Journal of Nonlinear and Convex Analysis
Volume 9, Number 2, 2008, 161–167

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

AN INVERSE OF THE BERGE MAXIMUM THEOREM FOR INFINITE DIMENSIONAL SPACES

TAKAMITSU YAMAUCHI

Abstract. An inverse of the Berge maximum theorem for paracompact domains and locally convex topological vector space ranges is proved. Cellina's approximate selection theorem for paracompact domains and locally convex topological vector space ranges is also obtained.

1. Introduction

Throughout this paper, all spaces are assumed to be Hausdorff. For a topological vector space Y, the set of all non-empty subsets (respectively, non-empty convex subsets, non-empty compact convex subsets) of Y is denoted by 2^Y (respectively, 2^Y_c, $\mathcal{C}_c(Y)$). For spaces X and Y, a mapping $\varphi : X \to 2^Y$ is called *lower semicontinuous* (respectively, *upper semicontinuous*) or *l.s.c.* (respectively, *u.s.c.*) if for every open subset V of Y, the set $\varphi^{-1}[V] = \{x \in X : \varphi(x) \cap V \neq \emptyset\}$ (respectively, $\varphi^{\#}[V] = \{x \in X : \varphi(x) \subset V\}$) is open in X. A mapping $\varphi : X \to 2^Y$ is called *continuous* if φ is both l.s.c. and u.s.c. For a vector space Y, a function $f : Y \to \mathbf{R}$ is *quasi-concave* if the set $\{y \in Y : f(y) \geq r\}$ is convex for each $r \in \mathbf{R}$.

The Berge maximum theorem [2, p. 116] is fundamental in mathematical economics and game theory. The following convex-version of the Berge maximum theorem is also well-known (cf. [5, §3.4]).

Theorem 1.1 (C. Berge [2]). *Let X be a topological space and Y a topological vector space. If $f : X \times Y \to \mathbf{R}$ is a continuous function such that $f(x, \cdot) : Y \to \mathbf{R}$ is quasi-concave for each $x \in X$ and $\varphi : X \to \mathcal{C}_c(Y)$ is a continuous mapping, then the mapping $\psi : X \to 2^Y$ defined by $\psi(x) = \{y \in \varphi(x) : f(x, y) = \max\{f(x, z) : z \in \varphi(x)\}\}$ for each $x \in X$ is u.s.c. and compact-and-convex-valued.*

In [13], Komiya posed an inverse problem of Theorem 1.1 for Euclidean spaces and gave an application to Kakutani's fixed point theorem. Here we state the problem in a general setting. For a mapping $\varphi : X \to 2^Y$, let $\mathrm{Gr}\varphi$ denote the graph $\{(x, y) \in X \times Y : y \in \varphi(x)\}$ of φ. Notice that, in Theorem 1.1, $\mathrm{Gr}\psi$ is a G_δ-set (that is, an intersection of countable many open subsets) of $\mathrm{Gr}\varphi$ since the marginal function $M : X \to \mathbf{R}$ defined by $M(x) = \max\{f(x, z) : z \in \varphi(x)\}$ is continuous (see [2, Theorem 1 and 2 in Chapter VI, §3]). Thus Komiya's inverse problem of Theorem 1.1 can be restated as follows.

Problem 1.2 (H. Komiya [13]). *Let X be a topological space, Y a topological vector space, $\varphi : X \to \mathcal{C}_c(Y)$ a continuous mapping and $\psi : X \to \mathcal{C}_c(Y)$ a u.s.c. set-valued*

2000 *Mathematics Subject Classification.* Primary 54C60, 54C65, 91B50.

Key words and phrases. the Berge maximum theorem, set-valued mapping, approximate selection.

selection of φ *such that* $\mathrm{Gr}\psi$ *is a* G_δ*-set of* $\mathrm{Gr}\varphi$*. Then does there exist a continuous mapping* $f : X \times Y \to \mathbf{R}$ *such that* $\psi(x) = \{y \in \varphi(x) : f(x, y) = \max\{f(x, z) : z \in \varphi(x)\}\}$ *and the mapping* $f(x, \cdot) : Y \to \mathbf{R}$ *is quasi-concave for each* $x \in X$*?*

Komiya [13, Theorem 2.1] answered Problem 1.2 affirmatively for every subset X of a Euclidean space and every Euclidean space Y. Park and Komiya [17, Theorem 2] answered Problem 1.2 affirmatively for every σ-selectionable set-valued mapping ψ from a topological space X to a locally convex metrizable topological vector space Y. Here we recall that a mapping $\psi : X \to 2^Y$ is called *σ-selectionable* (or said to satisfy *property* (σ)) if there exists a sequence $\{\varphi_n : n \in \mathbf{N}\}$ of continuous mappings $\varphi_n : X \to \mathcal{C}_c(Y)$ such that $\varphi_{n+1}(x) \subset \varphi_n(x)$ and $\psi(x) = \bigcap_{n\in\mathbf{N}} \psi_n(x)$ for each $n \in \mathbf{N}$ and $x \in X$. Note that every σ-selectionable mapping $\psi : X \to \mathcal{C}_c(Y)$ is u.s.c. Aoyama [1, Theorem 3.6] extended Park-Komiya's theorem to a convex metric space Y in the sense of Takahashi [20] with property (K) (see Remark 2.3). In this context, Komiya [14] asked whether the assumption of σ-selectionability of the mapping ψ can be removed even in the case that X and Y are subsets of Banach spaces.

The purpose of this paper is to answer Komiya's question by proving that Problem 1.2 is affirmative even if X is a paracompact space and Y is a locally convex topological vector space.

Theorem 1.3. *Let* X *be a paracompact space,* Y *a locally convex topological vector space,* $\varphi : X \to 2^Y$ *a mapping and* $\psi : X \to \mathcal{C}_c(Y)$ *a u.s.c. mapping such that* $\mathrm{Gr}\psi$ *is a* G_δ*-set of* $\mathrm{Gr}\varphi$*. Then there exists a continuous function* $f : X \times Y \to [0, 1]$ *such that* $\psi(x) = \{y \in \varphi(x) : f(x, y) = \max\{f(x, z) : z \in \varphi(x)\}\}$ *and the mapping* $f(x, \cdot) : Y \to \mathbf{R}$ *is quasi-concave for each* $x \in X$*.*

Proof of Theorem 1.3 is obtained in the next section. A key lemma is Lemma 2.2 below. In section 3, we obtain three applications of Lemma 2.2. The first one is to give another proof of Fan-Glicksberg's generalization of Kakutani's fixed point theorem ([7], [10]). The second one is to show Cellina's approximation theorem ([3], [4]) for paracompact domains and locally convex topological vector space ranges. The third one is to show that every u.s.c. compact- and convex-valued mapping from a metric space to a Banach space is σ-selectionable, which also answers Komiya's question in [14] affirmatively.

The closure of a subset S of a space X is denoted by $\mathrm{Cl}(S)$. A mapping $\varphi : X \to 2^Y$ is said to have an *open graph* if $\mathrm{Gr}\varphi$ is open in $X \times Y$. For undefined terminology, we refer to [6] and [11].

2. Proof of Theorem 1.3

Lemma 2.1. *Let* X *be a space,* Y *a locally convex topological vector space,* $\psi : X \to \mathcal{C}_c(Y)$ *a u.s.c. mapping and* O *an open subset of* $X \times Y$ *with* $\mathrm{Gr}\psi \subset O$*. Then for each* $x \in X$*, there exist a neighborhood* N *of* x *and a convex neighborhood* V *of the origin of* Y *such that* $\mathrm{Gr}\psi \cap (N \times Y) \subset N \times (\psi(x) + V) \subset N \times (\psi(x) + 2V) \subset O$*.*

Proof. Let $x \in X$. Since $\psi(x)$ is compact, there exist a neighborhood U of x and a convex neighborhood V of the origin such that $U \times (\psi(x) + 2V) \subset O$. Put $N = U \cap \psi^{\#}[\psi(x) + V]$. Then N and V are required neighborhoods. □

Lemma 2.2. *Let X be a paracompact space, Y a locally convex topological vector space, $\psi : X \to \mathcal{C}_c(Y)$ a u.s.c. mapping and $\{O_n : n \in \mathbf{N}\}$ is a decreasing sequence of open subsets of $X \times Y$ such that $\mathrm{Gr}\psi \subset \bigcap\{O_n : n \in \mathbf{N}\}$. Then there exists a continuous function $f : X \times Y \to [0,1]$ such that $f(\mathrm{Gr}\psi) = \{1\}$, $f(X \times Y \setminus O_n) \subset [0, 1 - 1/2^{n-1}]$ for each $n \in \mathbf{N}$ and the function $f(x, \cdot) : Y \to \mathbf{R}$ is quasi-concave for each $x \in X$.*

Proof. As in [13], our proof is based on the idea used in a proof of Urysohn's Lemma. Let $D = \{n/2^m : n, m \in \mathbf{N}\} \cap (0, 1]$. We first construct a family $\{\varphi_t : t \in D\}$ of mappings $\varphi_t : X \to 2^Y_c$ having open graphs such that $\mathrm{Gr}\psi \subset \mathrm{Gr}\varphi_s \subset \mathrm{Cl}(\mathrm{Gr}\varphi_s) \subset \mathrm{Gr}\varphi_t$ for each $s, t \in D$ with $s < t$ and $\mathrm{Gr}\varphi_t \subset O_n$ for each $t \in D$ and $n \in \mathbf{N}$ with $t < 1/2^{n-1}$ by induction on n.

Define $\varphi_1 : X \to 2^Y_c$ by putting $\varphi_1(x) = Y$ for each $x \in X$. Assume that $\{\varphi_t : t \in D,\ 1/2^{n-1} \le t\}$ has been obtained. By Lemma 2.1, for each $x \in X$, there exist a neighborhood N_x of x and a convex neighborhood V_x of the origin of Y such that $\mathrm{Gr}\psi \cap (N_x \times Y) \subset N_x \times (\psi(x) + V_x) \subset N_x \times (\psi(x) + 2V_x) \subset \mathrm{Gr}\psi_{1/2^{n-1}} \cap O_n$. Since X is paracompact, there exist locally finite open covers $\{W_\alpha : \alpha \in A\}$ and $\{U^{1/2^n}_\alpha : \alpha \in A\}$ of X such that the cover $\{\mathrm{Cl}(U^{1/2^n}_\alpha) : \alpha \in A\}$ refines $\{N_x : x \in X\}$ and $\mathrm{Cl}(W_\alpha) \subset U^{1/2^n}_\alpha$ for each $\alpha \in A$. By using the normality of X, take a family $\{\mathcal{U}_t : t \in D, 1/2^n < t < 1/2^{n-1}\}$ of locally finite open covers $\mathcal{U}_t = \{U^t_\alpha : \alpha \in A\}$ such that $\mathrm{Cl}(W_\alpha) \subset U^t_\alpha \subset \mathrm{Cl}(U^t_\alpha) \subset U^s_\alpha$ for each $\alpha \in A$ and each $s, t \in D$ with $1/2^n \le s < t < 1/2^{n-1}$. For each $\alpha \in A$, take $x_\alpha \in X$ so that $\mathrm{Cl}(U^{1/2^n}_\alpha) \subset N_{x_\alpha}$. Define a mapping $\varphi_t : X \to 2^Y$ by

$$\varphi_t(x) = \bigcap\{\psi(x_\alpha) + 2^n t V_{x_\alpha} : \alpha \in A, x \in \mathrm{Cl}(U^t_\alpha)\}$$

for each $t \in D$ with $1/2^n \le t < 1/2^{n-1}$. Then φ_t is convex-valued.

Let us show that $\mathrm{Gr}\psi \subset \mathrm{Gr}\varphi_s \subset \mathrm{Cl}(\mathrm{Gr}\varphi_s) \subset \mathrm{Gr}\varphi_t \subset \mathrm{Gr}\varphi_{1/2^{n-1}} \cap O_n$ for each $s, t \in D$ with $1/2^n \le s < t < 1/2^{n-1}$. The first, second and fourth inclusion is immediate from the definition of φ_t. To show the third one, let $s, t \in D$ with $1/2^n \le s < t < 1/2^{n-1}$ and take $(x, y) \in X \times Y \setminus \mathrm{Gr}\varphi_t$. Then there is $\alpha \in A$ such that $x \in \mathrm{Cl}(U^t_\alpha)$ and $y \notin \psi(x_\alpha) + 2^n t V_{x_\alpha}$. Since $\mathrm{Cl}(U^t_\alpha) \subset U^s_\alpha$, U^s_α is a neighborhood of x. If $(u, v) \in U^s_\alpha \times (y - 2^n(t - s)V_{x_\alpha})$, then $u \in \mathrm{Cl}(U^s_\alpha)$ and $v \notin \psi(x_\alpha) + 2^n s V_{x_\alpha}$, and hence $v \notin \varphi_t(u)$. Thus $(U^s_\alpha \times (y - 2^n(t - s)V_{x_\alpha})) \cap \mathrm{Gr}\varphi_s = \emptyset$, which proves $(x, y) \in X \times Y \setminus \mathrm{Cl}(\mathrm{Gr}\varphi_s)$. Therefore we have $\mathrm{Cl}(\mathrm{Gr}\varphi_s) \subset \mathrm{Gr}\varphi_t$.

To see that φ_t has an open graph, take $(x, y) \in \mathrm{Gr}\varphi_t$ and put $U = X \setminus \bigcup\{\mathrm{Cl}(U^t_\alpha) : x \notin \mathrm{Cl}(U^t_\alpha)\}$ and $V = \bigcap\{\psi(x_\alpha) + 2^n t V_{x_\alpha} : \alpha \in A, x \in \mathrm{Cl}(U^t_\alpha)\}$. Since $\{\mathrm{Cl}(U^t_\alpha) : \alpha \in A\}$ is locally finite, U and V are open neighborhoods of x and y, respectively. If $(u, v) \in U \times V$, then

$$\begin{aligned} v &\in \bigcap\{\psi(x_\alpha) + 2^n t V_{x_\alpha} : \alpha \in A, x \in \mathrm{Cl}(U^t_\alpha)\} \\ &\subset \bigcap\{\psi(x_\alpha) + 2^n t V_{x_\alpha} : \alpha \in A, u \in \mathrm{Cl}(U^t_\alpha)\} = \varphi_t(u). \end{aligned}$$

Hence φ_t has an open graph.

Thus, by induction on n, we obtain the desired family $\{\varphi_t : t \in D\}$. Define a function $g : X \times Y \to [0, 1]$ by $g(x, y) = \inf\{t \in D : (x, y) \in \mathrm{Gr}\varphi_t\}$. Then g is continuous and the set $\{y \in Y : f(x, y) \le s\} = \bigcap\{\varphi_t(x) : t > s\}$ is convex for each

$s \in \mathbf{R}$ and $x \in X$. Because $\mathrm{Gr}\psi \subset \bigcap\{\mathrm{Gr}\varphi_t : t \in D\}$ and $\bigcup\{\mathrm{Gr}\varphi_t : t \in D, 0 < t < 1/2^{n-1}\} \subset O_n$, we have $g(\mathrm{Gr}\psi) = \{0\}$ and $g(X \times Y \setminus O_n) \subset [1/2^{n-1}, 1]$ for each $n \in \mathbf{N}$. Hence, the function $f : X \times Y \to \mathbf{R}$ defined by $f(x, y) = 1 - g(x, y)$ for each $(x, y) \in X \times Y$ is the desired one. □

Proof of Theorem 1.3. Let X, Y, $\varphi : X \to 2^Y$ and $\psi : X \to \mathcal{C}_c(Y)$ be as in Theorem 1.3. Let $\{O_n : n \in \mathbf{N}\}$ be a decreasing sequence of open subsets of $X \times Y$ such that $\mathrm{Gr}\psi = \bigcap\{O_n : n \in \mathbf{N}\} \cap \mathrm{Gr}\varphi$. Then the mapping $f : X \times Y \to \mathbf{R}$ as in Lemma 2.2 is the desired function. □

Remark 2.3. A *convex metric space* (Y, d, W) ([20]) is a metric space (Y, d) together with a mapping $W : Y \times Y \times [0, 1] \to Y$ satisfying $d(z, W(x, y, \lambda)) \leq \lambda d(z, x) + (1 - \lambda)d(z, y)$ for each $x, y, z \in Y$ and $\lambda \in [0, 1]$. A subset C of a convex metric space (Y, d, W) is called *convex* if $W(x, y, \lambda) \in C$ for each $x, y \in C$ and $\lambda \in [0, 1]$. A convex metric space (Y, d, W) is said to have property (K) ([1]) if $d(W(x, y, \lambda), W(x', y', \lambda)) \leq \lambda d(x, x') + (1 - \lambda)d(y, y')$ for every $x, x', y, y' \in Y$ and $\lambda \in [0, 1]$. Aoyama [1, Theorem 3.5] extended Park-Komiya's theorem [17, Theorem 2] to a convex metric space Y with property (K). Concerning Theorem 1.3, we have the following: *Let X be a paracompact space, (Y, d, W) a convex metric space with property (K), $\varphi : X \to 2^Y$ a mapping and $\psi : X \to \mathcal{C}_c(Y)$ a u.s.c. mapping such that $\mathrm{Gr}\psi$ is a G_δ-set of $\mathrm{Gr}\varphi$. Then there exists a continuous function $f : X \times Y \to [0, 1]$ such that $\psi(x) = \{y \in \varphi(x) : f(x, y) = \max\{f(x, z) : z \in \varphi(x)\}\}$ and the mapping $f(x, \cdot) : Y \to \mathbf{R}$ is quasi-concave for each $x \in X$.* The proof is obtained by repeating the proofs of Lemma 2.2 and Theorem 1.3, and by replacing open neighborhoods of $\psi(x)$ such as "$\psi(x) + V_x$" with open neighborhoods "$B_d(\psi(x), \varepsilon_x)$" for some appropriate $\varepsilon_x > 0$, where $d(y, \psi(x)) = \inf\{d(y, z) : z \in \psi(x)\}$ and $B_d(\psi(x), \varepsilon_x) = \{y \in Y : d(y, \psi(x)) < \varepsilon_x\}$.

3. Applications

In this section, we show some applications of Lemma 2.2. The following proposition is immediate from Lemma 2.2.

Proposition 3.1. *Let X be a paracompact space, Y a locally convex topological vector space, $\psi : X \to \mathcal{C}_c(Y)$ a u.s.c. mapping and O is an open subset of $X \times Y$ such that $\mathrm{Gr}\psi \subset O$. Then there exists a continuous function $f : X \times Y \to [0, 1]$ such that $f(\mathrm{Gr}\psi) = \{1\}$, $f(X \times Y \setminus O) \subset \{0\}$ and the function $f(x, \cdot) : Y \to \mathbf{R}$ is quasi-concave for each $x \in X$. In particular, there exists a mapping $\varphi : X \to 2_c^Y$ having an open graph such that $\mathrm{Gr}\psi \subset \mathrm{Gr}\varphi \subset \mathrm{Cl}(\mathrm{Gr}\varphi) \subset O$.*

Applying the KKMF principle ([12], [8]), Fan [9, Theorem 1] proved the following minimax inequality.

Theorem 3.2 (K. Fan [9])**.** *Let X be a compact convex subset in a topological vector space. Let $f : X \times X \to \mathbf{R}$ be a function such that the function $f(\cdot, y) : X \to \mathbf{R}$ is lower semicontinuous for each $y \in Y$ and the function $f(x, \cdot) : X \to \mathbf{R}$ is quasi-concave for each $x \in X$. Then the inequality $\min_{x \in X} \sup_{y \in Y} f(x, y) \leq \sup_{x \in X} f(x, x)$ holds.*

From the Fan's minimax inequality and Proposition 3.1, we can derive Fan-Glicksberg's generalization of Kakutani's fixed point theorem ([7], [10]).

Theorem 3.3 (K. Fan [7], I. L. Glicksberg [10])**.** *Let C be a compact convex subset of a locally convex topological vector space X. Then every u.s.c. mapping $\psi : C \to \mathcal{C}_c(C)$ has a fixed point.*

Proof. Assume ψ does not have a fixed point. Then the closed subset $\Delta = \{(x,x) \in C \times X : x \in C\}$ of $C \times X$ does not meet $\mathrm{Gr}\,\psi$. By Proposition 3.1, there exists a continuous mapping $f : C \times X \to \mathbf{R}$ such that $f(\mathrm{Gr}\,\psi) = \{1\}$, $f(\Delta) = \{0\}$ and the function $f(c,\cdot) : X \to \mathbf{R}$ is quasi-concave for each $c \in C$. Then $\min_{c\in C} \sup_{x\in C} f(c,x) = 1 > 0 = \sup_{x\in C} f(x,x)$, which contradicts Theorem 3.2. □

Let $\mathrm{Gr}\,f$ denote the graph $\{(x,y) \in X \times Y : y = f(x)\}$ of a single-valued mapping $f : X \to Y$. For a mapping $\psi : X \to 2^Y$ and $S \subset X$, let $\psi(S) = \bigcup\{\psi(x) : x \in S\}$. For a subset S of a topological vector space Y, let $\mathrm{conv}\,S$ denote the convex hull of S. Cellina [3], [4] proved an approximate selection theorem for upper semicontinuous convex-valued mappings from metric spaces to metric locally convex topological vector spaces, and applied the theorem to Kakutani's fixed point theorem for Banach spaces. For upper semicontinuous compact- and convex-valued mappings, we can drop metrizability of both spaces as follows.

Theorem 3.4. *Let X be a paracompact space, Y a locally convex topological vector space, $\psi : X \to \mathcal{C}_c(Y)$ a u.s.c. mapping and O an open subset $X \times Y$ containing $\mathrm{Gr}\,\psi$. Then there exists a continuous mapping $f : X \to Y$ such that $\mathrm{Gr}\,f \subset O$ and $f(X) \subset \mathrm{conv}\,\psi(X)$.*

Note that an analogous approximation theorem was proved by Repovš, Semenov and Ščepin [18, Theorem 1.3]. A family $\{p_\alpha : \alpha \in A\}$ of continuous functions $p_\alpha : X \to [0,1]$ is called a *partition of unity* on X if $\sum_{\alpha\in A} p_\alpha(x) = 1$ for each $x \in X$. A partition of unity $\{p_\alpha : \alpha \in A\}$ on X is said to be *locally finite* if the cover $\{\{x \in X : p_\alpha(x) > 0\} : \alpha \in A\}$ of X is locally finite. For an open cover $\mathcal{U}$ of X, a partition of unity $\{p_\alpha : \alpha \in A\}$ on X is *subordinated to* $\mathcal{U}$ if the cover $\{\{x \in X : p_\alpha(x) > 0\} : \alpha \in A\}$ of X refines $\mathcal{U}$.

Proof of Theorem 3.4. By Proposition 3.1, there is a mapping $\varphi : X \to 2^Y_c$ having an open graph such that $\mathrm{Gr}\,\psi \subset \mathrm{Gr}\,\varphi \subset O$. Since the collection $\mathcal{U} = \{\varphi^{-1}[\{y\}] : y \in \psi(X)\}$ is an open cover of the paracompact space X, there exists a locally finite partition of unity $\{p_\alpha : \alpha \in A\}$ subordinated to $\mathcal{U}$. For each $\alpha \in A$, take $y_\alpha \in \psi(X)$ such that $\{x \in X : p_\alpha(x) > 0\} \subset \psi^{-1}[\{y_\alpha\}]$. Define a mapping $f : X \to Y$ by putting $f(x) = \sum_{\alpha\in A} p_\alpha(x) y_\alpha$ for each $x \in X$. Then f is continuous, $f(X) \subset \mathrm{conv}\,\psi(X)$ and $f(x) \in \varphi(x)$ for each $x \in X$. Thus $\mathrm{Gr}\,f \subset \mathrm{Gr}\,\varphi \subset O$. □

Remark 3.5. By applying the Schauder-Tychonoff fixed point theorem and Theorem 3.4, we can also obtain Theorem 3.3.

Finally, we consider σ-selectionability of u.s.c. mappings. The set of all non-empty closed convex subsets of a Banach space Y is denoted by $\mathcal{F}_c(Y)$. The following theorem is proved by Nepomnyashchiĭ [16] (see also [15, Theorem 1.2], [19, Theorem 1]).

Theorem 3.6 (G. M. Nepomnyashchiĭ [16]). *Let X be a paracompact space, Y a Banach space, $\varphi : X \to \mathcal{F}_c(Y)$ an l.s.c. mapping and $\psi : X \to \mathcal{C}_c(Y)$ a u.s.c. mapping. Then there exists a continuous mapping $\theta : X \to \mathcal{C}_c(Y)$ such that $\psi(x) \subset \theta(x) \subset \varphi(x)$ for each $x \in X$.*

Applying Theorem 3.6 and Proposition 3.1, we have the following.

Theorem 3.7. *Let X be a paracompact space, Y a Banach space and $\psi : X \to \mathcal{C}_c(Y)$ a u.s.c. mapping such that* $\mathrm{Gr}\psi$ *is a G_δ-set of $X \times Y$. Then ψ is σ-selectionable.*

Proof. There is a decreasing sequence $\{O_n : n \in \mathbf{N}\}$ of open subsets of $X \times Y$ such that $\mathrm{Gr}\psi = \bigcap_{n\in\mathbf{N}} O_n$. By Proposition 3.1, there exists a sequence $\{\theta_n : n \in \mathbf{N}\}$ of mappings $\theta_n : X \to 2^Y_c$ such that each θ_n has an open graph, $\mathrm{Gr}\psi \subset \mathrm{Gr}\theta_1 \subset \mathrm{Cl}(\mathrm{Gr}\theta_1) \subset O_1$ and $\mathrm{Gr}\psi \subset \mathrm{Gr}\theta_{n+1} \subset \mathrm{Cl}(\mathrm{Gr}\theta_{n+1}) \subset \mathrm{Gr}\theta_n \cap O_{n+1}$ for each $n \in \mathbf{N}$. We construct the required sequence $\{\varphi_n : n \in \mathbf{N}\}$ of continuous mappings $\varphi_n : X \to \mathcal{C}_c(Y)$ by induction on n. Since the mapping $\gamma_1 : X \to \mathcal{F}_c(Y)$ defined by $\gamma_1(x) = \mathrm{Cl}(\theta_1(x))$ for each $x \in X$ is l.s.c., by virtue of Theorem 3.6, there exists a continuous mapping $\varphi_1 : X \to \mathcal{C}_c(Y)$ such that $\psi(x) \subset \varphi_1(x) \subset \gamma_1(x)$ for each $x \in X$. Assume φ_n has been obtained. Then the mapping $\gamma_{n+1} : X \to \mathcal{F}_c(Y)$ defined by $\gamma_{n+1}(x) = \mathrm{Cl}(\varphi_n(x) \cap \theta_n(x))$ for each $x \in X$ is l.s.c. Thus, by virtue of Theorem 3.6 again, there exists a continuous mapping $\varphi_{n+1} : X \to \mathcal{C}_c(Y)$ such that $\psi(x) \subset \varphi_{n+1}(x) \subset \gamma_{n+1}(x)$ for each $x \in X$. Then $\{\varphi_n : n \in \mathbf{N}\}$ is the required sequence. □

Thus we have the following corollary which answers Komiya's question in [14] affirmatively.

Corollary 3.8. *Every u.s.c. compact-valued mapping from a metric space to a Banach space is σ-selectionable.*

Acknowledgment

The author would like to thank Professor Hidetoshi Komiya for introducing him to an inverse problem of the Berge maximum theorem.

References

[1] K. Aoyama, *Inverse of the Berge maximum theorem in convex metric spaces*, Sci. Math. Jpn. **58** (2003), 541–546.
[2] C. Berge, *Topological spaces, including a treatment of multi-valued functions, vector spaces and convexity*, translated from the French original by E. M. Patterson, reprint of the 1963 translation. Dover Publications, New York, 1997.
[3] A. Cellina, *Approximation of set valued functions and fixed point theorems*, Ann. Mat. Pura Appl. (4) **82** (1969), 17–24.
[4] A. Cellina, *A theorem on the approximation of compact multivalued mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **47** (1969), 429–433.
[5] G. Debreu, *Theory of value*, John Wiley & Sons, Inc., New York, 1959.
[6] R. Engelking, *General topology*, Heldermann Verlag, Berlin, 1989.
[7] K. Fan, *Fixed-point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. U. S. A. **38** (1952), 121–126.
[8] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann. **142** (1961), 305–310.

[9] K. Fan, *A minimax inequality and applications*, in Inequalities III (O. Shisha ed.), Academic Press, New York, 1972, 103–113.
[10] I. L. Glicksberg, *A further generalization of the Kakutani fixed theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. **3** (1952), 170–174.
[11] A. Granas and J. Dugundji, *Fixed point theory*, Springer-Verlag, New York, 2003.
[12] B. Knaster, C. Kuratowski and S. Mazurkiewicz, *Ein beweis des fixpunktsatszes fur n-dimensionale simplexes*, Fund. Math. **14** (1929), 132–137.
[13] H. Komiya, *Inverse of the Berge maximum theorem*, Economic Theory **9** (1997), 371–375.
[14] H. Komiya, *Inverse problems from economics and game theory*, Nonlinear analysis and convex analysis (Kyoto, 2001). Sūrikaisekikenkyūsho Kōkyūroku No. 1246 (2002), 45–49.
[15] E. Michael, *A theorem of Nepomnyashchii on continuous subset-selections*, Topology Appl. **142** (2004), 235–244.
[16] G. M. Nepomnyashchiĭ, *On the existence of intermediate continuous many-valued selections*, in Continuous functions on topological spaces (Russian), vol. XIII, Latv. Gos. Univ., Riga, 1986, pp. 111-122 (in Russian).
[17] S. Park and H. Komiya, *Another inverse of the Berge maximum theorem*, J. Nonlinear Convex Anal. **2** (2001), 105–109.
[18] D. Repovš, P. V. Semenov and E. V. Ščepin, *Approximation of upper semicontinuous maps on paracompact spaces*, Rocky Mountain J. Math. **28** (1998), 1089–1101.
[19] P. V. Semenov and E. V. Ščepin, *Selection approach to multivalued separation theorems*, Topol. Methods Nonlinear Anal. **14** (1999), 183–192.
[20] W. Takahashi, *A convexity in metric space and nonexpansive mappings*, Kōdai Math. Sem. Rep. **22** (1970), 142–149.

Manuscript received January 29, 2008
revised February 15, 2008

Takamitsu Yamauchi
Department of Mathematics, Shimane University, Matsue, 690-8504, Japan
Current address: Department of General Education, Kure National College of Technology, Kure, 737-8506, Japan
E-mail address: yamauchi@kure-nct.ac.jp