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APPLICATIONS OF FIXED POINT THEOREMS ON ALMOST CONVEX SETS

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Abstract. Our fixed point theorems on multimaps in the class $\mathfrak{B}$ defined on almost convex subsets are applied to deduce extension theorems of monotone sets, intersection theorems, minimax theorems, equilibrium theorems, and quasi-variational inequalities. Consequently, our new results generalize well-known works of von Neumann, Nash, Debreu, Fan, Browder, and others.

1. Introduction

The fixed point theory of multimaps in topological vector spaces has numerous applications in many fields in mathematical sciences. Recently, in [15], we obtained new fixed point theorems for the 'better' admissible class $\mathfrak{B}$ defined on *almost convex* subsets of topological vector spaces. It is essential to note that a multimap in the class has an almost fixed point whenever its range is *Klee approximable* as in [13].

In the present paper, our fixed point theorems are applied to deduce extension theorems of monotone sets, intersection theorems, minimax theorems, equilibrium theorems, and quasi-variational inequalities. Consequently, our new results generalize well-known works of von Neumann [16,17], Nash [7], Debreu [3], Fan [5,6], Debrunner and Flor [4], Browder [1], and others; for the related history, see the reference [9].

Section 2 deals with preliminaries on the better admissible classes $\mathfrak{B}$ for which we give basic fixed point theorems and a coincidence theorem in [15]. The coincidence theorem is applied in Section 3 to generalize the extension theorem of nonotone sets due to Debrunner and Flor [4]. Section 4 deals with generalizations of the von Neumann intersection lemma [17]. In Section 5, we apply such intersection theorems to deal with generalizations of the Nash equilibrium theorem [7], which are applied to various forms of the von Neumann minimax theorem [16] in Section 6. Finally, Section 7 deals with quasi-variational inequality problem as applications of our fixed point theorems.

2. Better admissible maps

All topological spaces are assumed to be Hausdorff. A t.v.s. means a topological vector space and $\mathcal{V}$ denotes a fundamental system of neighborhoods of the origin 0 of a t.v.s. E. We follow the terminology and notations in our previous work [15].

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Recall that a nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For topological spaces X and Y, a map $F : X \multimap Y$ is called a *Kakutani map* whenever Y is a subset of a t.v.s. and F is u.s.c. with nonempty compact convex values; and an *acyclic map* whenever F is u.s.c. with compact acyclic values.

Let $\mathbb{V}(X,Y)$ be the class of all acyclic maps $F : X \multimap Y$, and $\mathbb{V}_c(X,Y)$ all finite compositions of acyclic maps, where the intermediate spaces are arbitrary topological spaces.

A *polytope* P in a subset X of a t.v.s. E is a nonempty compact convex subset of X contained in a finite dimensional subspace of E.

Recall that a nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee) provided that, for every nonempty compact subset K of X and every $V \in \mathcal{V}$, there exists a continuous function $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E. Examples of admissible subsets can be seen in [8,9,13].

A nonempty subset K of E is said to be *Klee approximable* if for any $V \in \mathcal{V}$, there exists a continuous function $h : K \to E$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a polytope of E. Especially, for a subset X of E, K is said to be Klee approximable *into* X whenever the range $h(K)$ is contained in a polytope in X.

Examples of Klee approximable sets can be seen in [15].

We define a class $\mathfrak{B}$ of maps from a subset X of a t.v.s. E into a topological space Y as follows [15]:

$F \in \mathfrak{B}(X,Y) \iff F : X \multimap Y$ is a map such that, for each polytope P in X and for any continuous function $f : F(P) \to P$, the composition $f(F|_P) : P \multimap P$ has a fixed point.

We call $\mathfrak{B}$ the 'better' admissible class. Recently it is known that any u.s.c. map with compact values having *trivial shape* (that is, contractible in each neighborhood) belongs to $\mathfrak{B}(X,Y)$. Note that the class $\mathfrak{B}^p$ in [13,15] should be replaced by $\mathfrak{B}$.

The following results appeared in our previous work [15]:

Theorem 2.1 ([15, Corollary 2.3]). *Let X be a subset of a t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed map. If $F(X)$ is Klee approximable into X, then F has a fixed point.*

Theorem 2.2 ([15, Corollary 3.5.]). *Let X be an almost convex admissible subset of a t.v.s. E and $F \in \mathfrak{B}(X,X)$ a compact closed map. Then F has a fixed point.*

For a subset X of a t.v.s. E and a topological space Y, we define a class of multimaps as follows:

$T \in \mathbb{M}^*(Y,X) \iff T : Y \multimap X$ is a map such that $T|_K$ has a continuous selection $s : K \to X$ for each nonempty compact subset K of Y such that $s(K) \subset P$ for some polytope P of X.

Theorem 2.3 ([15, Theorem 4.5]). *Let X be a convex subset of a t.v.s. E and Y a topological space. Let $F \in \mathfrak{B}(X,Y)$ be a compact map and $T \in \mathbb{M}^*(Y,X)$. Then F and T have a coincidence point.*

Theorem 2.4 ([15, Theorem 6.1]). *Let X be an almost convex dense subset of an admissible subset Y of a t.v.s. E. Let $G : Y \multimap Y$ be a compact closed map such that $G(x)$ is acyclic (resp., has trivial shape) for all $x \in X$. Then G has a fixed point.*

3. Extensions of monotone sets

In this section, generalizations of the extension theorems on monotone sets due to Debrunner and Flor [4] and in [1,6,12] are obtained. In fact, Browder's extension theorem [1] involving Kakutani multimaps is extended to the one involving a large class of 'better' admissible maps.

Given two t.v.s. E and F, let $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ be a bilinear pairing which is continuous on compact subsets of $F \times E$. This assumption is quite natural in most applications, since the natural pairing between a locally convex t.v.s. E and its dual space E^* equipped with the strong topology enjoys this property.

A subset $M \subset E \times F$ is said to be *monotone* if for any two points (u, w) and (u', w') in M, we have $\langle w - w', u - u' \rangle \geq 0$; see Debrunner and Flor [4].

Browder [1, Theorem 8] obtained the following extension theorem of monotone sets:

Lemma 3.1 ([1]). *Let K be a compact convex subset of a t.v.s. E, and F a t.v.s. with a bilinear pairing $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ which is continuous on compact subsets of $F \times E$. Let $f : K \to F$ be continuous and M a monotone subset of $K \times F$. Then there exists a $u_0 \in K$ such that*

$$\langle f(u_0) - w, u_0 - u \rangle \geq 0 \quad \textit{for all } (u, w) \in M,$$

or equivalently, the set $M \cup \{(u_0, f(u_0))\}$ remains monotone.

This result sharpens corresponding result of Debrunner and Flor [4] for E locally convex and of Fan [6, Theorem 12] for F locally convex and quasi-complete.

We deduce the following equilibrium existence theorem from Theorem 2.3:

Theorem 3.2. *Let K be a compact convex subset of a t.v.s. E, K_1 a compact subset of a t.v.s. F, $T \in \mathfrak{B}(K, K_1)$ with closed graph, and $M \subset E \times F$. Let $\Phi : E \times F \to \mathbf{R} \cup \{-\infty\}$ be a function such that*

(1) *Φ is u.s.c. on compact subsets of $E \times F$;*
(2) *for each $x \in E$, $\Phi(x, \cdot)$ is l.s.c. on compact subsets of F;*
(3) *for each $w \in F$, $F(\cdot, w)$ is quasiconcave.*

Suppose that for each $y \in K_1$, there exists an $x \in K$ such that

$$\Phi(x - u, y - w) \geq 0 \quad \textit{for all } (u, w) \in M.$$

Then there exist a $u_0 \in K$ and a $w_0 \in T(u_0)$ such that

$$\Phi(u_0 - u, w_0 - w) \geq 0 \quad \textit{for all } (u, w) \in M.$$

Proof. For any $\varepsilon > 0$ and any nonempty finite subset N of M, we set

$$H_{(\varepsilon,N)} = \{(u_0, w_0) \in \mathrm{Gr}(T) \mid \Phi(u_0 - u, w_0 - w) \geq -\varepsilon \text{ for all } (u, w) \in N\}$$

and

$$H_0 := \{(u_0, w_0) \in \mathrm{Gr}(T) \mid \Phi(u_0 - u, w_0 - w) \geq 0 \text{ for all } (u, w) \in M\}$$

$$= \bigcap \{H_{(\varepsilon,N)} \mid \varepsilon > 0 \text{ and } N \text{ is a finite subset of } M\}.$$

Then we have to show $H_0 \neq \emptyset$.

By (1), each $H_{(\varepsilon,N)}$ is a closed subset of $\mathrm{Gr}(T)$. The intersection of each finite family of such sets is also a set of the form $H_{(\varepsilon',N')}$ for some $\varepsilon' > 0$ and a finite subset N' of M. Therefore, in order to show $H_0 \neq \emptyset$, it suffices to show that each $H_{(\varepsilon,N)}$ is nonempty.

Choose a given $\varepsilon > 0$ and a nonempty finite subset N of M. Define a map $S : K \multimap K_1$ by

$$S(x) := \{y \in K_1 \mid \Phi(x-u, y-w) > -\varepsilon,\ (u,w) \in N\}$$

for $x \in X$. Then $S(x)$ is open in K_1 by (2). Moreover,

$$S^-(y) = \{x \in K \mid \Phi(x-u, y-w) > -\varepsilon,\ (u,w) \in N\}$$

is nonempty by hypothesis and convex by (3). Therefore, by [15, Lemma 2.4], we have $S^- \in \mathbb{M}^*(K_1, K)$.

Now we apply Theorem 2.3. Then there exists a $(u_0, w_0) \in \mathrm{Gr}(T)$ such that $w_0 \in S(u_0)$; that is,

$$\Phi(u_0 - u, w_0 - w) > -\varepsilon \quad \text{for all } (u,w) \in N.$$

Therefore, $H_{(\varepsilon,N)}$ is nonempty. This completes our proof. □

Remarks. 1. In Theorem 3.2, instead of $T \in \mathfrak{B}(K, K_1)$ with closed graph, we can adopt $T \in \mathfrak{A}_c^\kappa(K, K_1)$ without affecting its conclusion.

2. In Theorem 3.2, since T has closed graph and K_1 is compact, T itself is actually u.s.c. with compact values.

3. For the subclass $\mathbb{C}$ of $\mathfrak{B}$, Theorem 3.2 reduces to Fan [6, Theorem 11], who assumed that F is locally convex and other restrictions.

4. For the subclass $\mathbb{K}$ of $\mathfrak{B}$, Theorem 3.2 reduces to Browder [1, Theorem 9], where F is locally convex.

The following is our theorem on extensions of monotone sets:

Theorem 3.3. *Let E be a t.v.s., F a t.v.s. with a bilinear pairing $\langle\ ,\ \rangle : F \times E \to \mathbf{R}$ which is continuous on compact subsets of $F \times E$, K a compact convex subset of E, and K_1 a compact subset of F. Let $T \in \mathfrak{B}(K, K_1)$ have closed graph and M a monotone subset of $K \times F$. Then there exist a $u_0 \in K$ and a $w_0 \in T(u_0)$ such that*

$$\langle w_0 - w, u_0 - u\rangle \geq 0 \quad \textit{for all} \quad (u,w) \in M.$$

Proof. We put $\Phi(x,w) = \langle w, x\rangle$ for $(x,w) \in E \times F$. Then Φ satisfies conditions (1)–(3) in Theorem 3.2. By Theorem 3.2, it suffices to show that for each $y \in K_1$, there exists an $x \in K$ such that

$$\langle y - w, x - u\rangle \geq 0 \quad \text{for all } (u,w) \in M.$$

Now, we define $f : K \to K_1$ by

$$f(v) = y \quad \text{for all } v \in K.$$

By applying Lemma 3.1 to f, such an $x \in K$ exists. This completes our proof. □

Remarks. 1. In Theorem 3.3, we can replace $T \in \mathfrak{B}(K, K_1)$ with closed graph by $T \in \mathfrak{A}_c^\kappa(K, K_1)$.

2. For the subclass $\mathbb{C}$ of $\mathfrak{B}$, Theorem 3.3 reduces to Browder [1, Theorem 8] or Lemma 3.1.

3. Even for the subclass $\mathbb{K}$ of $\mathfrak{B}$, Theorem 3.3 improves Browder [1, Theorem 9], where F is assumed to be locally convex.

From now on, we are mainly concerned with maps with acyclic values or values of trivial shape. Other types of appropriate values can also be adopted if necessary. Those results in the following sections have originated from the corresponding ones for Kakutani maps.

4. The von Neumann type intersection theorems

In this section, we deduce some collectively fixed point theorems for families of maps and, then, various von Neumann type intersection theorems.

Let $\{X_i\}_{i\in I}$ be a family of nonempty sets, and let $i \in I$ be fixed. Let

$$X := \prod_{j\in I} X_j \quad \text{and} \quad X^i := \prod_{j\in I\setminus\{i\}} X_j.$$

If $x^i \in X^i$ and $j \in I \setminus \{i\}$, let x^i_j denote the jth coordinate of x^i. If $x^i \in X^i$ and $x_i \in X_i$, let $[x^i, x_i] \in X$ be defined as follows: its ith coordinate is x_i and, for $j \neq i$, the jth coordinate is x^i_j. Therefore, any $x \in X$ can be expressed as $x = [x^i, x_i]$ for any $i \in I$, where x^i denotes the projection of x onto X^i.

For $A \subset X$, $x^i \in X^i$, and $x_i \in X_i$, let

$$A(x^i) := \{y_i \in X_i \mid [x^i, y_i] \in A\} \quad \text{and} \quad A(x_i) := \{y^i \in X^i \mid [y^i, x_i] \in A\}.$$

Theorem 4.1. *Let $\{E_i\}_{i=1}^n$ be a family of t.v.s. For each i, let X_i be a subset of E_i, K_i a nonempty compact subset of X_i, and $F_i : X \multimap K_i$ a closed map with acyclic values* (*resp., values of trivial shape*)*. If $K := \prod_{i=1}^n K_i$ is Klee approximable into X, then there exists an $\overline{x} = (\overline{x_i})_{i=1}^n \in X$ such that $\overline{x_i} \in F_i(\overline{x})$ for each i.*

Proof. Define $F : X \multimap K$ by $F(x) := \prod_{i=1}^n F_i(x)$ for each $x = (x_i)_{i=1}^n \in X$. Then it can be checked that F is a compact closed map having acyclic values (resp., values of trivial shape). Since $F(X) \subset K$ is Klee approximable into X, by Theorem 2.1, F has a fixed point $\overline{x} \in X$. This completes our proof. □

Remark. Recall that K is Klee approximable into X whenever one of the following holds:

(1) Each X_i is convex and X is admissible (see [10, Theorem 1]).
(2) Each X_i is almost convex and X is admissible.
(3) Each X_i is an almost convex dense subset of an admissible subset (see [15, Lemma 3.1]).
(4) Each K_i is Klee approximable into X_i.

From Theorem 4.1, we obtain the following von Neumann type intersection theorem:

Theorem 4.2. *Let $\{X_i\}_{i=1}^n$ be a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X such that $A_i(x^i)$ is an acyclic subset of K_i for each $x^i \in X^i$, where $1 \le i \le n$. If X is an almost convex admissible subset of E, then $\bigcap_{j=1}^n A_j \neq \emptyset$.*

Proof. We use Theorem 4.1 with $F_i : X \multimap K_i$ defined by $F_i(x) := A_i(x^i)$ for $x \in X$. Then, for each $x \in X$, we have

$$(x, y) \in \mathrm{Gr}(F_i) \iff (x_i, x^i) \in X_i \times X^i \text{ and } y \in A_i(x^i) \subset K_i$$
$$\iff (x_i, x^i, y) \in X_i \times (A_i \cap (X^i \times K_i)),$$

which implies that $\mathrm{Gr}(F_i)$ is closed in $X \times K_i$. Hence, each F_i is a compact closed map with acyclic values; that is, F_i is an acyclic map. Therefore, by Theorem 4.1, there exists an $\hat{x} \in K$ such that $\hat{x}_i \in F_i(\hat{x})$ for all i. Since $\hat{x}_i \in K_i \subset X_i$, we have $\hat{x} = [\hat{x}^i, \hat{x}_i] \in A_i$ for all i. This completes our proof. □

Similarly, we can obtain a more general result than Theorem 4.2 as follows:

Theorem 4.2′. *Let I be any index set, $\{X_i\}_{i\in I}$ a family of sets, each in a t.v.s. E_i, K_i a nonempty compact subset of X_i, and A_i a closed subset of X for each $i \in I$. Suppose that for each $x^i \in X^i$, $A_i(x^i)$ is a convex subset of K_i except a finite number of i's for which $A_i(x^i)$ is an acyclic subset of K_i. If X is an almost convex admissible subset of E, then $\bigcap_{j\in I} A_j \neq \emptyset$.*

Remark. If $I = \{1, 2\}$, E_i are Euclidean, $X_i = K_i$, and $A_i(x^i)$ are nonempty and convex, then Theorem 4.2 or 4.2′ reduces to the intersection lemma of von Neumann [17].

We have another intersection theorem:

Theorem 4.3. *Let X_0 be a topological space and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For each $i = 0, 1, 2, \cdots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n and $F_i \in \mathbb{V}_c(X^i, X_i)$. If K^0 is Klee approximable into X^0, then $\bigcap_{i=0}^n \mathrm{Gr}(F_i) \neq \emptyset$.*

Proof. For each $i \in \mathbb{Z}_{n+1}$, define a map $V_i : X^i \multimap X^{i+1}$ by letting

$$V_i(x^i) := F_i(x^i) \times \prod_{j\in\mathbb{Z}_{n+1}\setminus\{i,i+1\}} \{x_j^i\} \quad \text{for } x^i \in X^i.$$

Then $V_i \in \mathbb{V}_c(X^i, X^{i+1})$ for each $i \in \mathbb{Z}_{n+1}$. Hence, the composite map $V : X^0 \multimap X^0$ defined by $V := V_n V_{n-1} \cdots V_0$ belongs to $\mathbb{V}_c(X^0, X^0)$.

(1) We claim that V is compact. In fact, for each $i = 0, 1, 2, \cdots, n-1$, K_i is a compact subset satisfying $F_i(X^i) \subset K_i \subset X_i$. Note that

$$V_0(X^0) \subset K_0 \times X_2 \times \cdots \times X_n,$$
$$V_1 V_0(X^0) \subset K_0 \times K_1 \times X_3 \times \cdots \times X_n,$$

and finally,

$$V_{n-1} V_{n-2} \cdots V_0(X^0) \subset K_0 \times K_1 \times \cdots \times K_{n-1}.$$

Hence, $V(X^0)$ is contained in the compact set $V_n(K_0 \times K_1 \times \cdots \times K_{n-1})$. Thus V is compact.

(2) Note that $V(X^0) \subset K^0$ is Klee approximable into X^0.

Therefore, by Theorem 2.1, $V \in \mathbb{V}_c(X^0, X^0)$ has a fixed point $x^0 \in V(x^0)$. Hence, there exists $x^1 \in X^1, \cdots, x^n \in X^n$ such that $x^{i+1} \in V_i(x^i)$ for each $i \in \mathbb{Z}_{n+1}$, which implies

$$x_i^{i+1} \in F_i(x^i) \quad \text{for each} \quad i \in \mathbb{Z}_{n+1}$$

and

$$x_j^{i+1} = x_j^i \quad \text{for each} \quad j \in \mathbb{Z}_{n+1} \setminus \{i, i+1\}.$$

From this, it follows that $x_j^i = x_j^k$ for any $i, j, k \in \mathbb{Z}_{n+1}$ with $j \neq i, j \neq k$. Therefore, $[x^i, x_i^{i+1}] = [x^k, x_k^{k+1}]$ for any $i, k \in \mathbb{Z}_{n+1}$. Let us denote by x the point of X given by $x := [x^i, x_i^{i+1}]$ for any $i \in \mathbb{Z}_{n+1}$. Since $x_i^{i+1} \in F_i(x^i)$, we have $x \in \mathrm{Gr}(F_i)$ for each $i \in \mathbb{Z}_{n+1}$ and hence $\bigcap_{i=0}^n \mathrm{Gr}(F_i) \neq \emptyset$. This completes our proof. □

Remarks. 1. In case when each X_i is convex for $i \geq 1$ and X^0 is admissible in E^0, Theorem 4.3 reduces to [11, Theorem 4].

2. Particular forms of Theorem 4.3 were given by von Neumann, Fan, Lassonde, Chang, and Park; see [11]. The following is one of them:

Corollary 4.4. *Let X be a topological space, Y a subset in a t.v.s. E, and $F \in \mathbb{V}_c(X, Y)$ and $G \in \mathbb{V}_c(Y, X)$. If F is compact and $F(X)$ is Klee approximable into Y, then* $\mathrm{Gr}(F) \cap \mathrm{Gr}(G) \neq \emptyset$.

From Corollary 4.4, we have the following:

Corollary 4.5. *Let X be a topological space and Y a compact subset of a t.v.s. E. Let A and B be two closed subsets of $X \times Y$ such that*

(1) *for each $x \in X$, $A(x) := \{y \in Y \mid (x, y) \in A\}$ is acyclic; and*
(2) *for each $y \in Y$, $B(y) := \{x \in X \mid (x, y) \in B\}$ is acyclic.*

If $A(X) := \bigcup\{A(x) \mid x \in X\}$ is Klee approximable into Y, then $A \cap B \neq \emptyset$.

Remarks. 1. If Y is an admissible, compact, and almost convex subset of E, then $A(X)$ is Klee approximable into Y. Especially, for the particular case when X is compact and Y is convex, Corollary 4.5 was obtained in [10].

2. For other particular forms of Corollary 4.5, see [10].

5. The Nash type equilibrium theorems

From Theorem 4.3, we deduce the following generalized form of the quasi-equilibrium theorem or the social equilibrium existence theorem in the sense of Debreu [3]:

Theorem 5.1. *Let X_0 be a topological space, and $\{X_i\}_{i=1}^n$ a family of sets, each in a t.v.s. E_i. For $i = 0, 1, \cdots, n$, let K_i be a nonempty subset of X_i which is compact except possibly K_n, $S_i : X^i \multimap K_i$ be a closed map with compact values, and $f_i, g_i : X = X^i \times X_i \to \mathbf{R}$ u.s.c. real functions.*

Suppose that for each $i = 0, 1, \cdots, n$,

(i) *$g_i(x) \leq f_i(x)$ for each $x \in X$;*

(ii) *the real function* $M_i : X^i \to \mathbf{R}$ *defined by*

$$M_i(x^i) := \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad for \quad x^i \in X^i$$

is l.s.c.; and

(iii) *for each* $x^i \in X^i$, *the set*

$$\{y_i \in S_i(x^i) \mid f_i[x^i, y_i] \geq M_i(x^i)\}$$

is acyclic.

If K^0 *is Klee approximable into* X^0 *and if* S_n *is u.s.c., then there exists an equilibrium point* $\hat{x} \in X$*; that is,*

$$\hat{x}_i \in S_i(\hat{x}^i) \quad and \quad f_i(\hat{x}^i, \hat{x}_i) \geq \max_{y_i \in S_i(x^i)} g_i[x^i, y_i] \quad for\ each \quad i \in \mathbb{Z}_{n+1}.$$

Proof. For each $i \in \mathbb{Z}_{n+1}$, define a map $T_i : X^i \multimap X_i$ by

$$T_i(x^i) := \{y_i \in S_i(x^i) \mid f_i[x^i, y_i] \geq M_i(x^i)\}$$

for $x^i \in X^i$. Note that each $T_i(x^i)$ is nonempty by (ii) since $S_i(x^i)$ is compact and $g_i[x^i, \cdot]$ is u.s.c. on $S_i(x^i)$. We show that $\mathrm{Gr}(T_i)$ is closed in $X^i \times \overline{S_i(X^i)}$. In fact, let $[x^i_\alpha, y^\alpha_i] \in \mathrm{Gr}(T_i)$ and $[x^i_\alpha, y^\alpha_i] \to [x^i, y_i]$. Then

$$f_i[x^i, y_i] \geq \overline{\lim_\alpha} f_i[x^i_\alpha, y^\alpha_i] \geq \overline{\lim_\alpha}\ M_i(x^i_\alpha) \geq \underline{\lim_\alpha} M_i(x^i_\alpha) \geq M_i(x^i)$$

and, since $\mathrm{Gr}(S_i)$ is closed in $X^i \times \overline{S_i(X^i)}$, $y^\alpha_i \in S_i(x^i_\alpha)$ implies $y_i \in S_i(x^i)$. Hence, $[x^i, y_i] \in \mathrm{Gr}(T_i)$. Therefore, all T_i are closed.

Since T_i is compact for all $i \neq n$, each T_i is u.s.c. for $i \neq n$ by (iii). Moreover, S_n is u.s.c. with compact values by assumption and T_n is closed, $T_n = S_n \cap T_n$ is u.s.c. Hence we have $T_i \in \mathbb{V}(X^i, X_i)$. Therefore, by Theorem 4.3, there exists an $\hat{x} \in \bigcap_{i=0}^n \mathrm{Gr}(T_i)$; that is, $\hat{x}_i \in T_i(\hat{x}^i)$ for all $i \in \mathbb{Z}_{n+1}$. This completes our proof. □

Remarks. 1. For particular forms of Theorem 5.1, see [10,11].

2. If S_i are u.s.c., by Berge's theorem, M_i is automatically u.s.c. since g_i is u.s.c. If S_i and g_i are continuous, condition (ii) holds immediately by Berge's theorem, and hence, each M_i is continuous; see [11].

Therefore, from Theorem 5.1, we have the following particular form:

Theorem 5.2. *Let* X_0 *be a topological space, and* $\{X_i\}_{i=1}^n$ *a family of sets, each in a t.v.s.* E_i*. For* $i = 0, 1, \cdots, n$*, let* K_i *be a nonempty subset of* X_i *which is compact except possibly* K_n*,* $S_i : X^i \multimap K_i$ *be a continuous multimap with compact values, and* $f_i : X = X^i \times X_i \to \mathbf{R}$ *a continuous real function.*

Suppose that for each $i = 0, 1, \cdots, n$*, the following holds:*

(0) *for each* $x^i \in X^i$ *and each* $\alpha \in \mathbf{R}$*, the set*

$$\{x_i \in S_i(x^i) \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

If K^0 *is Klee approximable into* X^0*, there exists an equilibrium point* $\hat{x} \in X$*.*

Remarks. 1. If each X_i is convex and if X is admissible in E, then Theorem 5.2 reduces to [11, Theorem 6].

2. For other particular forms of Theorems 5.1 and 5.2, see [10,11].

The following generalizes the Nash theorem:

Corollary 5.3. *Let X_0 be a compact topological space, and $\{X_i\}_{i=1}^n$ a family of convex sets, each in a t.v.s. E_i, such that each X_i is compact except X_n. For $i = 0, 1, \cdots, n$, let $f_i : X = X^i \times X_i \to \mathbf{R}$ be a continuous real function such that*

(1) *for each $x^i \in X^i$ and each $\alpha \in \mathbf{R}$, the set*

$$\{x_i \in X_i \mid f_i[x^i, x_i] \geq \alpha\}$$

is empty or acyclic.

If X^0 is admissible, then there exists an equilibrium point $\hat{x} \in X$; that is,

$$f_i(\hat{x}) = \max_{y_i \in X_i} f_i[\hat{x}^i, y_i] \quad for\ all \quad i \in \mathbb{Z}_{n+1}.$$

Proof. Apply Theorem 5.2 with $S_i(x) := X_i$ for $x \in X$ and $K_i := X_i$ for each i. Then we have the conclusion. □

Remarks. 1. This slightly extends [10, Theorem 7].

2. If all X_i are compact convex subsets of Euclidean spaces and if $x_i \mapsto f_i[x^i, x_i]$ is quasiconcave for each $x^i \in X^i$, then Corollary 5.3 reduces to Nash [7, Theorem].

6. The von Neumann type minimax theorems

From Corollaries 4.5 or 5.3, we have the following von Neumann type minimax theorem:

Theorem 6.1. *Let X be a compact space and Y an admissible compact convex subset of a t.v.s., and $f : X \times Y \to \mathbf{R}$ a continuous real function. Suppose that for each $x_0 \in X$ and $y_0 \in Y$, the sets*

$$\{x \in X \mid f(x, y_0) = \max_{\zeta \in X} f(\zeta, y_0)\}$$

and

$$\{y \in Y \mid f(x_0, y) = \min_{\eta \in Y} f(x_0, \eta)\}$$

are acyclic. Then

(1) *f has a saddle point $(x_0, y_0) \in X \times Y$; that is,*

$$\min_{\eta \in Y} f(x_0, y) = f(x_0, y_0) = \max_{\zeta \in X} f(\zeta, y_0).$$

(2) *We have the minimax inequality*

$$\max_{x \in X} \min_{y \in Y} f(x, y) = \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. (1) Note that a saddle point is a particular case of an equilibrium point for two agents ($i = 0, 1$) in Corollary 5.3 for $X_0 = X, X_1 = Y$ and $f_0(x, y) = f(x, y), f_1(x, y) = -f(x, y)$.

The existence of a saddle point can be also shown by Corollary 4.5 as follows: Let

$$S(x) := \{y \in Y \mid f(x,y) = \min_{\eta \in Y} f(x,\eta)\}$$

and

$$T(y) := \{x \in X \mid f(x,y) = \max_{\zeta \in X} f(\zeta,y)\}.$$

Since f is continuous and X, Y are compact, each $S(x)$ and $T(y)$ are nonempty and closed for all $x \in X$ and $y \in Y$. Moreover, by Berge's theorem, $S : X \multimap Y$ and $T : Y \multimap X$ are u.s.c. with closed values. Therefore $A := \mathrm{Gr}(S)$ and $B := \mathrm{Gr}(T^-)$ are closed subsets in $X \times Y$. Moreover, $A(X) = S(X)$ is a compact subset of an admissible convex set Y, and hence, by Corollary 3.2 of [40], it is Klee approximable into Y. Therefore, by Corollary 4.6, there exists an $(x_0, y_0) \in A \cap B$; that is,

$$\max_{x \in X} f(x, y_0) = f(x_0, y_0) = \min_{y \in Y} f(x_0, y).$$

(2) This implies

$$\min_{y \in Y} \max_{x \in X} f(x,y) \le \max_{x \in X} f(x, y_0) = f(x_0, y_0) = \min_{y \in Y} f(x_0, y) \le \max_{x \in X} \min_{y \in Y} f(x,y).$$

On the other hand, we clearly have

$$\min_{y \in Y} \max_{x \in X} f(x,y) \ge \max_{x \in X} \min_{y \in Y} f(x,y).$$

This completes our proof. □

Remarks. 1. Theorem 6.1 includes [10, Theorem 4] and [11, Corollary 6.2], where some particular forms were noted.

2. For Euclidean spaces or locally convex t.v.s., if acyclicity is replaced by convexity, then Theorem 5.1 reduces to the von Neumann minimax theorem [16] or Fan [5, Theorem 3], resp.

The following generalization of the von Neumann minimax theorem is a simple consequence of Corollary 4.6 or Theorem 6.1:

Theorem 6.2. *Let X, Y, and f be the same as in Theorem 6.1. Suppose that*

(1) *for every $x \in X$ and $\alpha \in \mathbf{R}$, $\{y \in Y \mid f(x,y) \le \alpha\}$ is acyclic; and*

(2) *for every $y \in Y$ and $\beta \in \mathbf{R}$, $\{x \in X \mid f(x,y) \ge \beta\}$ is acyclic.*

Then we have

$$\max_{x \in X} \min_{y \in Y} f(x,y) = \min_{y \in Y} \max_{x \in X} f(x,y).$$

7. Quasi-Variational Inequalities

Let E be a t.v.s., $X \subset E$ and $x \in E$. The *inward set* of X at x (due to Halpern; see [9]) is defined by

$$I_X(x) = x + \bigcup_{r>0} r(X - x).$$

In this section, we show that our fixed point theorems can be applied to various types of quasi-variational inequalities on *almost convex sets.* For example, from Theorem 2.2, we have the following main result of this section:

Theorem 7.1. *Let X be an almost convex admissible subset of a t.v.s. E, $S : X \multimap X$ a compact map, F a t.v.s., $T \in \mathbb{V}(X, F)$ a compact map, and C an almost convex admissible subset of F containing $\overline{T(X)}$, and $\phi : X \times C \times X \to \overline{\mathbf{R}}$ a u.s.c. function. Suppose that*

(1) *$\phi(x, y, x) \leq 0$ for all $(x, y) \in X \times C$; and*
(2) *the map $M : X \times C \to X$ defined by*

$$M(x, y) = \{u \in S(x) \mid \phi(x, y, u) = \max_{s \in S(x)} \phi(x, y, s)\} \text{ for } (x, y) \in X \times C$$

belongs to $\mathbb{V}(X \times C, X)$.

Then there exist an $\overline{x} \in S(\overline{x})$ and a $\overline{y} \in T(\overline{x})$ such that

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \quad \text{for all} \quad x \in S(\overline{x}).$$

Further, if $\phi(x, y, x) = 0$ for all $(x, y) \in X \times C$, $S(\overline{x})$ is convex, and $x \mapsto \phi(\overline{x}, \overline{y}, x)$ is concave and l.s.c., then

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \text{ for all } x \in \overline{I}_{S(\overline{x})}(\overline{x}).$$

Proof. Define a map $G : X \times C \multimap X \times C$ by

$$G(x, y) := M(x, y) \times T(x) \text{ for } (x, y) \in X \times C.$$

Then $G \in \mathbb{V}(X \times C, X \times C)$. In fact, since M and T are compact-valued u.s.c. maps, their product G is also compact-valued and u.s.c. Note that each $G(x, y)$ is acyclic. Moreover, G is compact because

$$G(X \times C) \subset \overline{S(X)} \times \overline{T(X)} \subset X \times C.$$

Since $X \times C$ is an almost convex admissible subset of the t.v.s. $E \times F$, by Theorem 2.2, G has a fixed point $(\overline{x}, \overline{y}) \in X \times C$. Since

$$(\overline{x}, \overline{y}) \in G(\overline{x}, \overline{y}) = M(\overline{x}, \overline{y}) \times T(\overline{x}) \subset S(\overline{x}) \times T(\overline{x}),$$

we have $\overline{x} \in S(\overline{x})$ and $\overline{y} \in T(\overline{x})$. Moreover, since $\overline{x} \in M(\overline{x}, \overline{y})$, by (1), we have

$$0 \geq \phi(\overline{x}, \overline{y}, \overline{x}) = \max_{x \in S(\overline{x})} \phi(\overline{x}, \overline{y}, x).$$

For the last part, since $x \mapsto \phi(\overline{x}, \overline{y}, x)$ is l.s.c., it is sufficient to show that

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \text{ for all } x \in I_{S(\overline{x})}(\overline{x}) \backslash S\overline{x}.$$

For any $x \in I_{S(\overline{x})}(\overline{x}) \backslash S(\overline{x})$, there exist $z \in S(\overline{x})$ and $r > 0$ such that $x = \overline{x} + r(z - \overline{x})$. Then we must have $r > 1$; otherwise, $x = (1 - r)\overline{x} + rz \in S(\overline{x})$ since $0 < r \leq 1$ and $S(\overline{x})$ is convex. Suppose $\phi(\overline{x}, \overline{y}, x) > 0$. Since $r > 1$ and

$$\frac{1}{r}x + (1 - \frac{1}{r})\overline{x} = z \in S(\overline{x}),$$

we have

$$\phi(\overline{x}, \overline{y}, z) = \phi(\overline{x}, \overline{y}, \frac{1}{r}x + (1 - \frac{1}{r})\overline{x}) \geq \frac{1}{r}\phi(\overline{x}, \overline{y}, x) + (1 - \frac{1}{r})\phi(\overline{x}, \overline{y}, \overline{x}) = \frac{1}{r}\phi(\overline{x}, \overline{y}, x) > 0$$

by the concavity of $x \mapsto \phi(\overline{x}, \overline{y}, x)$ and condition (1). This is a contradiction. Therefore, we have $\phi(\overline{x}, \overline{y}, x) \leq 0$. This completes our proof. □

Remarks. 1. Theorem 7.1 has a number of particular cases by considering particular ones of S, T and ϕ.

2. Instead of $\mathbb{V}$, we can adopt the class $\mathbb{A}$ of u.s.c. approachable maps or other appropriate class; see [2, Theorem 1], which was stated for convex subsets X, C of locally convex t.v.s.

From Theorem 7.1, we obtain the following:

Theorem 7.2. *Let X, S, F, T and C be the same as in Theorem 7.1. Suppose that $S \in \mathbb{V}(X, X)$ is l.s.c. (hence continuous). Let $\phi : X \times C \times X \to \overline{\mathbf{R}}$ be continuous such that, for each $(x, y) \in X \times C$,*

(1) *$\phi(x, y, x) \leq 0$; and*
(2) *$M(x, y) = \{u \in S(x) \mid \phi(x, y, u) = \max_{s \in S(x)} \phi(x, y, s)\}$ is acyclic.*

Then there exist an $\overline{x} \in S(\overline{x})$ and a $\overline{y} \in T(\overline{x})$ such that

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \quad \text{for all} \quad x \in S(\overline{x}).$$

Further, if $\phi(x, y, x) = 0$ for all $(x, y) \in X \times C$, $S(\overline{x})$ is convex, and $x \mapsto \phi(\overline{x}, \overline{y}, x)$ is concave, then

$$\phi(\overline{x}, \overline{y}, x) \leq 0 \quad \text{for all} \quad x \in \overline{I}_{S(\overline{x})}(\overline{x}).$$

Remark. In our previous work [2], particular forms of Theorems 7.1 and 7.2 are applied to more than fifteen known variational or quasi-variational inequalities due to Hartman-Stampacchia, Browder, Lions-Stampacchia, Mosco, Saigal, and many others. Now all of them can be stated for almost convex sets instead of convex sets.

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