{\mathbb{R}} k(t)(T_\alpha(t)(f) - f)\,dt\Big\|_X \le \omega_\alpha^*(f,\delta) \times (A^2\|k\|_1/2 + AB\mu(k;1)/\delta + B^2\delta^{-2}\mu(k;2)/2). \tag{3.12}$$

Proof. Since, by (3.10),

$$\|T_\alpha(t)(f) + T_\alpha(-t)(f) - 2f\|_X \le (A + Bt/\delta)^2\omega_\alpha^*(f,\delta),$$

we have

$$\Big\|\int_{\mathbb{R}} k(t)(T_\alpha(t)(f) - f)\,dt\Big\|_X \le \omega_\alpha^*(f,\delta)\int_0^\infty (A + Bt/\delta)^2|k(t)|\,dt,$$

which immediately implies the desired inequality (3.12). □

In the rest of this section, we suppose that $k_{\alpha,\lambda}$ is an even function in $L^1(\mathbb{R})$ for all $\alpha \in D$ and all $\lambda \in \Lambda$.

Theorem 3.9. *For all $f \in X$ and all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \le \|f\|_X z_\alpha + \frac{1}{2}\Big(A\sqrt{x_\alpha} + \frac{B}{\delta_\alpha}\Big)^2\omega_\alpha^*(f,\delta_a y_{\alpha,2}). \tag{3.13}$$

In particular, if $\mathfrak{K}$ is positive and normal, then (3.13) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \le \frac{1}{2}\Big(A + \frac{B}{\delta_\alpha}\Big)^2\omega_\alpha^*(f,\delta_\alpha y_{\alpha,2}).$$

Proof. Taking $k = k_{\alpha,\lambda}$ and putting $\delta = \delta_\alpha y_{\alpha,2}$ in Lemma 3.8, we obtain (3.13) by (3.7) and $y_{\alpha,1} \le x_\alpha^{1/2} y_{\alpha,2}$ which follows from Hölder's inequality. □

Lemma 3.10. *Suppose that for each $\alpha \in D$, $\{T_\alpha(t) : t \in \mathbb{R}\}$ is a strongly continuous group of operators in $B[X]$ with its infinitesimal generator G_α with domain $\mathfrak{D}(G_\alpha)$ and $K = \sup\{\|T_\alpha(t)\|_{B[X]} : \alpha \in D, t \in \mathbb{R}\} < \infty$. Let k be an even function in $L^1(\mathbb{R})$ and $f \in \mathfrak{D}(G_\alpha)$. Then for all $\alpha \in D, \delta > 0$ and all $q \geq 1$,*

$$\Big\| \int_{\mathbb{R}} k(t)(T_\alpha(t)(f) - f)\, dt \Big\|_X \leq \Big(\mu(k; 1) + \frac{K\mu(k; q+1)}{\delta^q(q+1)}\Big)\omega_\alpha(G_\alpha(f), \delta). \tag{3.14}$$

Proof. Since

$$T_\alpha(s)(f) - f = \int_0^s T_\alpha(u)(G_\alpha(f))\, du$$

and k is even, we have

$$\begin{aligned}
\int_{\mathbb{R}} k(t)(T_\alpha(t)(f) - f)\, dt &= \int_0^\infty k(t)(T_\alpha(t)(f) + T_\alpha(-t)(f) - 2f)\, dt \\
&= \int_0^\infty k(t)\Big\{\int_0^t (T_\alpha(u)(G_\alpha(f)) - T_\alpha(-u)(G_\alpha(f)))\, du\Big\}\, dt \\
&= \int_0^\infty k(t)\Big\{\int_0^t (T_\alpha(u)(G_\alpha(f)) - G_\alpha(f))\, du\Big\}\, dt \\
&+ \int_0^\infty k(t)\Big\{\int_0^t (G_\alpha(f) - T_\alpha(-u)(G_\alpha(f)))\, du\Big\}\, dt = g + h,
\end{aligned}$$

say. Now, since

$$\|T_\alpha(u)(G_\alpha(f)) - G_\alpha(f)\|_X \leq (1 + K\delta^{-q}u^q)\omega_\alpha(G_\alpha(f), \delta)$$

(cf. Lemma 3.2, Remark 2 and the proof of Lemma 3.3), we obtain

$$\begin{aligned}
\|g\|_X &\leq \int_0^\infty |k(t)|\Big\{\int_0^t \Big(1 + \frac{K}{\delta^q}u^q\Big)\omega_\alpha(G_\alpha(f), \delta)\, du\Big\}\, dt \\
&= \omega_\alpha(G_\alpha(f), \delta)\int_0^\infty |k(t)|\Big(t + \frac{K}{\delta^q(q+1)}t^{q+1}\Big)\, dt \\
&= \frac{1}{2}\omega_\alpha(G_\alpha(f), \delta)\Big(\mu(k; 1) + \frac{K}{\delta^q(q+1)}\mu(k; q+1)\Big).
\end{aligned}$$

In a similar manner, we get the same estimate for $\|h\|_X$, and consequently the desired inequality (3.14) is obtained. □

Theorem 3.11. *Let $\{T_\alpha(t) : t \in \mathbb{R}\}$ be as in Lemma 3.10 for each $\alpha \in D$. Then for all $\alpha \in D, f \in \mathfrak{D}(G_\alpha)$ and all $q \geq 1$,*

$$\|L_\alpha(f) - f\|_\Lambda \leq \|f\|_X z_\alpha + y_{\alpha,q+1}\Big(x_\alpha^{q/(q+1)} + \frac{K}{\delta_\alpha^q(q+1)}\Big)\omega_\alpha(G_\alpha(f), \delta_\alpha y_{\alpha,q+1}). \tag{3.15}$$

In particular, if $\mathfrak{K}$ is positive and normal, then (3.15) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \leq y_{\alpha,q+1}\Big(1 + \frac{K}{\delta_\alpha^q(q+1)}\Big)\omega_\alpha(G_\alpha(f, \delta_\alpha y_{\alpha,q+1}).$$

Proof. Taking $k = k_{\alpha,\lambda}$ and putting $\delta = \delta_\alpha y_{\alpha,q+1}$ in Lemma 3.10, we get (3.15) by (3.7) and $y_{\alpha,1} \leq x_\alpha^{q/(q+1)} y_{\alpha,q+1}$ which follows from Hölder's inequality. □

Corollary 3.12. *Let $\{T_\alpha(t) : t \in \mathbb{R}\}$ be as in Lemma 3.10 for each $\alpha \in D$. Then for all $\alpha \in D$ and all $f \in \mathfrak{D}(G_\alpha)$,*

$$\|L_\alpha(f) - f\|_\Lambda \leq \|f\|_X z_\alpha + y_{\alpha,2}\Big(\sqrt{x_\alpha} + \frac{K}{2\delta_\alpha}\Big)\omega_\alpha(G_\alpha(f), \delta_\alpha y_{\alpha,2}). \tag{3.16}$$

In particular, if $\mathfrak{K}$ is positive and normal, then (3.16) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \leq y_{\alpha,2}\Big(1 + \frac{K}{2\delta_\alpha}\Big)\omega_\alpha(G_\alpha(f), \delta_\alpha y_{\alpha,2}).$$

Remark 4. Let $1 \leq p < \infty$ and let $L^p_{2\pi}(\mathbb{R})$ denote the Banach space of all 2π-periodic, pth power Lebesgue integrable functions h with the norm

$$\|h\|_p = \Big(\frac{1}{2\pi}\int_{-\pi}^{\pi} |h(t)|^p \, dt\Big)^{1/p}.$$

Let $\mathfrak{H} = \{h_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be a family of functions in $L^1_{2\pi}(\mathbb{R})$ having Fourier series expansions

$$h_{\alpha,\lambda}(t) \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{h_{\alpha,\lambda}}(j)e^{ijt}, \qquad \hat{h_{\alpha,\lambda}}(j) := \frac{1}{2\pi}\int_{-\pi}^{\pi} h_{\alpha,\lambda}(t)e^{-ijt}\, dt,$$

and we define

$$k_{\alpha,\lambda} = \frac{1}{2\pi}\chi_{[-\pi,\pi]}h_{\alpha,\lambda} \qquad (\alpha \in D, \lambda \in \Lambda),$$

which belongs to $L^1(\mathbb{R})$. Then the convolution type operator $L_{\alpha,\lambda}$ defined by (1.2) becomes

$$L_{\alpha,\lambda}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_{\alpha,\lambda}(t)T_\alpha(t)\, dt \qquad (f \in X). \tag{3.17}$$

Consquently, all the results hold for the family $\mathfrak{L} = \{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ of operators given by (3.17). Note that if $h_{\alpha,\lambda}$ is nonnegative, then

$$\mu(k_{\alpha,\lambda}; 2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 h_{\alpha,\lambda}(t)\, dt \leq \frac{\pi^2}{2}(\hat{h_{\alpha,\lambda}}(0) - \mathrm{Re}(\hat{h_{\alpha,\lambda}}(1))), \tag{3.18}$$

where $\mathrm{Re}(\hat{h_{\alpha,\lambda}}(1))$ denotes the real part of $\hat{h_{\alpha,\lambda}}(1)$. Indeed, by Jordan's inequality

$$\frac{2}{\pi}t \leq \sin t \leq t \qquad \Big(0 \leq t \leq \frac{\pi}{2}\Big), \tag{3.19}$$

we have

$$\int_{-\pi}^{\pi} t^2 h_{\alpha,\lambda}(t)\, dt \leq \pi^2 \int_{-\pi}^{\pi} h_{\alpha,\lambda}(t)\sin^2\frac{t}{2}\, dt = \frac{\pi^2}{2}\int_{-\pi}^{\pi}(1 - \cos t)h_{\alpha,\lambda}(t)\, dt,$$

which yields (3.18).

In the following, let $\mathfrak{L} = \{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be the family of operators defined by (3.17), where $h_{\alpha,\lambda}$ is nonnegative for all $\alpha \in D$ and all $\lambda \in \Lambda$, and we define

$$x_\alpha = \sup\{\hat{h_{\alpha,\lambda}}(0) : \lambda \in \Lambda\} < \infty \qquad (\alpha \in D),$$

$$y_\alpha = (\sup\{\hat{h_{\alpha,\lambda}}(0) - \mathrm{Re}(\hat{h_{\alpha,\lambda}}(1)) : \lambda \in \Lambda\})^{1/2} < \infty \qquad (\alpha \in D) \tag{3.20}$$

and

$$z_\alpha = \sup\{|\hat{h_{\alpha,\lambda}}(0) - 1| : \lambda \in \Lambda\} \qquad (\alpha \in D).$$

Theorem 3.13. *For all $f \in X$ and all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \le \|f\|_X z_\alpha + \Big(Cx_\alpha + \frac{K\pi}{\sqrt{2}} \min\Big\{\frac{\pi}{\sqrt{2}\delta_\alpha^2},\ \frac{\sqrt{x_\alpha}}{\delta_\alpha}\Big\}\Big)\omega_\alpha(f, \delta_\alpha y_\alpha). \tag{3.21}$$

In particular, if $\hat{h_{\alpha,\lambda}}(0) = 1$ for all $\alpha \in D$ and all $\lambda \in \Lambda$, then (3.21) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \le \Big(C + \frac{K\pi}{\sqrt{2}} \min\Big\{\frac{\pi}{\sqrt{2}\delta_\alpha^2},\ \frac{1}{\delta_\alpha}\Big\}\Big)\omega_\alpha(f, \delta_\alpha y_\alpha) \tag{3.22}$$

and

$$y_\alpha = (\sup\{1 - Re(\hat{h_{\alpha,\lambda}}(1)) : \lambda \in \Lambda\})^{1/2}.$$

Proof. In view of (3.18), we have

$$y_{\alpha,2} \le \frac{\pi}{\sqrt{2}} y_\alpha \qquad (\alpha \in D), \tag{3.23}$$

and so $\omega_\alpha(f, \delta_\alpha y_{\alpha,2}) \le \omega_\alpha(f, (\pi/\sqrt{2})\delta_\alpha y_\alpha)$. Therefore, putting $(\sqrt{2}/\pi)\delta_\alpha$ instead of δ_α, the desired result follows from Corollary 3.5. □

Theorem 3.14. *Suppose that $h_{\alpha,\lambda}$ is even for all $\alpha \in D$ and all $\lambda \in \Lambda$. Then for all $f \in X$ and all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \le \|f\|_X z_\alpha + \frac{1}{2}\Big(A\sqrt{x_\alpha} + \frac{B\pi}{\sqrt{2}\delta_\alpha}\Big)^2 \omega_\alpha^*(f, \delta_\alpha y_\alpha). \tag{3.24}$$

In particular, if $\hat{h_{\alpha,\lambda}}(0) = 1$ for all $\alpha \in D$ and all $\lambda \in \Lambda$, then (3.24) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \le \frac{1}{2}\Big(A + \frac{B\pi}{\sqrt{2}\delta_\alpha}\Big)^2 \omega_\alpha^*(f, \delta_\alpha y_\alpha) \tag{3.25}$$

and

$$y_\alpha = (\sup\{1 - \hat{h_{\alpha,\lambda}}(1) : \lambda \in \Lambda\})^{1/2}. \tag{3.26}$$

Proof. By (3.23) we have $\omega_\alpha^*(f, \delta_\alpha y_{\alpha,2}) \le \omega_\alpha^*(f, (\pi/\sqrt{2})\delta_\alpha y_\alpha)$. Therefore, putting $(\sqrt{2}/\pi)\delta_\alpha$ instead of δ_α, the desired result follows from Theorem 3.9. □

Theorem 3.15. *Let $\{T_\alpha(t) : t \in \mathbb{R}\}$ be as in Lemma 3.10 for each $\alpha \in D$ and suppose that $h_{\alpha,\lambda}$ is even for all $\alpha \in D$ and all $\lambda \in \Lambda$. Then for all $\alpha \in D$ and all $f \in \mathfrak{D}(G_\alpha)$,*

$$\|L_\alpha(f) - f\|_\Lambda \le \|f\|_X z_\alpha + \frac{\pi}{\sqrt{2}} y_\alpha\Big(\sqrt{x_\alpha} + \frac{K\pi}{2\sqrt{2}\delta_\alpha}\Big)\omega_\alpha(G_\alpha(f), \delta_\alpha y_\alpha). \tag{3.27}$$

In particular, if $\hat{h_{\alpha,\lambda}}(0) = 1$ for all $\alpha \in D$ and all $\lambda \in \Lambda$, then (3.27) reduces to

$$\|L_\alpha(f) - f\|_\Lambda \le \frac{\pi}{\sqrt{2}} y_\alpha\Big(1 + \frac{K\pi}{2\sqrt{2}\delta_\alpha}\Big)\omega_\alpha(G_\alpha(f), \delta_\alpha y_\alpha), \tag{3.28}$$

where y_α is given by (3.26).

Proof. By (3.23) we have $\omega_\alpha(G_\alpha(f), \delta_\alpha y_{\alpha,2}) \le \omega_\alpha(G_\alpha(f), (\pi/\sqrt{2})\delta_\alpha y_\alpha)$. Therefore, putting $(\sqrt{2}/\pi)\delta_\alpha$ instead of δ_α, the desired result follows from Corollary 3.12. □

4. Applications

Put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $\mathcal{A} = \{a^{(\lambda)}_{\alpha,m} : \alpha \in D, m \in \mathbb{N}_0\}$ be a family of scalars such that $\sum_{m=0}^{\infty} |a^{(\lambda)}_{\alpha,m}| < \infty$ for each $\alpha \in D$ and $\lambda \in \Lambda$. Let $\{k_m\}_{m\in\mathbb{N}_0}$ be a bounded sequence of functions in $L^1(\mathbb{R})$ and we define

$$k_{\alpha,\lambda} = \sum_{m=0}^{\infty} a^{(\lambda)}_{\alpha,m} k_m \qquad (\alpha \in D, \lambda \in \Lambda),$$

which belongs to $L^1(\mathbb{R})$ and

$$K_{\alpha,m}(f) = \int_{\mathbb{R}} k_m(t) T_\alpha(t)(f)\, dt \qquad (f \in X),$$

which exists as a Bochner integral. Then the convolution type operator $L_{\alpha,\lambda}$ defined by (1.2) becomes

$$L_{\alpha,\lambda}(f) = \sum_{m=0}^{\infty} a^{(\lambda)}_{\alpha,m} K_{\alpha,m}(f) \qquad (f \in X). \tag{4.1}$$

Let $\mathfrak{L} = \{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ be the family of operators given by (4.1). The family $\{K_{\alpha,m} : \alpha \in D, m \in \mathbb{N}_0\}$ is called an $\mathcal{A}$-summation process on X if the family $\mathfrak{L}$ is an approximation process on X. Consequently, the results obtained in the preceding sections are applicable to the family $\mathfrak{L}$. In the following, we especially restrict ourselves to the case where $\mathcal{A}$ is stochastic, i.e.,

$$a^{(\lambda)}_{\alpha,m} \geq 0 \qquad (\alpha \in D, m \in \mathbb{N}_0, \lambda \in \Lambda), \qquad \sum_{m=0}^{\infty} a^{(\lambda)}_{\alpha,m} = 1 \qquad (\alpha \in D, \lambda \in \Lambda)$$

and k_m is a probability density function in $L^1(\mathbb{R})$ for each $m \in \mathbb{N}_0$.

Theorem 4.1. *Let $q \geq 1$ and let*

$$y_{\alpha,q} := \Big(\sup\Big\{\sum_{m=0}^{\infty} a^{(\lambda)}_{\alpha,m} \mu_m(q) : \lambda \in \Lambda\Big\}\Big)^{1/q} < \infty \qquad (\alpha \in D),$$

where $\mu_m(q) := \mu(k_m; q) < \infty$. Then the following statements hold:

(a) *For all $f \in X$ and all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \leq (C + K \min\{\delta_\alpha^{-1},\ \delta_\alpha^{-q}\})\omega_\alpha(f, \delta_\alpha y_{\alpha,q}).$$

(b) *If k_m is an even function for each $m \in \mathbb{N}_0$, then for all $f \in X$ and all $\alpha \in D$,*

$$\|L_\alpha(f) - f\|_\Lambda \leq \frac{1}{2}\Big(A + \frac{B}{\delta_\alpha}\Big)^2 \omega^*_\alpha(f, \delta_\alpha y_{\alpha,2}).$$

(c) *Let $\{T_\alpha(t) : t \in \mathbb{R}\}$ be as in Lemma 3.10 for each $\alpha \in D$. If k_m is an even function for each $m \in \mathbb{N}_0$, then for all $\alpha \in D$ and all $f \in \mathfrak{D}(G_\alpha)$,*

$$\|L_\alpha(f) - f\|_\Lambda \leq y_{\alpha,q+1}\Big(1 + \frac{K}{\delta_\alpha^q (q+1)}\Big)\omega_\alpha(G_\alpha(f), \delta_\alpha y_{\alpha,q+1}).$$

Proof. $\mathfrak{K}$ is positive and normal. Therefore, (a), (b) and (c) follow from Theorems 3.4, 3.9 and 3.11, respectively. □

Here we mention several concrete examples of functions $k_m, m \in \mathbb{N}_0$, which are induced by various probability density functions defined as follows:

Let $\{\alpha_m\}_{m\in\mathbb{N}_0}$ and $\{\beta_m\}_{m\in\mathbb{N}_0}$ be sequences of positive real numbers, and let $q > 0$.

(1°) *Gauss type distribution*:

$$k_m(t) := \sqrt{\frac{1}{\pi\alpha_m}} \exp\big(-\frac{t^2}{\alpha_m}\big) \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_m(q) = \frac{1}{\sqrt{\pi}} \Gamma\big(\frac{q+1}{2}\big)\alpha_m^{q/2},$$

where

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt \qquad (x > 0)$$

is the gamma function. In particular, we have $\mu_m(1) = \sqrt{\alpha_m/\pi}$, $\mu_m(2) = \alpha_m/2$.

(2°) *Laplace type distribution*:

$$k_m(t) := \frac{1}{2\alpha_m} \exp\big(-\frac{|t|}{\alpha_m}\big) \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_m(q) = q\Gamma(q)\alpha_m^q,$$

and so $\mu_m(1) = \alpha_m$ and $\mu_m(2) = 2\alpha_m^2$.

(3°) *Student (t) type distribution*:

$$k_m(t) := \sqrt{\frac{\alpha_m}{\pi}} \frac{\Gamma(\beta_m)}{\Gamma\big(\beta_m - \frac{1}{2}\big)} (1 + \alpha_m t^2)^{-\beta_m} \qquad (t \in \mathbb{R}).$$

Then we have

$$\mu_m(q) = \frac{\Gamma\big(\frac{q+1}{2}\big)}{\sqrt{\pi}} \Big(\frac{1}{\sqrt{\alpha_m}}\Big)^q \frac{\Gamma\big(\beta_m - \frac{q+1}{2}\big)}{\Gamma\big(\beta_m - \frac{1}{2}\big)},$$

and so

$$\mu_m(1) = \frac{1}{\sqrt{\pi\alpha_m}} \frac{\Gamma(\beta_m - 1)}{\Gamma\big(\beta_m - \frac{1}{2}\big)}, \qquad \mu_m(2) = \frac{1}{\alpha_m(2\beta_m - 3)}.$$

(4°) *Gamma type distribution*:

$$k_m(t) := \begin{cases} \frac{\beta_m^{\alpha_m}}{\Gamma(\alpha_m)} t^{\alpha_m - 1} e^{-\beta_m t} & (t > 0) \\ 0 & (t \le 0). \end{cases}$$

Then we have

$$\mu_m(q) = \frac{1}{\beta_m^q} \frac{\Gamma(q + \alpha_m)}{\Gamma(\alpha_m)},$$

and so

$$\mu_m(1) = \frac{\alpha_m}{\beta_m}, \qquad \mu_m(2) = \frac{\alpha_m(\alpha_m + 1)}{\beta_m^2}.$$

(5°) *Beta type distribution*:

$$k_m(t) := \begin{cases} \frac{1}{B(\alpha_m, \beta_m)} t^{\alpha_m - 1}(1 - t)^{\beta_m - 1} & (0 < t < 1) \\ 0 & (t \le 0 \text{ or } 1 \le t), \end{cases}$$

where

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad (x,y>0)$$

is the beta function. Then we have

$$\mu_m(q) = \frac{B(\alpha_m+q,\beta_m)}{B(\alpha_m,\beta_m)} = \frac{\Gamma(\alpha_m+\beta_m)}{\Gamma(\alpha_m)}\frac{\Gamma(\alpha_m+q)}{\Gamma(\alpha_m+\beta_m+q)},$$

and so

$$\mu_m(1) = \frac{\alpha_m}{\alpha_m+\beta_m}, \qquad \mu_m(2) = \frac{\alpha_m(\alpha_m+1)}{(\alpha_m+\beta_m)(\alpha_m+\beta_m+1)}.$$

(6°) *Landau type distribution*:

$$k_m(t) := \begin{cases} \frac{\alpha_m}{2B(1/\alpha_m,\beta_m)}(1-|t|^{\alpha_m})^{\beta_m-1} & (|t|\le 1) \\ 0 & (|t|>1). \end{cases}$$

Then we have

$$\mu_m(q) = \frac{\Gamma\big(\frac{q+1}{\alpha_m}\big)}{\Gamma\big(\frac{1}{\alpha_m}\big)}\frac{\Gamma\big(\beta_m+\frac{1}{\alpha_m}\big)}{\Gamma\big(\beta_m+\frac{q+1}{\alpha_m}\big)},$$

and so

$$\mu_m(1) = \frac{\Gamma\big(\frac{2}{\alpha_m}\big)}{\Gamma\big(\frac{1}{\alpha_m}\big)}\frac{\Gamma\big(\beta_m+\frac{1}{\alpha_m}\big)}{\Gamma\big(\beta_m+\frac{2}{\alpha_m}\big)}, \qquad \mu_m(2) = \frac{\Gamma\big(\frac{3}{\alpha_m}\big)}{\Gamma\big(\frac{1}{\alpha_m}\big)}\frac{\Gamma\big(\beta_m+\frac{1}{\alpha_m}\big)}{\Gamma\big(\beta_m+\frac{3}{\alpha_m}\big)}.$$

In particular, if $\alpha_m = 2$ for all $m \in \mathbb{N}_0$, then

$$\mu_m(q) = \frac{\Gamma\big(\frac{q+1}{2}\big)}{\sqrt{\pi}}\frac{\Gamma\big(\beta_m+\frac{1}{2}\big)}{\Gamma\big(\beta_m+\frac{q+1}{2}\big)},$$

and so

$$\mu_m(1) = \frac{1}{\sqrt{\pi}}\frac{\Gamma\big(\beta_m+\frac{1}{2}\big)}{\beta_m\Gamma(\beta_m)}, \qquad \mu_m(2) = \frac{1}{2\beta_m+1}.$$

(7°) *Weibull type distribution*:

$$k_m(t) := \begin{cases} \frac{\beta_m}{\alpha_m}t^{\beta_m-1}\exp\big(-\frac{t^{\beta_m}}{\alpha_m}\big) & (t>0) \\ 0 & (t\le 0). \end{cases}$$

Then we have

$$\mu_m(q) = \frac{q\alpha_m^{q/\beta_m}}{\beta_m}\Gamma\big(\frac{q}{\beta_m}\big),$$

and so

$$\mu_m(1) = \frac{\alpha_m^{1/\beta_m}}{\beta_m}\Gamma\big(\frac{1}{\beta_m}\big), \qquad \mu_m(2) = \frac{2\alpha_m^{2/\beta_m}}{\beta_m}\Gamma\big(\frac{2}{\beta_m}\big).$$

Let $\{h_m\}_{m\in\mathbb{N}_0}$ be a bounded sequence of nonnegative functions in $L^1_{2\pi}(\mathbb{R})$ having Fourier series expansions

$$h_m(t) \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{h_m}(j)e^{ijt},$$

and we define

$$H_{\alpha,m}(f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_m(t)T_\alpha(t)(f)\,dt \qquad (f \in X),$$

which exists as a Bochner integral. Suppose that $a_{\alpha,m}^{(\lambda)} \geq 0$ for all $\alpha \in D, m \in \mathbb{N}_0$ and all $\lambda \in \Lambda$, and we define

$$L_{\alpha,\lambda}(f) = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} H_{\alpha,m}(f) \qquad (f \in X), \tag{4.2}$$

which converges in X. Furthermore, for each $\alpha \in D$ we define

$$x_\alpha = \sup\Big\{\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \hat{h_m}(0) : \lambda \in \Lambda\Big\} < \infty,$$

$$y_\alpha = \Big(\sup\Big\{\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} (\hat{h_m}(0) - \mathrm{Re}(\hat{h_m}(1)) : \lambda \in \Lambda\Big\}\Big)^{1/2} < \infty \tag{4.3}$$

and

$$z_\alpha = \sup\Big\{\Big|\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \hat{h_m}(0) - 1\Big| : \lambda \in \Lambda\Big\}.$$

Theorem 4.2. *The following statements hold:*

(a) *The inequality (3.21) holds for all $f \in X$ and all $\alpha \in D$. In particular, if $\mathcal{A}$ is stochastic and if $\hat{h_m}(0) = 1$ for all $m \in \mathbb{N}_0$, then (3.21) reduces to (3.22) and*

$$y_\alpha = \sup\Big\{1 - \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} Re(\hat{h_m}(1)) : \lambda \in \Lambda\Big\}.$$

(b) *Suppose that h_m is an even function for each $m \in \mathbb{N}_0$. Then (3.24) holds for all $f \in X$ and all $\alpha \in D$. In particular, if $\mathcal{A}$ is stochastic and if $\hat{h_m}(0) = 1$ for all $m \in \mathbb{N}_0$, (3.24) reduces to (3.25) and*

$$y_\alpha = \Big(\sup\Big\{1 - \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} \hat{h_m}(1) : \lambda \in \Lambda\Big\}\Big)^{1/2}. \tag{4.4}$$

(c) *Let $\{T_\alpha(t) : t \in \mathbb{R}\}$ be as in Lemma 3.10 for each $\alpha \in D$ and suppose that h_m is an even function for each $m \in \mathbb{N}_0$. Then (3.27) holds for all $\alpha \in D$ and all $f \in \mathfrak{D}(G_\alpha)$. In particular, if $\mathcal{A}$ is stochastic and if $\hat{h_m}(0) = 1$ for all $m \in \mathbb{N}_0$, then (3.27) reduces to (3.28) and y_α is given by (4.4).*

Proof. We define

$$h_{\alpha,\lambda} = \sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)} h_m \qquad (\alpha \in D, \lambda \in \Lambda),$$

which is nonnegative and belongs to $L^1_{2\pi}(\mathbb{R})$. Then (3.17) and (3.20) turn out (4.2) and (4.3), respectively (cf. Remark 4). Therefore, (a), (b) and (c) follow from Theorems 3.13, 3.14 and 3.15, respectively. □

In the following, we especially restrict ourselves to the case where $\mathcal{A}$ is stochastic and

$$h_0(t) = 1, \quad h_m(t) = 1 + 2\sum_{j=1}^{m} \theta_m(j)\cos jt \geq 0 \qquad (m \in \mathbb{N}, t \in \mathbb{R}),$$

where $(\theta_m(j))$ $(m, j = 1, 2, \ldots,)$ is a lower triangular infinite matrix of real numbers. Therefore, $\hat{h_m}(0) = 1$ for all $m \in \mathbb{N}_0$, and so (4.3) reduces to

$$y_\alpha = \Big(\sup\Big\{\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}(1 - \theta_m(1)) : \lambda \in \Lambda\Big\}\Big)^{1/2}, \quad \theta_0(1) := 0.$$

Remark 5. Applying the Abel's transfomation twice to the function $h_m(t)$, we have

$$h_m(t) = \sum_{j=0}^{m-1}(j+1)F_j(t)\Delta^2\theta_m(j) + (m+1)\theta_m(m)F_m(t), \quad \theta_m(0) := 1,$$

where

$$F_n(t) = \sum_{j=-n}^{n}\Big(1 - \frac{|j|}{n+1}\Big)e^{ijt} = \frac{1}{n+1}\Big\{\frac{\sin\frac{1}{2}(n+1)t}{\sin\frac{t}{2}}\Big\}^2$$

is the nth Fejér kernel and

$$\Delta^2\theta_m(j) = \theta_m(j) - 2\theta_m(j+1) + \theta_m(j+2).$$

Therefore, if $\theta_m(m) \geq 0$ and $\{\theta_m(j)\}_{j\in\mathbb{N}_0}$ is convex, i.e., $\Delta^2\theta_m(j) \geq 0$ for all $j \in \mathbb{N}_0$, then $h_m(t)$ is a nonnegative, even trigonometric polynomial of degree at most m.

Several examples of $\theta_m(j)$ produce important positive summability kernels given as follows:

(8°) *Fejér*:

$$\theta_m(j) = \begin{cases} 1 - \frac{j}{m+1} & (1 \leq j \leq m) \\ 0 & (j > m). \end{cases}$$

(9°) *de la Vallée-Poussin*:

$$\theta_m(j) = \begin{cases} \frac{(m!)^2}{(m-j)!(m+j)!} & (1 \leq j \leq m) \\ 0 & (j > m). \end{cases}$$

(10°) *Fejér-Korovkin*:

$$\theta_m(j) = \begin{cases} A_m\sum_{n=0}^{m-j} a_n a_{j+n} & (1 \leq j \leq m) \\ 0 & (j > m), \end{cases}$$

where

$$a_n = \sin\Big(\frac{n+1}{m+2}\Big)\pi \qquad (n = 0, 1, 2, \ldots m), \qquad A_m = \Big(\sum_{n=0}^{m} a_n^2\Big)^{-1}.$$

In this case, we have

$$h_m(t) = A_m\Big|\sum_{n=0}^{m} a_n e^{int}\Big|^2, \qquad \theta_m(1) = \cos\Big(\frac{\pi}{m+2}\Big).$$

(11°) *Nörlund*:

$$\theta_m(j) = \begin{cases} \frac{Q_{m-j}}{Q_m} & (1 \le j \le m) \\ 0 & (j > m), \end{cases}$$

where

$$0 < q_0 \le q_m \le q_{m+1}, \qquad Q_m = \sum_{n=0}^{m} q_n \qquad (m \in \mathbb{N}_0).$$

Obviously, if $q_m = 1$ for all $m \in \mathbb{N}_0$, then the Nörlund kernel reduces to the Fejér kernel.

(12°) *Cesàro*:

$$\theta_m(j) = \begin{cases} \frac{C_{m-j}^{(\beta)}}{C_m^{(\beta)}} & (1 \le j \le n) \\ 0 & (j > m), \end{cases} \qquad (\beta \ge 1)$$

where $\tau > -1$ and

$$C_0^{(\tau)} = 1, \quad C_n^{(\tau)} = \binom{n+\tau}{n} = \frac{(\tau+1)(\tau+2)\cdots(\tau+n)}{n!} \qquad (n \in \mathbb{N}).$$

Note that if $q_m = C_m^{(\beta-1)}$ for all $m \in \mathbb{N}_0$, then the Nörlund kernel reduces to the Cesàro kernel. In particular, if $\beta = 1$, then the Cesàro kernel turns out the Fejér kernel.

Other important examples of the sequences $\{h_m\}_{m\in\mathbb{N}_0}$ of nonnegative, even functions in $C_{2\pi}(\mathbb{R})$ with $\hat{h_m}(0) = 1$ are the following, where $C_{2\pi}(\mathbb{R})$ denotes the Banach space of all 2π-periodic, continuous functions h on $\mathbb{R}$ with the norm

$$\|h\|_\infty = \max\{|h(t)| : |t| \le \pi\}.$$

(13°) *Jackson*:

$$h_m(t) = h_{m,s}(t) = c_{m,s} \begin{cases} \left(\frac{\sin((m+1)t/2)}{\sin(t/2)}\right)^{2s} & \text{if } t \text{ is not a multiple of } 2\pi \\ (m+1)^{2s} & \text{if } t \text{ is a multiple of } 2\pi, \end{cases}$$

where $s \in \mathbb{N}$ and the normalizing constant $c_{m,s} > 0$ is taken in such a way that

$$\frac{1}{\pi}\int_0^\pi h_m(t)\,dt = 1.$$

Since $h_m(t) = c_{m,s}(m+1)^s F_m(t)^s$, $h_{m,s}(t)$ is a nonnegative, even triginometric polynomial of degree ms and we have $c_{m,1} = 1/(m+1)$ for $s = 1$, and so $h_{m,1}(t)$ becomes the Fejér kernel. Also, we have

$$c_{m,2} = \frac{3}{(m+1)(2(m+1)^2+1)}, \qquad \hat{h_{m,2}}(1) = \frac{2m(m+2)}{2m(m+2)+3}.$$

Furthermore, making use of Jordan's inequality (3.19) we have that for $s \ge 3$,

$$\left(\frac{\pi}{2}\right)^{1-2s}\frac{2s-1}{2s}(m+1)^{1-2s} < c_{m,s} \le \left(\frac{\pi}{2}\right)^{2s}(m+1)^{1-2s}$$

and

$$0 < 1 - \hat{h_{m,s}}(1) < \left(\frac{\pi}{2}\right)^{2(2s-1)}\frac{8s}{3\pi(2s-3)}(m+1)^{-2}.$$

(14°) *Abel-Poisson*:

$$h_m(t) = 1 + 2\sum_{n=1}^{\infty} r_m^n \cos nt \qquad (t \in \mathbb{R}),$$

where $\{r_m\}_{m\in\mathbb{N}_0}$ is a sequence of real numbers converging to one such that $0 \leq r_m < 1$ for all $m \in \mathbb{N}_0$. Note that

$$h_m(t) = \frac{1 - r_m^2}{(1 - r_m)^2 + 4r_m \sin^2(t/2)}$$

and (4.3) becomes

$$y_\alpha = \left(\sup\left\{\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}(1 - r_m) : \lambda \in \Lambda\right\}\right)^{1/2}.$$

(15°) *Gauss-Weierstrass*:

$$h_m(t) = \sqrt{\frac{\pi}{\rho_m}} \sum_{n=-\infty}^{\infty} \exp\left\{-\frac{(t - 2\pi n)^2}{4\rho_m}\right\} \qquad (t \in \mathbb{R}),$$

where $\{\rho_m\}_{m\in\mathbb{N}_0}$ is a sequence of positive real numbers converging to zero. We can rewrite $h_m(t)$ as

$$h_m(t) = 1 + 2\sum_{n=1}^{\infty} e^{-\rho_m n^2} \cos nt,$$

and so (4.3) becomes

$$y_\alpha = \left(\sup\left\{\sum_{m=0}^{\infty} a_{\alpha,m}^{(\lambda)}(1 - e^{-\rho_m}) : \lambda \in \Lambda\right\}\right)^{1/2}.$$

Let $\mathbb{Z}$ denote the set of all integers, and let $\mathfrak{P} = \{P_j\}_{j\in\mathbb{Z}}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

(P-1) $\mathfrak{P}$ is orthogonal, i.e., $P_j P_n = \delta_{j,n} P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.

(P-2) $\mathfrak{P}$ is fundamental, i.e., the linear span of the set $\cup_{j\in\mathbb{Z}} P_j(X)$ is dense in X.

(P-3) $\mathfrak{P}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

For any $f \in X$, we associate its (formal) Fourier series expansion

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f). \tag{4.5}$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j\}_{j\in\mathbb{Z}}$ of scalars such that for every $f \in X$,

$$T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f), \tag{4.6}$$

and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j$$

(cf. [4], [11], [12], [16]). Let $M[X]$ denote the set of all multiplier operators on X, which is a commutative closed subalgebra of $B[X]$ containing I.

Remark 6. The expansion (4.5) represents a generalization of the concept of Fourier series in a Banach space X associated with a fundamental, total, biorthogonal system $\mathfrak{F} = \{f_j, f_j^*\}_{j\in\mathbb{Z}}$. Here, $\{f_j\}_{j\in\mathbb{Z}}$ and $\{f_j^*\}_{j\in\mathbb{Z}}$ are sequences of elemets in X and X^* (the dual space of X), respectively, such that the linear span of the set $\{f_j : j \in \mathbb{Z}\}$ is dense in X (fundamental), $f \in X$ and $f_j^*(f) = 0$ for all $j \in \mathbb{Z}$ imply $f = 0$ (total), and $f_j^*(f_n) = \delta_{j,n}$ for all $j, n \in \mathbb{Z}$ (biorthogonal). Then (4.5) and (4.6) read

$$(4.7) \qquad f \sim \sum_{j=-\infty}^{\infty} f_j^*(f) f_j \quad \text{and} \quad T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j f_j^*(f) f_j,$$

respectively (cf. [10], [14]).

Let $\mathfrak{T} = \{T_\alpha(t) : \alpha \in D, t \in \mathbb{R}\}$ be a family of operators in $M[X]$ having the expansions

$$(4.8) \qquad T_\alpha(t) \sim \sum_{j=-\infty}^{\infty} v_{\alpha,j}(t) P_j \qquad (\alpha \in D, t \in \mathbb{R})$$

with

$$(4.9) \qquad \sup\{\|T_\alpha(t)\|_{B[X]} : t \in \mathbb{R}\} < \infty$$

for each $\alpha \in D$, where $\mathfrak{V} = \{v_{\alpha,j} : \alpha \in D, j \in \mathbb{Z}\}$ is a family of scalar-valued continuous functions on $\mathbb{R}$ such that

$$(4.10) \qquad c_{\alpha,\lambda,j} = c_{\alpha,\lambda,j}(\mathfrak{K}, \mathfrak{V}) := \int_{\mathbb{R}} k_{\alpha,\lambda}(t) v_{\alpha,j}(t)\, dt < \infty$$

for each $\alpha \in D, \lambda \in \Lambda$ and $j \in \mathbb{Z}$. Then Condition (P-2) and (4.9) imply that for each $\alpha \in D$ and $f \in X$, the mapping $t \mapsto T_\alpha(t)(f)$ is bounded and strongly continuous on $\mathbb{R}$. Also, the convolution type opetator $L_{\alpha,\lambda}$ defined by (1.2) belongs to $M[X]$ and

$$(4.11) \qquad L_{\alpha,\lambda} \sim \sum_{j=-\infty}^{\infty} c_{\alpha,\lambda,j} P_j \qquad (\alpha \in D, \lambda \in \Lambda).$$

Furthermore, if for each $j \in \mathbb{Z}$,

$$(4.12) \qquad \lim_{(\alpha,t)_0} v_{\alpha,j}(t) = 1$$

and if

$$(4.13) \qquad \limsup_{(\alpha,t)_0} \|T_\alpha(t)\|_{B[X]} < \infty,$$

then Condition (P-2) implies the limit relation (2.11) for every $f \in X$. Consequently, the results obtained in the preceding sections are applicable to the family $\mathfrak{L} = \{L_{\alpha,\lambda} : \alpha \in D, \lambda \in \Lambda\}$ of multiplier operators having the expansions given by (4.11).

In the following, we especially restrict ourselves to the case where $T_\alpha(t)$ is induced by a uniformly bounded family $\mathcal{T} = \{T(t) : t \in \mathbb{R}\}$ of operators in $M[X]$ having the expansions

$$T(t) \quad \sim \quad \sum_{j=-\infty}^{\infty} v_j(t) P_j \qquad (t \in \mathbb{R}), \tag{4.14}$$

where $\{v_j\}_{j\in\mathbb{Z}}$ is a sequence of scalar-valued continuous functions on $\mathbb{R}$ such that $v_j(0) = 1$ for all $j \in \mathbb{Z}$. Let $\{\epsilon_\alpha\}_{\alpha\in D}$ be a net of positive real numbers such that $\limsup_\alpha \epsilon_\alpha < \infty$ and let $T_\alpha(t) = T(t\epsilon_\alpha)$ for all $\alpha \in D$ and all $t \in \mathbb{R}$. Then (4.8) holds with $v_{\alpha,j}(t) = v_j(t\epsilon_\alpha)$ and (4.11) holds with

$$c_{\alpha,\lambda.j} = \int_{\mathbb{R}} k_{\alpha,\lambda}(t) v_j(t\epsilon_\alpha)\, dt < \infty \qquad (\alpha \in D, \lambda \in \Lambda, j \in \mathbb{Z}). \tag{4.15}$$

Furthermore, (4.12) holds for each $j \in \mathbb{Z}$ and since

$$\sup\{\|T_\alpha(t)\|_{B[X]} : \alpha \in D, t \in \mathbb{R}\} \leq \sup\{\|T(s)\|_{B[X]} : s \in \mathbb{R}\} < \infty,$$

(4.13) holds. In particular, if $\{\xi_j\}_{j\in\mathbb{Z}}$ is a sequence of scalars and $v_j(t) = e^{\xi_j t}$ for all $j \in \mathbb{Z}$ and all $t \in \mathbb{R}$, then $\mathcal{T}$ becomes a strongly continuous group of operators in $M[X]$ and (4.15) reduces to

$$c_{\alpha,\lambda,j} = \int_{\mathbb{R}} k_{\alpha,\lambda}(t) e^{\xi_j t\epsilon_\alpha}\, dt \qquad (\alpha \in D, \lambda \in \Lambda, j \in \mathbb{Z}).$$

Furthermore, the infinitesimal genetator G of $\mathcal{T}$ with domain $\mathfrak{D}(G)$ satisfies

$$G(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \xi_j P_j(f) \qquad (f \in \mathfrak{D}(G)),$$

and if $\{S_n\}_{n\in\mathbb{N}_0}$ denotes the sequence of the nth partial sum operators of the Fourier series expansion (4.5), that is,

$$S_n = \sum_{j=-n}^{n} P_j \qquad (n \in \mathbb{N}_0)$$

and if the sequence $\{\sigma_n\}_{n\in\mathbb{N}_0}$ of the Cesàro mean operators defined by

$$\sigma_n = \frac{1}{n+1} \sum_{m=0}^{n} S_m = \sum_{j=-n}^{n} \Big(1 - \frac{|j|}{n+1}\Big) P_j \qquad (n \in \mathbb{N}_0)$$

is uniformly bounded, then

$$\mathfrak{D}(G) = \{f \in X : g \ \sim \sum_{j=-\infty}^{\infty} \xi_j P_j \quad \text{for some } g \in X\}$$

(cf. [11, Proposition 2]). Note that $\mathfrak{D}(G_\alpha) = \mathfrak{D}(G), G_\alpha = \epsilon_\alpha G$ for each $\alpha \in D$. Also, we have

$$\omega_\alpha(f, \delta) = \omega(f, \delta\epsilon_\alpha) \qquad (f \in X, \alpha \in D, \delta > 0),$$

where

$$\omega(f, \xi) := \sup\{\|T(t)(f) - f\|_X : 0 < |t| \leq \xi\} \qquad (\xi > 0)$$

denotes the modulus of continuity of f associated with $\mathcal{T}$, and

$$\omega_\alpha^*(f,\delta) = \omega^*(f,\delta\epsilon_\alpha) \qquad (f \in X, \alpha \in D, \delta > 0),$$

where

$$\omega^*(f,\xi) := \sup\{\|T(t)(f) + T(-t)(f) - 2f\|_X : 0 < t \le \xi\} \qquad (\xi > 0)$$

denotes the generalized modulus of continuity of f associated with $\mathcal{T}$.

Finally, we restrict ourselves to the case where X is a homogeneous Banach space on $\mathbb{R}$ (cf. [15], [17]). That is, $(X, \|\cdot\|_X)$ is a Banach space of Lebesgue measurable functions on $\mathbb{R}$ which satisfies the following conditions:

(H-1) The right translation oprator T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \qquad (f \in X)$$

belongs to $B[X]$ and it is isometric on X for each $t \in \mathbb{R}$.

(H-2) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on $\mathbb{R}$.

Typical examples of homogeneous Banach spaces on $\mathbb{R}$ are $BUC(\mathbb{R})$ and $L^p(\mathbb{R})$ $(1 \le p < \infty)$.

If a homogeneous Banach space $(X, \|\cdot\|_X)$ on $\mathbb{R}$ is a linear subspace of $L^1_{2\pi}(\mathbb{R})$ and if it is continuously embedded in $L^1_{2\pi}(\mathbb{R})$, i.e., there exists a constant $K > 0$ such that

$$\|f\|_1 \le K\|f\|_X$$

for all $f \in X$, then X is called a 2π-periodic homogeneous Banach space on $\mathbb{R}$ (cf. [9], [11]). Typical examples of 2π-periodic homogeneous Banach spaces on $\mathbb{R}$ are $C_{2\pi}(\mathbb{R})$ and $L^p_{2\pi}(\mathbb{R})$ $(1 \le p < \infty)$. For other examples, see [11] (cf. [9], [15], [17]).

Now let X be a 2π-periodic homogeneous Banach space on $\mathbb{R}$. Let $\mathcal{T} = \{T_t : t \in \mathbb{R}\}$ be the family of right translation operators on X, which is an isometric strongly continuous group in $B[X]$. We define

$$f_j(t) = e^{ijt} \qquad (j \in \mathbb{Z}, t \in \mathbb{R}) \quad \text{and} \quad f_j^*(f) = \hat{f}(j) \qquad (j \in \mathbb{Z}, f \in X).$$

Then $\mathfrak{F} = \{f_j, f_j^*\}_{j\in\mathbb{Z}}$ becomes a fundamental, total, biorthogonal system (cf. [9, Theorems 2.7, 2.11 and 2.12]) and (4.7) reads

$$f \quad \sim \quad \sum_{j=-\infty}^{\infty} \hat{f}(j)e^{ij\cdot} \quad \text{and} \quad T(f) \quad \sim \quad \sum_{j=-\infty}^{\infty} \tau_j \hat{f}(j)e^{ij\cdot}.$$

Furthermore, we have

$$T(t)(f) \quad \sim \sum_{j=-\infty}^{\infty} \hat{f}(j)e^{ij(\cdot - t)} \qquad (t \in \mathbb{R}, f \in X)$$

(cf. (4.14)),

$$\omega(f,\xi) = \sup\{\|f(\cdot - t) - f(\cdot)\|_X : 0 < |t| \le \xi\} \qquad (f \in X, \xi > 0)$$

and

$$\omega^*(f,\xi) = \sup\{\|f(\cdot + t) + f(\cdot - t) - 2f(\cdot)\|_X : 0 < t \le \xi\} \qquad (f \in X, \xi > 0).$$

References

[1] H. T. Bell, *Order summability and almost convergence*, Proc. Amer. Math. Soc. **38** (1973), 548-552.

[2] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Springer-Verlag, Berlin/Heidelberg/New York, 1967.

[3] P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Vol. **I**, Birkhäuser-Verlag, Basel/Academic Press, New York, 1971.

[4] P. L. Buzter, R. J. Nessel and W. Trebels, *On summation processes of Fourier expansions in Banach spaces. I. Comparison theorems*, Tôhoku Math. J. **24** (1972), 127-140; II. Saturation theorems, ibid. 551-569; III. Jackson- and Zamansky-type inequalities for Abel-bounded expansions, ibid. **27** (1975), 213-223.

[5] E. M. Davies, *One-Parameter Semigroups*, Academic Press, New York, 1980.

[6] K.-J. Engel and R. Nagel, *One-Parameter Semi-Groups for Linear Evolution Equations*, Springer-Verlag, New York/Berlin/Heidelberg, 2000.

[7] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, New York, 1985.

[8] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, Amer. Math. Soc., Providence, R. I., 1957.

[9] Y. Katznelson, *An Introduction to Harmonic Analysis*, John Wiley, New York, 1968.

[10] V. D. Milman, *Geometric theory of Banach spaces I, The Theory of bases and minimal systems*, Russian Math. Sueveys **25** (1970), 111-170.

[11] T. Nishishiraho, *Quantitative theorems on linear approximation processes of convolution operators in Banach spaces*, Tôhoku Math. J. **33** (1981), 109-126.

[12] T. Nishishiraho, *Saturation of multiplier operators in Banach spaces*, Tõhoku Math. J. **34** (1982), 13-26.

[13] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York/Berlin/Heidelberg, 1983.

[14] I. Singer, *Bases in Banach Spaces I*, Springer-Verlag, Berlin/Heidelberg/New York, 1970.

[15] H. S. Shapiro, *Topics in Approximation Theory*, Lecture Notes in Math. **187**, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[16] W. Trebels, *Multiplier for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory*, Lecture Notes in Math. Vol. **329**, Springer-Verlag, Berlin/Heidelberg/New York, 1973.

[17] H. C. Wang, *Homogeneous Banach Algebras*, Marcel Dekker Inc., New York/Basel, 1977.

Manuscript received September 28, 2007
revised May 23, 2008

Toshihiko Nishishiraho
Department of Mathematical Sciences, Faculty of Science, University of the Ryukyus, Nishihara-cho, Okinawa 903-0213, Japan
E-mail address: nisiraho@sci.u-ryukyu.ac.jp