Journal of Nonlinear and Convex Analysis
Volume 9, Number 2, 2008, 181–204
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

FIXED POINTS OF GENERALIZED CONTRACTIVE MAPPINGS

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Abstract. We give an improved version of a theorem of Kincses and Totik concerning fixed points of a very general class of mappings of contractive type. By isolating the requirements on the mapping, specifically on the contractivity condition in question, we give an extension of the theorem from the compact case to the setting of arbitrary metric spaces. We also supply numerical information concerning the convergence of the Picard iteration sequence to the fixed point.

Using the uniformity features of the Cauchy rate exhibited we in addition show that any continuous selfmapping on a compact metric space satisfying one of the conditions (1)-(50) treated by B.E. Rhoades in the paper [B.E. Rhoades, *A comparison of various definitions of contractive mappings*, Transactions of the American Mathematical Society **226** (1977), 257-290] is an asymptotic contraction in the sense of Kirk.

The results were derived with the help of techniques and insights from proof mining.

1. Introduction

If a function $f : X \to X$ on a nonempty compact metric space (X, d) is contractive, i.e. satisfies

$$\forall x, y \in X(x \neq y \to d(f(x), f(y)) < d(x, y)),$$

then it has a unique fixed point, and for every starting point $x_0 \in X$ the iteration sequence $(f^n(x_0))$ converges to this fixed point. This well-known theorem due to Edelstein has led to the study of many generalizations of the notion of contractivity. (For a simple proof of Edelstein's theorem, see e.g. [17].) The hope when considering such generalizations is then to obtain corresponding generalizations of the fixed point theorem. These generalized contraction properties are also considered as conditions on functions $f : X \to X$ on complete metric spaces, or on metric spaces in general. In [27], B.E. Rhoades compares 25 contraction conditions, most of them previously considered in the literature, and also considers generalizations of the 25 basic conditions to the cases where the condition holds for various iterates of the function. The basic conditions are numbered (1)-(25). Functions satisfying one of these conditions are called functions of contractive type. P. Collaço and J. Carvalho e Silva completes the comparison of the 25 conditions in [11]. That is, the implications that hold between the different conditions are completely determined. Specifically, it is known that condition (25),

$$\forall x, y \in X(x \neq y \to d(f(x), f(y)) < \text{diam}\{x, y, f(x), f(y)\}),$$

2000 *Mathematics Subject Classification.* 47H10, 47H09, 03F10.

Key words and phrases. Metric fixed point theory, contractive type mappings, generalized p-contractive mappings, asymptotic contractions, proof mining.

is the most general. So if f satisfies one of the conditions (1)-(24), then it also satisfies condition (25). Hence a fixed point theorem for functions satisfying (25) would entail as corollaries corresponding fixed point theorems for conditions (1)-(24). However, a function on a complete metric space satisfying (25) need not have a fixed point. In [19] it is proved by J. Kincses and V. Totik that if one in addition assumes that f is continuous and X compact, then f has a unique fixed point, and for any $x_0 \in X$ the Picard iteration $(f^n(x_0))$ converges to this fixed point. In [19] it is also proved that this result extends to the case where (25) holds for an iterate of the function, i.e. if there exists $p \in \mathbb{N}$ such that

$$\forall x, y \in X(x \neq y \to d(f^p(x), f^p(y)) < \operatorname{diam}\{x, y, f^p(x), f^p(y)\}).$$

These conditions, where we require that for some $p \in \mathbb{N}$ we should have that f^p satisfies respectively (1)-(25) are numbered respectively (26)-(50). The fixed point theorem does not hold for the other standard generalizations treated. For example, a continuous function on a compact space satisfying the condition that there exists $p : X \times X \to \mathbb{N}$ such that

$$\forall x, y \in X(x \neq y \to d(f^{p(x,y)}(x), f^{p(x,y)}(y)) < \operatorname{diam}\{x, y, f^{p(x,y)}(x), f^{p(x,y)}(y)\}),$$

or satisfying the condition that there exists $p, q \in \mathbb{N}$ such that

$$\forall x, y \in X(x \neq y \to d(f^p(x), f^q(y)) < \operatorname{diam}\{x, y, f^p(x), f^q(y)\}),$$

does not necessarily have a fixed point. The second of these cases involves two iterates, the first involves one iterate which is not uniform in x and y. We will call a function *generalized p-contractive* if it satisfies (25) for an iterate $p \in \mathbb{N}$.

With the help of techniques and insights from proof mining, as developed in recent years by U. Kohlenbach, we develop a quantitative version of the theorem of Kincses and Totik mentioned above. This involves finding a rate of convergence for Picard iteration sequences to the unique fixed point. Kohlenbach has developed methods from that part of mathematical logic known as proof theory which under general conditions allow one to find ("extract") quantitative information, not earlier visible, from ordinary mathematical proofs. The new quantitative information can e.g. be effective bounds with strong uniformity features. This can in many cases also be used to infer a qualitative improvement of the original mathematical theorem. When the logical metatheorems guarantee that a mathematical theorem can be strenghtened, then they also supply an algorithm which can be used as a guideline when transforming a proof of the original theorem into a proof of the new theorem. The result of such a process is again an ordinary mathematical proof, with no trace of logic left. For more on proof mining, see e.g. [22], [15] and [23]. In [4] we give a novel application of one of Kohlenbach's logical metatheorems, thereby explaining how we were able to find a full rate of convergence for the Picard iteration sequence in Kincses and Totik's theorem without in addition assuming that the mapping should be nonexpansive.

Compared to Kincses and Totik's theorem we also obtain new qualitative information, insofar as we show that the convergence of the iteration sequence $(f^n(x_0))$ depends on conditions which are satisfied if the space is compact, but conditions which we can also single out and see satisfied in other cases. Namely, we require uniform continuity and a uniform version of generalized p-contractivity, and also the existence of a bounded iteration sequence for some starting point. Furthermore, we show that the rate of convergence is highly uniform in the sense that it only depends on the starting point x_0 and the function f through suitable moduli expressing uniform continuity and uniform generalized p-contractivity and a bound on the iteration sequence $(f^n(x_0))$. If the space is not complete we still get a Cauchy rate for the iteration sequence.

By using the uniformity of the Cauchy rate for the Picard iteration sequence we also show that any continuous selfmapping on a compact metric space which satisfies one of the conditions (1)-(50) is an asymptotic contraction in the sense of Kirk. Asymptotic contractions were introduced by W.A. Kirk in [20], and have in recent years been quite extensively studied. See for example [1], [3], [2], [5], [6], [7], [9], [13], [18], [30], [31], [32] and [33].

2. Preliminaries

Definition 2.1. Let (X, d) be a metric space, let $p \in \mathbb{N}$ and let $f : X \to X$. We say that f is *generalized p-contractive* if for all $x, y \in X$ with $x \neq y$ we have

$$d(f^p(x), f^p(y)) < diam\{x, y, f^p(x), f^p(y)\}.$$

Notice that a generalized p-contractive function is not necessarily nonexpansive, where $f : X \to X$ being nonexpansive means that

$$\forall x, y \in X(d(f(x), f(y)) \leq d(x, y)).$$

Take for instance $f : (0, \infty) \to (0, \infty)$ defined by $f(x) := 2x$. Then f is generalized 1-contractive.

Theorem 2.2 (Kincses, Totik [19]). *Let (X, d) be a compact metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be continuous and generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n\to\infty} f^n(x_0) = z.$$

To give a quantitative version of this theorem, we express the requirements on f by the following moduli.

Definition 2.3. Let (X, d) be a metric space, and let $f : X \to X$. We say that $\omega : (0, \infty) \to (0, \infty)$ is a *modulus of uniform continuity* for f if for all $\varepsilon \in (0, \infty)$ and for all $x, y \in X$ with $d(x, y) < \omega(\varepsilon)$ we have $d(f(x), f(y)) < \varepsilon$.

Definition 2.4. Let (X, d) be a metric space, and let $f : X \to X$. We say that $\eta : (0, \infty) \to (0, \infty)$ is a *modulus of uniform generalized p-contractivity* for f if for all $\varepsilon \in (0, \infty)$ and for all $x, y \in X$ with $d(x, y) > \varepsilon$ we have

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < diam\{x, y, f^p(x), f^p(y)\}.$$

When X is a compact metric space, f having such moduli coincides with f being continuous and generalized p-contractive.

Proposition 2.5. *Let (X, d) be a compact metric space, and let $f : X \to X$ be continuous and generalized p-contractive. Then f has moduli ω and η of uniform continuity and uniform generalized p-contractivity.*

Proof. We can without loss of generality assume that $\mathrm{diam}(X) > 0$, since otherwise everything is trivial. Existence of a modulus of uniform continuity follows since f is uniformly continuous. For the other modulus, consider for $\varepsilon > 0$ such that $\mathrm{diam}(X) > \varepsilon$ the set

$$A_\varepsilon := \{(x, y) \in X \times X : d(x, y) \geq \varepsilon\}.$$

Then A_ε is closed and therefore compact, and the continuous function $g : X \times X \to \mathbb{R}$ defined by $g(x, y) := \mathrm{diam}\{x, y, f^p(x), f^p(y)\} - d(f^p(x), f^p(y))$ assumes its infimum on A_ε. That is, there exists $(x, y) \in A_\varepsilon$ such that $g(x, y) = \inf g[A_\varepsilon]$. Therefore $\inf g[A_\varepsilon] \neq 0$, since we otherwise would have

$$\mathrm{diam}\{x, y, f^p(x), f^p(y)\} = d(f^p(x), f^p(y)),$$

contradicting the fact that f is generalized p-contractive and $d(x, y) \geq \varepsilon$. So we can define a modulus of uniform generalized p-contractivity η by for $0 < \varepsilon < \mathrm{diam}(X)$ letting $\eta(\varepsilon)$ be some positive real number smaller than $\inf g[A_\varepsilon]$ and for $\varepsilon \geq \mathrm{diam}(X)$ letting $\eta(\varepsilon)$ be e.g. 1. □

The following is just a way of rephrasing the statement that f has a modulus of uniform generalized p-contractivity.

Definition 2.6. Let (X, d) be a metric space, let $p \in \mathbb{N}$ and let $f : X \to X$. We say that f is *uniformly generalized p-contractive* if for all real $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) > \varepsilon$ we have

$$diam\{x, y, f^p(x), f^p(y)\} - d(f^p(x), f^p(y)) > \delta.$$

We note that for a metric space (X, d) examples of uniformly generalized p-contractive mappings $f : X \to X$ are e.g. the mappings f such that for some positive integer p we have that f^p fulfills condition (24) from [27], i.e. the condition that there should exist $0 \leq h < 1$ such that

$$d(f(x), f(y)) \leq h \cdot \mathrm{diam}\{x, y, f(x), f(y)\}$$

holds for all $x, y \in X$. This condition was introduced by L.B. Ćirić in [10], and a mapping on a complete metric space satisfying this condition is called there a quasi-contraction. Likewise mappings satisfying one of those conditions (1)-(23) which in [11] are listed as stronger than condition (24) are uniformly generalized p-contractive. Ćirić proved the following theorem concerning mappings satisfying condition (24).

Theorem 2.7 (Ćirić)**.** *Let (X, d) be a complete metric space, and let $f : X \to X$ and $h \in [0, 1)$ be such that*

$$d(f(x), f(y)) \leq h \cdot \mathrm{diam}\{x, y, f(x), f(y)\}$$

holds for all $x, y \in X$. Let $x_0 \in X$. Then f has a unique fixed point z, and $f^n(x_0) \to z$.

This theorem is different from Kincses and Totik's theorem in much the same way that Banach's contraction mapping principle is different from Edelstein's theorem on contractive mappings.

A quantitative version of Kincses and Totik's theorem will involve a rate of convergence to the fixed point.

Definition 2.8. Let (X, d) be a metric space, let $z \in X$ and let (x_n) be a sequence in X converging to z. We say that $\phi : (0, \infty) \to \mathbb{N}$ is a *rate of convergence* for (x_n) if for all $\varepsilon > 0$ and for all $n \geq \phi(\varepsilon)$ we have

$$d(z, x_n) \leq \varepsilon.$$

We say that ϕ is a *rate of convergence* for $f : X \to X$ (to $z \in X$) if for all $x_0 \in X$ and all $\varepsilon > 0$ we have that $n \geq \phi(\varepsilon)$ gives $d(z, x_n) \leq \varepsilon$, where (x_n) is defined by $x_{n+1} := f(x_n)$, with x_0 as starting point.

Since we will be considering situations where a sequence could be Cauchy without converging, we include the following notion.

Definition 2.9. Let (X, d) be a metric space, and let (x_n) be a sequence in X. We say that $\rho : (0, \infty) \to \mathbb{N}$ is a *Cauchy rate* for (x_n) if for all $\varepsilon > 0$ and all $m, n \geq \rho(\varepsilon)$ we have

$$d(x_m, x_n) \leq \varepsilon.$$

We say that $\rho : (0, \infty) \to \mathbb{N}$ is a *Cauchy rate* for f if for all $x_0 \in X$ and for all $\varepsilon > 0$ we have that $m, n \geq \rho(\varepsilon)$ implies $d(x_m, x_n) \leq \varepsilon$, where (x_n) is defined by $x_{n+1} := f(x_n)$, with x_0 as starting point.

We include also the following.

Definition 2.10. Given a metric space (X, d) and a mapping $f : X \to X$ we say that a sequence (x_n) is an *approximate fixed point sequence* if for all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that for all $m \geq n$ we have $d(x_m, f(x_m)) < \varepsilon$.

3. Main results

Our theorem will concern arbitrary metric spaces instead of compact ones.

Theorem 3.1. *Let (X, d) be a metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus ω of uniform continuity, and a modulus η of uniform generalized p-contractivity. Let $x_0 \in X$ be the starting point of a sequence (x_n) defined by $x_{n+1} := f(x_n)$. Suppose (x_n) is bounded, and let b be a bound on d when restricted to (x_n). Let $\rho : (0, \infty) \to (0, \infty)$ be defined by*

$$\rho(\varepsilon) := \min \{\eta(\varepsilon), \varepsilon/2, \eta(1/2 \cdot \omega^p(\varepsilon/2))\}.$$

Let $\phi : (0, \infty) \to \mathbb{N}$ be defined by

$$\phi(\varepsilon) := \begin{cases} p \lceil (b - \varepsilon)/\rho(\varepsilon) \rceil & \text{if } b > \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

Then ϕ is a Cauchy rate for (x_n). Given p, ω, η and b we will denote this Cauchy rate also by $\Phi(p, \omega, \eta, b, \cdot)$, so that given $\varepsilon > 0$ we get that

$$m, n \geq \Phi(p, \omega, \eta, b, \varepsilon)$$

gives $d(x_n, x_m) \leq \varepsilon$.

So the appropriate moduli, together with the existence of a bounded iteration sequence, guarantee the existence of a Cauchy sequence which is an approximate fixed point sequence. If the space is complete, then (x_n) converges to a fixed point z, and ϕ is a rate of convergence for the sequence. The fixed point is unique if it exists, for if x and y were fixed points with $x \neq y$, we would have $d(x, y) = d(f^p(x), f^p(y))$ and $d(f^p(x), f^p(y)) = \text{diam}\{x, y, f^p(x), f^p(y)\}$, contradicting the fact that f is generalized p-contractive. The rate ϕ only depends on the function f and the starting point $x_0 \in X$ through p and the moduli ω and η, and also through a bound b on (x_n). If b is a bound on the whole space then the rate does not depend on x_0, and gives if the fixed point exists a rate of convergence for f, or else a Cauchy rate for f.

We note in passing that the moduli in Definition 2.3 and Definition 2.4 might be equivalently given as functions $\omega : \mathbb{N} \to \mathbb{N}$ and $\eta : \mathbb{N} \to \mathbb{N}$ with conditions of the form that e.g. $d(x, y) < 2^{-\omega(k)}$ should give $d(f(x), f(y)) < 2^{-k}$. Likewise the Cauchy rate in Theorem 3.1 can be given as a function $\Phi : \mathbb{N} \to \mathbb{N}$. In this case we have that with b an integer and with ω and η computable, then Φ is computable.In fact, it is clear that a Cauchy rate as in Theorem 3.1 could be given as an effectively computable function $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ taking ω and η as two of its arguments.

Before proving this theorem we give some corollaries and a definition, and we also prove some lemmas.

Corollary 3.2. *Let (X, d) be a bounded, complete metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be uniformly continuous and uniformly generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n \to \infty} f^n(x_0) = z.$$

Together with Proposition 2.5, this corollary implies the theorem of Kincses and Totik as a special case.

Corollary 3.3 (Kincses and Totik's Theorem)**.** *Let (X, d) be a compact metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ be continuous and generalized p-contractive. Then f has a unique fixed point z, and for every $x_0 \in X$ we have*

$$\lim_{n \to \infty} f^n(x_0) = z.$$

Notice that if (X, d) is a compact metric space and $f : X \to X$ is continuous and satisfies one of the conditions (1)-(24) from [27], then f has moduli of uniform continuity and uniform generalized 1-contractivity, and hence also a rate of convergence as given in Theorem 3.1. As an application of Theorem 3.1 we note also the following relationship with asymptotic contractions as introduced by W.A. Kirk in [20]. The following two corollaries already appeared in [6], but the proofs there made reference to and were dependent on Theorem 3.1 and Proposition 2.5 in the present paper.

Corollary 3.4. *Let (X, d) be a bounded, complete metric space, and let $f : X \to X$ be uniformly generalized p-contractive and uniformly continuous. Then f is an asymptotic contraction in the sense of Kirk.*

Proof. See the proof of Corollary 3.8 in [6]. □

Corollary 3.5. *Let (X, d) be a compact metric space. Let $f : X \to X$ be continuous and such that it satisfies one of the conditions* (1)-(50) *from* [27]*. Then f is an asymptotic contraction in the sense of Kirk.*

Proof. See the proof of Corollary 3.9 in [6]. □

We will in the following let X, b, f, ω and η be as in Theorem 3.1.

Definition 3.6. We say that $\rho : (0, \infty) \to (0, \infty)$ is a *modulus of modified uniform generalized p-contractivity* for f if for all $\varepsilon > 0$ and for all $x, y \in X$ with

$$diam\{x, y, f^p(x), f^p(y)\} > \varepsilon$$

we have

$$d(f^p(x), f^p(y)) + \rho(\varepsilon) < diam\{x, y, f^p(x), f^p(y)\}.$$

Lemma 3.7. *Define* $\rho : (0, \infty) \to (0, \infty)$ *by*

$$\rho(\varepsilon) := \min\left\{\eta(\varepsilon), \frac{\varepsilon}{2}, \eta\left(1/2 \cdot \omega^p(\varepsilon/2)\right)\right\}.$$

Then ρ *is a modulus of modified uniform generalized p-contractivity for* f*.*

Proof. We consider the different cases.

(1) If $d(x, y) > \varepsilon$ then

$$d(f^p(x), f^p(y)) + \rho(\varepsilon) < \text{diam}\{x, y, f^p(x), f^p(y)\}, \tag{1}$$

since $\rho(\varepsilon) \le \eta(\varepsilon)$.

(2) If $d(f^p(x), x) > \varepsilon$ we again look at the different cases.

(a) If $d(x, y) < \omega^p(\varepsilon/2)$, then

$$d(f^p(x), f^p(y)) < \varepsilon/2,$$

and (1) holds since $\rho(\varepsilon) \le \varepsilon/2$ and

$$\text{diam}\{x, y, f^p(x), f^p(y)\} > \varepsilon.$$

(b) If $d(x, y) \ge \omega^p(\varepsilon/2)$, then by definition of η we have

$$d(f^p(x), f^p(y)) + \eta(1/2 \cdot \omega^p(\varepsilon/2)) < \text{diam}\{x, y, f^p(x), f^p(y)\}.$$

Then (1) holds since $\rho(\varepsilon) \le \eta(1/2 \cdot \omega^p(\varepsilon/2))$. (This holds in fact whether $d(f^p(x), x) > \varepsilon$ or not.)

The cases where $d(f^p(y), y) > \varepsilon$, $d(f^p(x), y) > \varepsilon$ or $d(f^p(y), x) > \varepsilon$ are treated in exactly the same way as the case $d(f^p(x), x) > \varepsilon$. □

Lemma 3.8. *Let* (X, d) *be a metric space, and let* $x_0 \in X$ *be such that* b *is a bound on* (x_n)*. Let* ρ *be a modulus of modified uniform generalized p-contractivity for* f*. Let* $\phi : (0, \infty) \to \mathbb{N}$ *be defined by*

$$\phi(\varepsilon) := \begin{cases} p\lceil(b - \varepsilon)/\rho(\varepsilon)\rceil & \text{if } b > \varepsilon, \\ 1 & \text{otherwise.} \end{cases}$$

Then ϕ *satisfies*

$$\forall \varepsilon > 0 \forall m, n \ge \phi(\varepsilon)(d(x_m, x_n) \le \varepsilon).$$

Proof. The proof of this lemma comes essentially from the proof of the first theorem in [19]. If $\varepsilon \geq b$, then

$$\forall \varepsilon > 0 \forall m, n \geq \phi(\varepsilon)(d(x_m, x_n) \leq \varepsilon).$$

So let $\varepsilon < b$. Let $x_0 \in X$, and let $n, k, l \in \mathbb{N}$. Let $n_0 := np + k$, $m_0 := np + l$. For $0 \leq i < n$ we define n_{i+1} and m_{i+1} inductively so that

$$n_{i+1}, m_{i+1} \in \{n_i, n_i - p, m_i, m_i - p\},$$
$$d(x_{n_{i+1}}, x_{m_{i+1}}) = \mathrm{diam}\{x_{n_i}, x_{n_i - p}, x_{m_i}, x_{m_i - p}\}.$$

We write d_i for $\mathrm{diam}\{x_{n_i}, x_{n_i-p}, x_{m_i}, x_{m_i-p}\}$ for $i < n$. If for some i we have $d_i = 0$, then

$$d(x_{np+k}, x_{np+l}) = 0.$$

So suppose not. Since ρ is a modulus of modified uniform generalized p-contractivity we have

$$d(x_{n_0}, x_{m_0}) + \rho(\varepsilon_0) < d_0$$

for all $\varepsilon_0 > 0$ with $\varepsilon_0 < d_0$. Furthermore, we have

$$d_0 + \rho(\varepsilon_1) < d_1$$

for all $\varepsilon_1 > 0$ with $\varepsilon_1 < d_1$. And in general

$$d_i + \rho(\varepsilon_{i+1}) < d_{i+1}$$

for all $\varepsilon_{i+1} > 0$ with $\varepsilon_{i+1} < d_{i+1}$. Therefore, for $0 \leq i < n$,

$$d(x_{n_0}, x_{m_0}) < d_i - \sum_{j=0}^{i} \rho(\varepsilon_j),$$

for $\varepsilon_j > 0$ with $\varepsilon_j < d_j$ for $j \leq i$. If for some $0 \leq i < n$ we have $d_i \leq \varepsilon$, then

$$d(x_{np+k}, x_{np+l}) = d(x_{n_0}, x_{m_0}) < \varepsilon.$$

If on the other hand we have $d_i > \varepsilon$ for all $0 \leq i < n$, then we get

$$d(x_{n_0}, x_{m_0}) < d_i - \sum_{j=0}^{i} \rho(\varepsilon).$$

Thus

$$d(x_{np+k}, x_{np+l}) < b - n\rho(\varepsilon).$$

Now let

$$n := \lceil (b - \varepsilon)/\rho(\varepsilon) \rceil .$$

Then $d(x_{np+k}, x_{np+l}) < \varepsilon$. And this n does not depend on x_0, except through the bound b. By letting

$$m := p \lceil (b - \varepsilon)/\rho(\varepsilon) \rceil ,$$

we get $d(x_{m+k}, x_{m+l}) < \varepsilon$. And since $\varepsilon < b$ we have

$$\phi(\varepsilon) = p \lceil (b - \varepsilon)/\rho(\varepsilon) \rceil .$$

Since k and l were arbitrary, we get

$$\forall \varepsilon > 0 \forall m, n \geq \phi(\varepsilon)(d(x_m, x_n) < \varepsilon).$$

□

Proof of Theorem 3.1. The lemmas give directly that ϕ as defined in the theorem is a Cauchy rate for (x_n). □

The following provides an example with an unbounded complete metric space (X, d) and a selfmapping $f : X \to X$ where the conditions in Theorem 3.1 are satisfied, where the fixed point is an element of a noncompact closed and bounded set, and where we cannot remove either $d(x, y)$, $d(y, f(x))$, $d(x, f(y))$, $d(x, f(x))$ or $d(y, f(y))$ in the formulation of the condition that for all real $\varepsilon > 0$ there should exist $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) > \varepsilon$ we have

$$\max\{d(x, y), d(x, f(y)), d(y, f(x)), d(x, f(x)), d(y, f(y))\} - d(f^p(x), f^p(y)) > \delta.$$

In addition the mapping f in the example does not satisfy the condition (24) from [27]. Thus theorem 2.7 does not apply.

Example 3.9. Let $a, b, c, d \notin \mathbb{R}$ be pairwise distinct. Let $Y = \{0, a, b, c, d\}$ and $Y' = \{3k + 1 : k \geq 0\} \cup \{-3k - 1 : k \geq 0\} \cup \{2^{-k} : k \geq 0\} \cup \{-2^{-k} : k \geq 0\} \cup \{0\}$. Equip Y' with the natural metric, and define a metric d_Y on Y such that $d_Y(0, a) = 3$, $d_Y(0, b) = 3$, $d_Y(0, c) = 1$, $d_Y(0, d) = 1$, $d_Y(c, d) = 2$, $d_Y(a, c) = 2$, $d_Y(b, c) = 2$, $d_Y(a, d) = 2$, $d_Y(b, d) = 2$ and $d_Y(a, b) = 3$. Let X be the set of sequences $(x_n)_{n\geq 0}$ with $x_0 \in Y$ and with $x_n \in Y'$ for $n \geq 1$ such that $\{|x_n| : n \geq 1\}$ is bounded. Define a metric on X by for $x, y \in X$ with $x = (x_n)_n$, $y = (y_n)_n$ letting $d(x, y) = \max\{d_Y(x_0, y_0), \sup\{|x_n - y_n| : n \geq 1\}\}$. Given $x = (x_n)_n \in X$ and x_n with $n \geq 1$, consider the condition:

$$\text{There is } m \geq 1 \text{ with } x_m > x_n. \tag{2}$$

Define $f : X \to X$ by for $x = (x_n)_n \in X$ letting $f(x) = (y_n)_n$ be given by

$$y_n = \begin{cases} 0 & \text{if } x_n = 0,\ x_n = c \text{ or } x_n = d, \\ 3k - 2 & \text{if } x_n = 3k + 1,\ k > 0 \text{ an integer}, \\ -3k + 2 & \text{if } x_n = -3k - 1,\ k > 0 \text{ an integer}, \\ -2^{-k-1} & \text{if } x_n = -2^{-k},\ k \geq 0 \text{ an integer}, \\ 2^{-k} & \text{if } x_n = 2^{-k} \text{ for an integer } k \geq 0 \text{ and (2) holds}, \\ -2^{-k} & \text{if } x_n = 2^{-2k} \text{ for an integer } k \geq 0 \text{ and (2) does not hold}, \\ -2^{-k} & \text{if } x_n = 2^{-2k-1} \text{ for an integer } k \geq 0 \text{ and (2) does not hold}, \\ c & \text{if } x_n = a, \\ d & \text{if } x_n = b. \end{cases}$$

Then $\{x \in X : d(x, 0) \leq 1\}$ is not compact, where 0 denotes the sequence which is constant 0, and it is easy to see that f is uniformly continuous. We leave out the verification that f is uniformly generalized 1-contractive. We have the following.

(1) For $x = (x_n)_n$ and $y = (y_n)_n$ with $x_0 = 0$, $y_0 = 0$ and with $x_n = 1$ and $y_n = 0$ for all $n \geq 1$ we have

$$d(f(x), f(y)) < d(x, f(x)),$$

but

$$d(f(x), f(y)) \geq d(x, y), d(y, f(y)), d(x, f(y)), d(y, f(x)).$$

(2) For $x = (x_n)_n$ and $y = (y_n)_n$ with $x_0 = 0$, $y_0 = 0$ and with $x_n = 4$ and $y_n = 7$ for all $n \geq 1$ we have
$$d(f(x), f(y)) < d(y, f(x)),$$
but
$$d(f(x), f(y)) \geq d(x, y), d(x, f(x)), d(y, f(y)), d(x, f(y)).$$

(3) For $x = (x_n)_n$ and $y = (y_n)_n$ with $x_0 = a$, $y_0 = b$ and with $x_n = 0$ and $y_n = 0$ for all $n \geq 1$ we have
$$d(f(x), f(y)) < d(x, y),$$
but
$$d(f(x), f(y)) \geq d(x, f(x)), d(y, f(y)), d(x, f(y)), d(y, f(x)).$$

Furthermore, f does not satisfy the condition (24) from [27], i.e. there does not exist $0 \leq h < 1$ such that
$$d(f(x), f(y)) \leq h \cdot \mathrm{diam}\{x, y, f(x), f(y)\}$$
holds for all $x, y \in X$. For given $0 \leq h < 1$ we can let $m \in \mathbb{N}$ and consider $x = (x_n)_n$ and $y = (y_n)_n$ with $x_0 = 0$, $y_0 = 0$ and with $x_n = 2^{-2m}$ and $y_n = 0$ for all $n \geq 1$. Then $d(f(x), f(y)) = 2^{-m}$ and
$$\mathrm{diam}\{x, y, f(x), f(y)\} = 2^{-m} + 2^{-2m}.$$
So for $m \in \mathbb{N}$ large enough we have $d(f(x), f(y)) > h \cdot \mathrm{diam}\{x, y, f(x), f(y)\}$.

We note that contrary to the case where f is contractive and we are given a modulus of uniform contractivity (see [14]), we cannot in Theorem 3.1 replace the bound b on (x_n) by a bound on $d(x_0, x_1)$. Even if we have a b which for all $x \in X$ bounds $d(x, f(x))$, we are not guaranteed to have a fixed point. Take for instance $X = \mathbb{R}$, $p = 1$ and $f(x) := x + 1$. Then the identity is a modulus of uniform continuity for f, and the function $\eta : (0, \infty) \to (0, \infty)$ defined by $\eta(\varepsilon) := 1/2$ is a modulus of uniform generalized 1-contractivity for f. Now $d(x, f(x))$ is bounded by 1, but the function has no fixed point, and no Picard iteration is a Cauchy sequence. It is also easy to see that given a uniformly continuous and uniformly generalized p-contractive f and bounded iteration sequences, we cannot in general construct a common Cauchy rate involving only the moduli of uniform continuity and uniform generalized p-contractivity. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) := \frac{x}{2}$.

Furthermore, as the following example shows, we cannot do without the modulus of uniform generalized p-contractivity. Let $X := \{x_n : n \geq 1\}$ and $d(x_i, x_j) = 1 + \frac{1}{i \cdot j}$ for $i \neq j$. Let $f : X \to X$ be defined by $f(x_i) := x_{i+1}$. Then X is bounded (complete, separable) and f is uniformly continuous and generalized 1-contractive, but no Picard iteration sequence is Cauchy. This example is taken from [19], where it is used to show that a function satisfying condition (25) need not have a fixed point. Notice that f in this case is not uniformly generalized 1-contractive. Now consider uniformly continuous and uniformly generalized p-contractive functions with the same modulus of uniform continuity, and bounded Picard iteration sequences (x_n) with a common bound. We cannot construct a common Cauchy rate for all the (x_n) involving only the bound b and the modulus of uniform continuity ω, as the following example shows. Let $X_k := \{x_n : 1 \le n \le k\}$ and $d_k(x_i, x_j) := 1 + \frac{1}{i \cdot j}$ for $i \neq j$. Let $f_k : X_k \to X_k$ be defined by

$$f_k(x_i) := \begin{cases} x_{i+1} & \text{for } i < k, \\ x_k & \text{for } i = k. \end{cases}$$

Then for all k we have the same bound b on $(f_k^n(x_1))_n$, and we can find a modulus of uniform continuity which is the same for all f_k, but no common Cauchy rate for all the $(f_k^n(x_1))_n$ exists.

Also, as we show in the following proposition, the modulus of uniform continuity contributes in an essential way to the Cauchy rate.

Proposition 3.10. *There exists a bounded metric space (X, d), a family of uniformly continuous functions $f_i : X \to X$ with $i \in \mathbb{N}$, and an $\eta : (0, \infty) \to (0, \infty)$ which is a modulus of uniform generalized 1-contractivity for all the f_i, such that for some $x_0 \in X$ the Picard iterations with starting point x_0 do not have a common Cauchy rate.*

Proof. Consider $X := \{(\frac{1}{2})^n : n \ge 0\} \bigcup \{-(\frac{1}{2})^n : n \ge 0\}$ with the natural metric, and define $f_i : X \to X$ by

$$f_i(x) := \begin{cases} -(\frac{1}{2})^{n+1} & \text{if } x = -(\frac{1}{2})^n, \\ (\frac{1}{2})^{n+1} & \text{if } x = (\frac{1}{2})^n \text{ and } n \neq i, \\ -1 & \text{if } x = (\frac{1}{2})^n \text{ and } n = i. \end{cases}$$

Then each f_i is uniformly continuous. And $\eta : (0, \infty) \to (0, \infty)$ defined by $\eta(\varepsilon) := \frac{\varepsilon}{2}$ is a modulus of uniform generalized 1-contractivity for each f_i. To see this, we fix i and consider different cases. If $x, y \in X$ with $d(x, y) > \varepsilon$, and neither is equal to $(\frac{1}{2})^i$, then

$$d(f_i(x), f_i(y)) = \frac{d(x, y)}{2}.$$

Therefore

$$\operatorname{diam}\{x, y, f_i(x), f_i(y)\} - d(f_i(x), f_i(y)) > \frac{\varepsilon}{2}.$$

If $x, y \in X$ with $d(x, y) > \varepsilon$, and $x = (\frac{1}{2})^i$, we have one of the following.

(1) If $y = -(\frac{1}{2})^n$, then

$$d(f_i(x), x) - d(f_i(x), f_i(y)) = \left(\frac{1}{2}\right)^i + \left(\frac{1}{2}\right)^{n+1} > \left(\frac{1}{2}\right)^{i+1} + \left(\frac{1}{2}\right)^{n+1} > \frac{\varepsilon}{2}.$$

(2) If $y = (\frac{1}{2})^n$ and $n < i$, then

$$d(f_i(x), y) - d(f_i(x), f_i(y)) = \left(\frac{1}{2}\right)^{n+1} > \frac{(\frac{1}{2})^n - (\frac{1}{2})^i}{2} > \frac{\varepsilon}{2}.$$

(3) If $y = (\frac{1}{2})^n$ and $n > i$, then

$$d(f_i(x), x) - d(f_i(x), f_i(y)) = \left(\frac{1}{2}\right)^i - \left(\frac{1}{2}\right)^{n+1} > \left(\frac{1}{2}\right)^{i+1} - \left(\frac{1}{2}\right)^{n+1} > \frac{\varepsilon}{2}.$$

So in all cases we have

$$\operatorname{diam}\{x, y, f_i(x), f_i(y)\} - d(f_i(x), f_i(y)) > \frac{\varepsilon}{2},$$

and η is a modulus of uniform generalized 1-contractivity. Let $x_0 := 1$. Then there does not exist a Cauchy rate valid for all the sequences $(f_i^n(x_0))_n$. □

As the following lemma shows, a function with a modulus of uniform generalized p-contractivity has what has been called a modulus of uniqueness. This notion was defined in full generality by U. Kohlenbach in [21].

Lemma 3.11. *Let (X, d) be a metric space. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity. Define $\psi : (0, \infty) \to (0, \infty)$ by $\psi(\varepsilon) := \eta(\varepsilon)/2$. Then for all $\varepsilon \in (0, \infty)$ and for all $x, y \in X$, if $d(x, f^p(x)) \leq \psi(\varepsilon)$ and $d(y, f^p(y)) \leq \psi(\varepsilon)$, then $d(x, y) \leq \varepsilon$.*

Proof. Since η is a modulus of uniform generalized p-contractivity, it follows that if $d(x, y) > \varepsilon$ then we have one of the following:

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(x, y), \tag{3}$$

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(x), y), \tag{4}$$

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(y), x), \tag{5}$$

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(x), x), \tag{6}$$

$$d(f^p(x), f^p(y)) + \eta(\varepsilon) < d(f^p(y), y). \tag{7}$$

We show that if $d(x, f^p(x)) \leq \eta(\varepsilon)/2$ and $d(y, f^p(y)) \leq \eta(\varepsilon)/2$, then $d(x, y) \leq \varepsilon$. So let $d(x, f^p(x)) \leq \eta(\varepsilon)/2$ and $d(y, f^p(y)) \leq \eta(\varepsilon)/2$. Then it is obvious that (6) and (7) do not hold. Furthermore, we have

$$d(x, y) \leq d(f^p(x), f^p(y)) + d(f^p(x), x) + d(f^p(y), y) \leq d(f^p(x), f^p(y)) + \eta(\varepsilon),$$

so (3) does not hold. In the same way it follows by the triangle inequality that (4) and (5) do not hold. It follows that we have $d(x, y) \leq \varepsilon$. □

Corollary 3.12. *Let (X, d) be a metric space. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity. If the sequences (x_n) and (y_n) satisfy*

$$\forall \varepsilon > 0 \exists n \forall m \geq n (d(x_m, f^p(x_m)) < \varepsilon) \tag{8}$$

and

$$\forall \varepsilon > 0 \exists n \forall m \geq n (d(y_m, f^p(y_m)) < \varepsilon), \tag{9}$$

then the sequence $(d(x_n, y_n))_n$ converges to 0, and in addition the sequences (x_n) and (y_n) are in fact Cauchy sequences.

Proof. Suppose the sequences (x_n) and (y_n) satisfy (8) and (9). Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ be such that for all $m \geq n$ we have $d(x_m, f^p(x_m)) < \eta(\varepsilon)/2$. Let $m_1, m_2 \geq n$. Then $d(x_{m_1}, f^p(x_{m_1})) < \eta(\varepsilon)/2$ and $d(x_{m_2}, f^p(x_{m_2})) < \eta(\varepsilon)/2$. And so by Lemma 3.11 it follows that $d(x_{m_1}, x_{m_2}) \leq \varepsilon$. Thus we have that (x_n) is a Cauchy sequence. In the same way it follows that (y_n) is a Cauchy sequence. Let $n' \in \mathbb{N}$ be such that for all $m \geq n'$ we have $d(y_m, f^p(y_m)) < \eta(\varepsilon)/2$. Then for $m \geq \max\{n, n'\}$ we have $d(x_m, f^p(x_m)) < \eta(\varepsilon)/2$ and $d(y_m, f^p(y_m)) < \eta(\varepsilon)/2$. So by Lemma 3.11 it follows that $d(x_m, y_m) \leq \varepsilon$. Hence the sequence $(d(x_n, y_n))_n$ converges to 0. □

We now prove that if the iteration sequence $(f^n(x_0))$ is bounded for one $x_0 \in X$, then it is bounded for any $x_0 \in X$.

Theorem 3.13. *Let (X, d) be a metric space, and let $f : X \to X$ be uniformly generalized p-contractive and uniformly continuous. For $x_0 \in X$ define the iteration sequence (x_n) by $x_{n+1} := f(x_n)$. Suppose for some $x_0 \in X$ the iteration sequence is bounded. Then for every choice of $x_0 \in X$ the iteration sequence (x_n) is bounded, and in fact Cauchy. Also, for all $x_0, y_0 \in X$ we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. If (X, d) is complete all Picard iteration sequences converge to the unique fixed point of f.*

Proof. We prove first the special case where $p = 1$ and the space is complete. We know from Lemma 3.7 that f has a modulus of modified uniform generalized 1-contractivity. Call this modulus ρ. Likewise from Theorem 3.1 we know that f has a unique fixed point z. Let $x_0 \in X$ be arbitrary, and consider $\operatorname{diam}\{x_0, \ldots, x_n\}$. Assume $x_n \neq z$, for else (x_n) is bounded. Of course $x_n \neq z$ also implies $z \neq x_i$ for $i < n$, and hence $d(x_{i-1}, x_i) > 0$ for $0 < i \leq n$. For some $0 \leq i \leq n$ we have

$$\operatorname{diam}\{x_0, \ldots, x_n\} = d(x_0, x_i),$$

for if we for $0 < i, j \leq n$ had $\operatorname{diam}\{x_0, \ldots, x_n\} = d(x_i, x_j)$, then we would have

$$\operatorname{diam}\{x_i, x_j, x_{i-1}, x_{j-1}\} > d(x_i, x_j) = \operatorname{diam}\{x_0, \ldots, x_n\}.$$

In the same way, for $i > 0$ we have

$$\operatorname{diam}\{z, x_0, \ldots, x_n\} \neq d(z, x_i),$$

for we have for such i

$$d(z, x_i) < \max\{d(z, x_{i-1}), d(x_i, x_{i-1})\}.$$

Assume

$$\operatorname{diam}\{z, x_0, \ldots, x_n, x_{n+1}\} > \operatorname{diam}\{z, x_0, \ldots, x_n\}.$$

By the above we have

$$\operatorname{diam}\{z, x_0, \ldots, x_{n+1}\} = d(x_0, x_{n+1}).$$

Assume $d(x_0, x_{n+1}) > 2d(x_0, z)$. Since

$$d(z, x_{n+1}) + d(x_0, z) \geq d(x_0, x_{n+1}),$$

we have

$$d(z, x_{n+1}) > d(x_0, z).$$

Let $\varepsilon > 0$ satisfy $\varepsilon \leq d(x_0, z)$. Since ρ is a modulus of modified uniform generalized 1-contractivity, we have either

$$d(z, x_n) > d(z, x_{n+1}) + \rho(\varepsilon)$$

or

$$d(x_{n+1}, x_n) > d(z, x_{n+1}) + \rho(\varepsilon).$$

Let $m_0 := n + 1$ and $m'_0 := -1$. We will let x_{-1} denote z. For $0 \leq i < n$ we define m_{i+1} and m'_{i+1} inductively such that the following holds.

(1) If $m'_i = -1$, then $m'_{i+1} \in \{m_i, m_i - 1, m'_i\}$ and $m_{i+1} \in \{m_i, m_i - 1\}$ such that

$$d(x_{m_{i+1}}, x_{m'_{i+1}}) = \operatorname{diam}\{x_{m_i}, x_{m_i - 1}, z\}.$$

(2) If $m'_i \neq -1$, then $m'_{i+1}, m_{i+1} \in \{m_i, m_i - 1, m'_i, m'_i - 1\}$ such that

$$d(x_{m_{i+1}}, x_{m'_{i+1}}) = \operatorname{diam}\{x_{m_i}, x_{m_i - 1}, x_{m'_i}, x_{m'_i - 1}\}.$$

Then since $d(x_{m_0}, x_{m'_0}) > \varepsilon$ we can prove by induction on i that

$$d(x_{m_{i+1}}, x_{m'_{i+1}}) > d(x_{m_i}, x_{m'_i}) + \rho(\varepsilon)$$

and $d(x_{m_i}, x_{m'_i}) > \varepsilon$ for all $0 \leq i < n$. And so

$$d(x_{m_i}, x_{m'_i}) > d(x_{m_0}, x_{m'_0}) + i\rho(\varepsilon)$$

for $0 < i < n$. Hence, if n satisfies $n\rho(\varepsilon) > d(x_0, z)$ then we get

$$d(x_{m_n}, x_{m'_n}) > d(x_{n+1}, z) + d(x_0, z) \geq d(x_{n+1}, x_0) = \operatorname{diam}\{z, x_0, \ldots, x_{n+1}\}.$$

Specifically, we may take

$$n := \left\lceil \frac{b}{\rho(\varepsilon)} \right\rceil,$$

for any $b > d(x_0, z)$. But we have

$$d(x_{m_n}, x_{m'_n}) \leq \operatorname{diam}\{z, x_0, \ldots, x_{n+1}\},$$

and hence for large enough n we have $d(x_0, x_{n+1}) \leq 2d(x_0, z)$ or

$$\operatorname{diam}\{z, x_0, \ldots, x_n, x_{n+1}\} \leq \operatorname{diam}\{z, x_0, \ldots, x_n\}.$$

Thus for the special case where the space is complete and $p = 1$ we have proved that if one Picard iteration sequence is bounded, then any Picard iteration sequence is bounded. And so in this case it follows by Theorem 3.1 that all Picard iteration sequences converge to the unique fixed point z.

Now let $p \neq 1$. Then by the above f^p has a unique fixed point z and moreover for any $x_0 \in X$ we have $\lim_{n\to\infty} f^{np}(x_0) = z$. A trivial argument now shows that the same is true for f. So $(f^n(x_0))$ converges to z for any $x_0 \in X$, and in particular it is bounded.

Next suppose the space X is not complete. We consider the completion of X and the canonical extension of the uniformly continuous function f. Then the extension of f still has moduli of uniform continuity and uniform generalized p-contractivity, and the bounded Picard iteration sequence we presupposed stays the same. So by the above every Picard iteration sequence in the completion of X converges to the unique fixed point z. And so for all $x_0, y_0 \in X$ we have that (x_n) and (y_n) are Cauchy and in particular bounded, and furthermore that $\lim_{n\to\infty} d(x_n, y_n) = 0$. This ends the proof. □

Corollary 3.14. *Let (X, d) be a complete metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let z be the unique fixed point of f. Let $x_0 \in X$ and for $0 \leq i < p$ let*

$b_i > 0$ and $\delta_i > 0$ satisfy $\delta_i \le d(x_i, z) < b_i$. Let $\varepsilon > 0$. Let Φ be as in Theorem 3.1 and let

$$N_i = \left\lceil \frac{b_i}{\rho(\delta_i)} \right\rceil,$$

where $\rho : (0, \infty) \to (0, \infty)$ is defined by

$$\rho(\gamma) = \min\left\{\eta(\gamma), \frac{\gamma}{2}, \eta(1/2 \cdot \omega^p(\gamma/2))\right\}.$$

Let

$$M_i = \max\left\{2b_i, \operatorname{diam}\{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{N_i p}(x_i)\}\right\},$$

and let

$$M = \max\{M_0, \ldots, M_{p-1}\}.$$

Then for all $m, n \in \mathbb{N}$ we have that

$$m, n \ge \Phi(p, \omega, \eta, 2M, \varepsilon)$$

gives

$$d(x_n, x_m) \le \varepsilon$$

and so

$$d(x_n, z) \le \varepsilon.$$

Proof. We first note that since f^p has moduli η and ω^p of uniform generalized 1-contractivity and uniform continuity, we have by Lemma 3.7 that ρ is a modulus of modified uniform generalized 1-contractivity for f^p. Then by the proof of Theorem 3.13 we can infer that for $0 \le i < p$ the iteration sequence $(f^{pn}(x_i))_{n\in\mathbb{N}}$ is bounded by M_i. Namely, we proved that

$$\operatorname{diam}\{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{(n+1)p}(x_i)\} = \operatorname{diam}\{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{np}(x_i)\}$$

if

$$\operatorname{diam}\{z, x_i, f^p(x_i), f^{2p}(x_i), \ldots, f^{(n+1)p}(x_i)\} > 2d(x_i, z)$$

and $n \ge N_i$. Thus $(f^n(x_0))_{n\in\mathbb{N}}$ is bounded by $2M$. Now the claim follows by Theorem 3.1. □

Corollary 3.15. *Let (X, d) be a complete metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let z be the unique fixed point of f. Let $x_0 \in X$ and let $\delta, b > 0$ be such that $\delta \le d(x_0, z) < b$. Let $\varepsilon > 0$. Let Φ be as in Theorem 3.1. Let*

$$N = \left\lceil \frac{b}{\rho(\delta)} \right\rceil,$$

where ρ is as in Corollary 3.14. Let

$$M = \max\{2b, \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\}\},$$

and let

$$K = \Phi(1, \omega^p, \eta, M, 1/2 \cdot \min\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\}).$$

Let

$$M' = \operatorname{diam}\{x_n : 0 \le n \le Kp\} + 2.$$

Then for all $m, n \in \mathbb{N}$ we have that

$$m, n \ge \Phi(p, \omega, \eta, M', \varepsilon)$$

gives

$$d(x_n, x_m) \le \varepsilon$$

and so

$$d(x_n, z) \le \varepsilon.$$

Proof. As in the proof of Corollary 3.14 we note that f^p has moduli η of uniform generalized 1-contractivity, ω^p of uniform continuity and ρ of modified uniform generalized 1-contractivity. Furthermore, as in the proof of Corollary 3.14 we get that $(f^{pn}(x_0))_n$ is bounded by M. Then for $m, n \ge K$ we have

$$d(x_{mp}, x_{np}) \le 1/2 \cdot \min\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\}$$

and

$$d(z, x_{np}) \le 1/2 \cdot \min\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\}.$$

Since ω is a modulus of uniform continuity for f we have in particular that

$$\begin{aligned} d(x_{np}, z) &< 1, \\ d(x_{np+1}, z) &< 1, \\ d(x_{np+2}, z) &< 1, \\ &\vdots \\ d(x_{np+(p-1)}, z) &< 1, \end{aligned}$$

for $n \ge K$. And so for $n \ge Kp$ we have in fact $d(x_n, z) < 1$. Let now m, n be nonnegative integers. We distinguish three cases:

(1) If $m, n \le Kp$, then

$$d(x_n, x_m) \le \operatorname{diam}\{x_k : 0 \le k \le Kp\} < M'.$$

(2) If $m, n \ge Kp$, then

$$d(x_n, x_m) \le d(x_n, z) + d(x_m, z) < 2 \le M'.$$

(3) If $m < Kp$ and $n > Kp$, then

$$d(x_n, x_m) \le d(x_m, x_{Kp}) + d(x_n, x_{Kp}) < \operatorname{diam}\{x_k : 0 \le k \le Kp\} + 2 = M'.$$

It follows that M' is a bound on $(x_n)_n$. Now Theorem 3.1 gives the conclusion. □

In the previous two corollaries we gave rates of convergence which were dependent on strictly positive upper and lower bounds on $d(z, x_i)$ for some i. We will now improve this as follows.

Corollary 3.16. *Let (X,d) be a complete metric space, and let $p \in \mathbb{N}$. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let $x_0 \in X$ and let $\delta > 0$ be such that $\delta \leq d(x_0, x_1)$. Let $b, c, \varepsilon > 0$. Let Φ be as in Theorem 3.1. Assume that there is $y \in X$ such that $d(x_0, y) < b$ and $d(x_1, y) < b$, and such that either*

$$d(y, f^p(y)) < \frac{\eta(c)}{2}$$

or $(f^n(y))_n$ is bounded by c. Let

$$N = \left\lceil \frac{b+c}{\rho(\delta/2)} \right\rceil,$$

where ρ is as in Corollary 3.14. Let

$$M_0 = \max\{2(b+c), \operatorname{diam}\{x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} + b + c\},$$

$$M_1 = \max\{2(b+c), \operatorname{diam}\{x_1, f^p(x_1), f^{2p}(x_1), \ldots, f^{Np}(x_1)\} + b + c\},$$

and let

$$K = \Phi(1, \omega^p, \eta, \max\{M_0, M_1\}, 1/2 \cdot \min\{1, \omega(1), \omega^2(1), \ldots, \omega^{p-1}(1)\}).$$

Let

$$M' = \operatorname{diam}\{x_n : 0 \leq n \leq Kp + 1\} + 2.$$

Then for all $m, n \in \mathbb{N}$ we have that

$$m, n \geq \Phi(p, \omega, \eta, M', \varepsilon)$$

gives

$$d(x_n, x_m) \leq \varepsilon.$$

And so

$$d(x_n, z) \leq \varepsilon,$$

where z is the unique fixed point.

Proof. By Lemma 3.11 it follows that $d(y,z) \leq c$, where z is the unique fixed point. Thus by the triangle inequality $d(x_0, z) < b + c$ and $d(x_1, z) < b + c$. Furthermore, either $\delta/2 \leq d(x_0, z)$ or $\delta/2 \leq d(x_1, z)$. As in the proof of Corollary 3.14 we get that either $(f^{pn}(x_0))_n$ is bounded by M_0 or $(f^{pn}(x_1))_n$ is bounded by M_1. So we have that $d(f^n(x_0), z) < 1$ for all $n \geq K$ or $d(f^n(x_1), z) < 1$ for all $n \geq K$, and so we have $d(f^n(x_0), z) < 1$ for all $n \geq K + 1$. Hence, M' is a bound on $(x_n)_n$, and the conclusion follows by Theorem 3.1. □

The Cauchy rates appearing in the last three corollaries depend heavily on f. If the space satisfies a further structural condition we may find Cauchy rates with uniformity properties with respect to f. This will include for instance spaces of hyperbolic type in the sense of [16], as well as hyperbolic spaces in the sense of [26] and hyperbolic spaces in the sense of [22], and therefore e.g. normed linear spaces, Hadamard manifolds and CAT(0)-spaces.

Definition 3.17. Let (X,d) be a metric space. Let $\varepsilon > 0$ and $x, y \in X$. We say that x is *ε-step-equivalent to y* if there exist points $x_0 = x, x_1, \dots, x_n = y$, belonging to X, with $d(x_i, x_{i+1}) \leq \varepsilon$ for $i < n$. This defines for each $\varepsilon > 0$ an equivalence relation on X. We call the equivalence classes *ε-step-territories.*

The notions in Definition 3.17 are taken from [24]. The condition on a metric space which in the terminology of Definition 3.17 amounts to requiring that the space should be an ε-step-territory was already treated by M. Edelstein. We will employ a uniform version of ε-step-territories.

Definition 3.18. Let (X,d) be a metric space, and let $\varepsilon > 0$. A subset T_ε of X is a *uniform ε-step-territory* if there exists $\alpha_\varepsilon : \mathbb{N} \to \mathbb{N}$ such that for all $x, y \in T_\varepsilon$ and all $n \in \mathbb{N}$, if $d(x,y) < n\varepsilon$, then there exist $x_0 = x, x_1, \dots, x_{\alpha_\varepsilon(n)} = y \in T_\varepsilon$ with $d(x_i, x_{i+1}) < \varepsilon$ for $i < \alpha_\varepsilon(n)$.

Definition 3.19. Let (X,d) be a metric space. A subset T of X is called a *territory* if it is an ε-step-territory for each $\varepsilon > 0$. A subset T of X is called a *uniform territory* if it is a uniform ε-step-territory for each $\varepsilon > 0$.

Definition 3.20. Let (X,d) be a metric space, and let T be a subset of X. A function $\alpha : \mathbb{R} \times \mathbb{N} \to \mathbb{N}$ is called a *uniform territory modulus* for T if for each $\varepsilon > 0$ and for all $x, y \in T$ and $n \in \mathbb{N}$ such that $d(x,y) < n\varepsilon$, there exist $x_0 = x, x_1, \dots, x_{\alpha(\varepsilon,n)} = y \in T$ with $d(x_i, x_{i+1}) < \varepsilon$ for $i < \alpha(\varepsilon, n)$.

We note that if T has a uniform territory modulus then T is a uniform territory.

Corollary 3.21. *Let (X,d) be a complete metric space with a uniform territory modulus α. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let z be the unique fixed point of f. Let $x_0 \in X$ and let $b > 0$ satisfy $d(x_0, z) < b$. Let $\varepsilon > 0$. Then for all $n \in \mathbb{N}$,*

$$d(z, f^n(x_0)) < K^{p-1}(b + K^{Np}(b)),$$

where $K : (0,\infty) \to (0,\infty)$ is defined by

$$K(\gamma) := \max\left\{\alpha\left(\omega(\varepsilon), \left\lceil \frac{\gamma}{\min\{\varepsilon, \omega(\varepsilon)\}} \right\rceil\right) \cdot \varepsilon, \gamma\right\},$$

$$N := \left\lceil \frac{b}{\rho(\delta)} \right\rceil,$$

$\rho : (0,\infty) \to (0,\infty)$ is defined by

$$\rho(\gamma) := \min\left\{\eta(\gamma), \frac{\gamma}{2}, \eta\left(\frac{1}{2}\omega^p\left(\frac{\gamma}{2}\right)\right)\right\},$$

and $\delta := \min\{b, \omega(b)\}$. Let $\varepsilon' > 0$ and let Φ be as in Theorem 3.1. Then

$$m, n \geq \Phi(p, \omega, \eta, 2K^{p-1}(b + K^{Np}(b)), \varepsilon')$$

gives $d(x_n, x_m) \leq \varepsilon'$.

Proof. Since f^p has moduli η and ω^p of uniform generalized 1-contractivity and uniform continuity, we have that ρ is a modulus of modified uniform generalized 1-contractivity for f^p. Now, if $d(x_0, z) < \delta$, then if we do not for all $n \in \mathbb{N}$ have $d(f^n(x_0), z) < \delta$, it follows from the definition of δ that we for some $m \in \mathbb{N}$ have $\delta \leq d(f^m(x_0), z) < b$. For if $f^m(x_0)$ is the first member of the sequence which is not an element of the set $\{x \in X : d(x, z) < \delta\}$, then $d(f^{m-1}(x_0), z) < \delta \leq \omega(b)$. So since ω is a modulus of uniform continuity for f we have $d(f^m(x_0), z) < b$. So in total $\delta \leq d(f^m(x_0), z) < b$. We can take $f^m(x_0)$ as the starting point x'_0 of a new Picard iteration sequence. If we can establish the bound on $d(f^n(x'_0), z)$ for this sequence, then it is also proved for our original sequence, since $d(f^i(x_0), z) < \delta < K^{p-1}(b + K^{Np}(b))$ for $i < m$. The last inequality follows since $K(\gamma) \geq \gamma$ for $\gamma > 0$. Hence, we may assume

$$\delta \leq d(x_0, z) < b.$$

Then as in the proof of Corollary 3.14 we can infer that the iteration sequence $(f^{pn}(x_0))_{n \in \mathbb{N}}$ is bounded by

$$M := \max\{2d(x_0, z), \operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\}\}.$$

From the proof of Theorem 3.13 it follows that

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} = d(x_0, z)$$

or

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} = d(x_0, f^{ip}(x_0)),$$

for some $i \leq N$. Therefore

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} \leq d(x_0, z) + d(z, f^{ip}(x_0)), \tag{10}$$

for some $i \leq N$. Since

$$d(z, x_0) < \left\lceil \frac{b}{\min\{\varepsilon, \omega(\varepsilon)\}} \right\rceil \cdot \omega(\varepsilon),$$

we have by definition of K and by the assumed property of the space that $d(z, f(x_0)) < K(b)$. This follows since with

$$m := \alpha\left(\omega(\varepsilon), \left\lceil \frac{b}{\min\{\varepsilon, \omega(\varepsilon)\}} \right\rceil\right),$$

we have that there exist $x'_0 = x_0, x'_1, \ldots, x'_m = z \in X$ with

$$d(x'_i, x'_{i+1}) < \omega(\varepsilon)$$

for $i < m$. And so $d(f(x_0), z) < K(b)$. Furthermore,

$$d(z, f(x_0)), d(z, f^2(x_0)) < K^2(b),$$

since $K(\gamma) \geq \gamma$. And in general,

$$d(z, f(x_0)), d(z, f^2(x_0)), \ldots, d(z, f^k(x_0)) < K^k(b).$$

So by (10) we have

$$\operatorname{diam}\{z, x_0, f^p(x_0), f^{2p}(x_0), \ldots, f^{Np}(x_0)\} < b + K^{Np}(b).$$

Thus $M < b + K^{Np}(b)$. Hence for any $n \in \mathbb{N}$ we have

$$d(z, f^{np}(x_0)) < b + K^{Np}(b),$$

and so

$$d(z, f^{np}(x_0)), d(z, f^{np+1}(x_0)) < K(b + K^{Np}(b)).$$

For all $n \in \mathbb{N}$ we have

$$d(z, f^{np}(x_0)), d(z, f^{np+1}(x_0)), \ldots, d(z, f^{np+p-1}(x_0)) < K^{p-1}(b + K^{Np}(b)).$$

That is, for all $n \in \mathbb{N}$ we have

$$d(z, f^n(x_0)) < K^{p-1}(b + K^{Np}(b)).$$

Hence, $2K^{p-1}(b + K^{Np}(b))$ is a bound on (x_n), and the conclusion follows by Theorem 3.1. □

Notice that the Cauchy rate in the preceeding corollary only depends on p, ω, η, α, b and ε. Given these the rate is uniform in the space, the mapping and the starting point.

We can treat the situation where the space is not complete as follows. We consider a metric space (X, d) and a function $f : X \to X$ with moduli ω and η of uniform continuity and uniform generalized p-contractivity. We denote by f also the canonical extension of f to the completion of X. We can then define e.g. $\omega' : (0, \infty) \to (0, \infty)$ by $\omega'(\varepsilon) := \omega(\varepsilon/2)$ and $\eta' : (0, \infty) \to (0, \infty)$ by $\eta'(\varepsilon) := \eta(\varepsilon)/2$. It is easy to see that ω' and η' are moduli of uniform continuity and uniform generalized p-contractivity for f considered as a function on the completion of X. We can thus find Cauchy rates for $(x_n)_n$ with $x_0 \in X$ by considering the completion and the suitably modified moduli.

We will now improve Corollary 3.21 similarly to the way Corollary 3.16 is an improvement of Corollary 3.15, and at the same time spell out the details for what happens in this case when the space is not complete.

Corollary 3.22. *Let (X, d) be a metric space with a uniform territory modulus α. Let $f : X \to X$ have a modulus η of uniform generalized p-contractivity and a modulus ω of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let ω' and η' be defined as above, and let $\varepsilon > 0$. Let $x_0 \in X$, and let $b, c \in (0, \infty)$ be such that there is $y \in X$ with*

$$d(y, f^p(y)) < \frac{\eta'(c)}{2},$$

such that $d(x_0, y) < b$. Then $(f^n(x_0))$ is bounded by

$$2K^{p-1}(b + c + K^{Np}(b + c)),$$

where $K : (0, \infty) \to (0, \infty)$ is defined by

$$K(\gamma) := \max\left\{\alpha\left(\frac{1}{2} \cdot \omega'(\varepsilon), \left\lceil \frac{\gamma}{1/2 \cdot \min\{\varepsilon, \omega'(\varepsilon)\}} \right\rceil\right) \cdot \varepsilon, \gamma\right\},$$

$$N := \left\lceil \frac{b + c}{\rho'(\delta)} \right\rceil,$$

$\rho' : (0, \infty) \to (0, \infty)$ is defined by

$$\rho'(\gamma) := \min\left\{\eta'(\gamma), \frac{\gamma}{2}, \eta'\left(\frac{1}{2}\omega'^p\left(\frac{\gamma}{2}\right)\right)\right\},$$

and $\delta := \min\{b + c, \omega'(b + c)\}$*. Let* $\varepsilon' > 0$ *and let* Φ *be as in Theorem 3.1. Then*

$$m, n \geq \Phi(p, \omega, \eta, 2K^{p-1}(b + c + K^{Np}(b + c)), \varepsilon')$$

gives $d(x_n, x_m) \leq \varepsilon'$.

Proof. We consider the completion $(\widehat{X}, \widehat{d})$ of (X, d) and the canonical extension of f, which we also denote f. We have that ω' and η' are moduli of uniform continuity and uniform generalized p-contractivity for f. Now $(\widehat{X}, \widehat{d})$ satisfies the condition that for each $\varepsilon > 0$ and for all $x, y \in \widehat{X}$ and $n \in \mathbb{N}$, if $\widehat{d}(x, y) < n\varepsilon$, then there exist $x'_0 = x, x'_1, \ldots, x'_{\alpha(\varepsilon,n)} = y \in \widehat{X}$ with $\widehat{d}(x'_i, x'_{i+1}) < 2\varepsilon$ for $i < \alpha(\varepsilon, n)$. Let z be the unique fixed point. By assumption we have $d(y, f^p(y)) < \eta'(c)/2$, and so by Lemma 3.11 we get that $\widehat{d}(y, z) \leq c$. And since we assume that $d(x_0, y) < b$ we get $\widehat{d}(x_0, z) < b + c$. Our new definition of K serves the same purpose as the version in Corollary 3.21, i.e. for $x \in \widehat{X}$ and $b' > 0$ with $\widehat{d}(x, z) < b'$, we get $\widehat{d}(f(x), z) < K(b')$. This follows since

$$\left\lceil \frac{b'}{1/2 \cdot \min\{\varepsilon, \omega'(\varepsilon)\}} \right\rceil \cdot 1/2 \cdot \omega'(\varepsilon) > \widehat{d}(x, z),$$

so with

$$m := \alpha\left(1/2 \cdot \omega'(\varepsilon), \left\lceil \frac{b'}{1/2 \cdot \min\{\varepsilon, \omega'(\varepsilon)\}} \right\rceil\right),$$

we have that there exist $x'_0 = x, x'_1, \ldots, x'_m = z \in \widehat{X}$ with

$$\widehat{d}(x'_i, x'_{i+1}) < 2 \cdot 1/2 \cdot \omega'(\varepsilon)$$

for $i < m$. And so $\widehat{d}(f(x), z) < K(b')$. Now by identical reasoning as in Corollary 3.21 we get that for all $n \in \mathbb{N}$ we have

$$\widehat{d}(f^n(x_0), z) < K^{p-1}(b + c + K^{Np}(b + c)).$$

Thus $2K^{p-1}(b+c+K^{Np}(b+c))$ is a bound on $(f^n(x_0))$ in $\widehat{X}$, and hence also in X. The conclusion follows by Theorem 3.1. □

Corollary 3.23. *Let* (X, d) *be a metric space with a uniform territory modulus* α*. Let* $f : X \to X$ *have a modulus* η *of uniform generalized p-contractivity and a modulus* ω *of uniform continuity. Suppose for some starting point the Picard iteration sequence is bounded. Let* $\varepsilon > 0$*. Let* $x_0, y_0 \in X$*, and let* $b, c \in (0, \infty)$ *be such that* $d(x_0, y_0) < b$ *and such that* c *is a bound on* $(f^n(y_0))$*. Then* $(f^n(x_0))$ *is bounded by*

$$2K^{p-1}(b + c + K^{Np}(b + c)),$$

where K*,* N*,* ρ' *and* δ *are defined as in Corollary 3.22. Let* $\varepsilon' > 0$ *and let* Φ *be as in Theorem 3.1. Then*

$$m, n \geq \Phi(p, \omega, \eta, 2K^{p-1}(b + c + K^{Np}(b + c)), \varepsilon')$$

gives $d(x_n, x_m) \leq \varepsilon'$.

Proof. We have in the completion $(\widehat{X}, \widehat{d})$ of (X, d) that $\widehat{d}(x_0, z) < b + c$, where z is the unique fixed point. Now the result follows as in Corollary 3.22. □

Finally we include some remarks on applications of fixed point theorems for mappings satisfying contractive type conditions more general than the one due to Banach or the one due to Edelstein. Such contractive type conditions have been extensively studied as part of an attempt to conceptually understand the fixed point theory of selfmappings of abstract metric spaces, but they are often difficult to apply in other areas of mathematics. We will now consider how we can formulate a more general version of Picard's theorem for differential equations using Theorem 2.7. This will illustrate the fact that often, when one wishes to apply fixed point theorems for more general contractive conditions to obtain in turn more general versions of theorems outside of metric fixed point theory, the results are indeed more general, but also seemingly not very practical.

Picard's theorem (for a proof, see e.g. [8]) tells us that given a bounded, continuous real-valued function $f : G \to \mathbb{R}$ defined on an open subset G of $\mathbb{R}^2$, if f satisfies a Lipschitz condition with respect to the second variable, i.e. if there exists $M \geq 0$ such that

$$|f(x, y_1) - f(x, y_2)| \leq M|y_1 - y_2|$$

holds for all $(x, y_1), (x, y_2) \in G$, then for any $(x_0, y_0) \in G$ the differential equation $y' = f(x, y)$ with initial condition $y(x_0) = y_0$ has a unique solution ϕ in some interval $I = [x_0 - \delta, x_0 + \delta]$. Here $\delta > 0$ is chosen such that $M\delta < 1$ and such that

$$\{(x, y) : |x - x_0| \leq \delta, |y - y_0| \leq K\delta\} \subseteq G,$$

where $K > 0$ is such that $|f(x, y)| \leq K$ for all $(x, y) \in G$. The proof involves considering the complete metric space (X, d) of all continuous functions $g : I \to [y_0 - K\delta, y_0 + K\delta]$, with the metric d defined by $d(g, h) = \max_{t\in I} |g(t) - h(t)|$, and the mapping $T : X \to X$ defined by

$$(Tg)(x) = y_0 + \int_{x_0}^{x} f(t, g(t))\, dt$$

for all $g \in X$ and $x \in I$, and then showing that T is a contraction. As a consequence we also get that $(T^n g)_n$ converges to the unique solution ϕ for any $g \in X$. A crucial step in the proof involves showing that $d(Tg, Th) \leq M\delta \cdot d(g, h)$ by showing that

$$|(Tg)(x) - (Th)(x)| \leq \left| \int_{x_0}^{x} [f(t, g(t)) - f(t, h(t))]\, dt \right| \leq M\delta \cdot d(g, h)$$

for all $g, h \in X$ and all $x \in I$. This follows since f is Lipschitzian with constant M with respect to the second variable. Now from Ćirić's theorem we can deduce that if we remove the condition that f is Lipschitzian with respect to the second variable (but still assume that f is continuous and bounded) then for $(x_0, y_0) \in G$ and initial condition $y(x_0) = y_0$ we can still conclude the existence of a unique solution ϕ in $I = [x_0 - \delta, x_0 + \delta]$, with $\delta > 0$ such that

$$\{(x, y) : |x - x_0| \leq \delta, |y - y_0| \leq K\delta\} \subseteq G,$$

if for some $M \geq 0$ with $M\delta < 1$ we for all $x \in I$ and all $g, h \in X$ have

$$\left| \int_{x_0}^{x} [f(t, g(t)) - f(t, h(t))]\, dt \right| \leq M\delta \cdot \max_{t\in I} |g(t) - h(t)|,$$

$$\left|\int_{x_0}^{x}[f(t,g(t))-f(t,h(t))]\,dt\right| \le M\delta\cdot\max_{t\in I}\left|y_0+\int_{x_0}^{t}f(u,h(u))\,du-h(t)\right|,$$

$$\left|\int_{x_0}^{x}[f(t,g(t))-f(t,h(t))]\,dt\right| \le M\delta\cdot\max_{t\in I}\left|y_0+\int_{x_0}^{t}f(u,g(u))\,du-g(t)\right|,$$

$$\left|\int_{x_0}^{x}[f(t,g(t))-f(t,h(t))]\,dt\right| \le M\delta\cdot\max_{t\in I}\left|y_0+\int_{x_0}^{t}f(u,g(u))\,du-h(t)\right|$$

or

$$\left|\int_{x_0}^{x}[f(t,g(t))-f(t,h(t))]\,dt\right| \le M\delta\cdot\max_{t\in I}\left|y_0+\int_{x_0}^{t}f(u,h(u))\,du-g(t)\right|.$$

We have also in this situation that $(T^n g)_n$ converges to the unique solution ϕ for any $g \in X$.

Acknowledgements

I am grateful to Dag Normann who supervised my Master Thesis, from which many of the above results are taken. I am also grateful to Ulrich Kohlenbach for fruitful discussions concerning the subject matter and many helpful suggestions concerning the presentation of the material. Finally, I wish to thank the referee for many useful suggestions for how to substantially improve the paper.

References

[1] I. D. Arandelović, *On a fixed point theorem of Kirk*, J. Math. Anal. Appl. **301** (2005), 384-385.
[2] M. Arav, F. E. C. Santos, S. Reich and A. J. Zaslavski, *A note on asymptotic contractions*, Fixed Point Theory and Applications **2007** (2007), 6 pages, Article ID 39465.
[3] M. Arav, S. Reich and A. J. Zaslavski, *Uniform Convergence of Iterates for a Class of Asymptotic Contractions*, Fixed Point Theory, **8** (2007), 3-9.
[4] E. M. Briseid, *Logical aspects of rates of convergence in metric spaces*, in preparation
[5] E. M. Briseid, *Addendum to the paper: Some results on Kirk's asymptotic contractions*, Fixed Point Theory **8** (2007), 321-322.
[6] E. M. Briseid, *A rate of convergence for asymptotic contractions*, J. Math. Anal. Appl. **330** (2008), 364-376.
[7] E. M. Briseid, *Some results on Kirk's asymptotic contractions*, Fixed Point Theory **8** (2007), 17-27.
[8] A. Browder, *Mathematical Analysis: An Introduction*, Springer-Verlag, New York, Berlin and Heidelberg, 2001.
[9] Y.-Z. Chen, *Asymptotic fixed points for nonlinear contractions*, Fixed Point Theory and Applications, **2005** (2005), 213-217.
[10] L. B. Ćirić, *A generalization of Banach's contraction principle*, Proc. Amer. Math. Soc. **45** (1974), 267-273.
[11] P. Collaço and J. Carvalho e Silva, *A complete comparison of 25 contraction conditions*, Nonlinear Analysis **30** (1997), 471-476.
[12] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74-79.
[13] P. Gerhardy, *A quantitative version of Kirk's fixed point theorem for asymptotic contractions*, J. Math. Anal. Appl. **316** (2006), 339-345.
[14] P. Gerhardy and U. Kohlenbach, *Strongly uniform bounds from semi-constructive proofs*, Ann. Pure Appl. Logic **141** (2006), 89-107.

[15] P. Gerhardy and U. Kohlenbach, *General logical metatheorems for functional analysis*, Transactions of the American Mathematical Society **360** (2008), 2615-2660.
[16] K. Goebel and W. A. Kirk, *Iteration processes for nonexpansive mappings*, in Topological Methods in Nonlinear Functional Analysis, S. P. Singh, S. Thomeier and B. Watson, (eds.), Contemporary Mathematics 21, AMS, pp. 115-123, 1983.
[17] K. Goebel and S. Reich, *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*, Marcel Dekker, New York and Basel, 1984.
[18] J. R. Jachymski and I. Jóźwik, *On Kirk's asymptotic contractions*, J. Math. Anal. Appl. **300** (2004), 147-159.
[19] J. Kincses and V. Totik, *Theorems and counterexamples on contractive mappings*, Mathematica Balkanica, New Series **4** (1990), 69-90.
[20] W. A. Kirk, *Fixed points of asymptotic contractions*, J. Math. Anal. Appl. **277** (2003), 645-650.
[21] U. Kohlenbach, *Effective moduli from ineffective uniqueness proofs: An unwinding of de La Vallée Poussin's proof for Chebycheff approximation*, Ann. Pure Appl. Logic **64** (1993), 27-94.
[22] U. Kohlenbach, *Some logical metatheorems with applications in functional analysis*, Transactions of the American Mathematical Society **357** (2005), 89-128.
[23] U. Kohlenbach and P. Oliva, *Proof mining: a systematic way of analyzing proofs in mathematics*, Proc. Steklov Inst. Math. **242** (2003), 136-164.
[24] A. G. O'Farrell, *When uniformly-continuous implies bounded*, Irish Math. Soc. Bulletin **53** (2004), 53-56.
[25] T. M. Rassias (ed.), *Nonlinear Analysis*, World Sci. Publishing, Singapore, 1987.
[26] S. Reich and I. Shafrir, *Nonexpansive iterations in hyperbolic spaces*, Nonlinear Anal. **15** (1990), 537-558.
[27] B. E. Rhoades, *A comparison of various definitions of contractive mappings*, Transactions of the American Mathematical Society **226** (1977), 257-290.
[28] B. E. Rhoades, *Contractive definitions*, in Nonlinear Analysis, T. M. Rassias (ed.), World Sci. Publishing, Singapore, 1987, pp. 513-526.
[29] S. P. Singh, S. Thomeier and B. B. Watson (eds.), *Topological Methods in Nonlinear Functional Analysis*, Contemporary Mathematics, Vol. 21, AMS, 1983.
[30] T. Suzuki, Fixed-point theorem for asymptotic contractions of Meir-Keeler type in complete metric spaces, Nonlinear Anal. **64** (2006), 971-978.
[31] T. Suzuki, A definitive result on asymptotic contractions, J. Math. Anal. Appl. **335** (2007), 707-715.
[32] K. Włodarczyk, D. Klim and R. Plebaniak, *Existence and uniqueness of endpoints of closed set-valued asymptotic contractions in metric spaces*, J. Math. Anal. Appl. **328** (2007), 46-57.
[33] H.-K. Xu, *Asymptotic and weakly asymptotic contractions*, *Indian J. Pure Appl. Math.* **36** (2005), 145-150.

Manuscript received March 22, 2006
revised February 7, 2008

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