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ON ABSTRACT CONVEXITY AND SET VALUED ANALYSIS

REGINA SANDRA BURACHIK AND ALEX RUBINOV

Abstract. Given a set $L \subset \mathbb{R}^X$ of functions defined on X, we consider abstract monotone (or, for short, L-monotone) multivalued operators $T : X \rightrightarrows L$. We extend the definition of enlargement of monotone operators to this framework and study semicontinuity properties of these mappings. We prove that sequential outer semicontinuity, which holds for maximal monotone operators and their enlargements in the classical case (i.e., when $L = X^*$ and X is a Banach space), holds also in our abstract setting. We also show through examples that some properties, known to hold in the classical case, may no longer be valid in the abstract setting. One of these properties is the maximality of the subdifferential and another one is the lack of inner semicontinuity of (point-to-set) monotone operators in the interior of their domain. We also focus on the structure of both the abstract subdifferential and the abstract ε-subdifferential. This is a key question in abstract convexity because these sets may be very large for certain choices of L and therefore it is important to be able to represent them by means of some special elements of the set of "affine" functions induced by L.

1. Introduction

Abstract convexity opens the way for extending some main ideas and results from classical convex analysis to much more general classes of functions, mappings and sets. It is well-known that every convex, proper and lower semicontinuous function is the upper envelope of a set of affine functions. Therefore, affine functions play a crucial role in classical convex analysis. In abstract convexity, the role of the set of affine functions is taken by an alternative set H of functions, and their upper envelopes constitute the set of abstract convex functions. Different choices of the set H generate variants of the classical concepts, and have shown important applications, especially in global optimization (see e.g., [19, 18, 17, 20]). Moreover, if a family of functions is abstract-convex for a specific choice of H, then we can use some key ideas of convex analysis in order to gain new insight on these functions. On the other hand, by using an alternative set for affine functions, we identify those facts in classical convex analysis which depend on the specific properties of affine functions.

Abstract convexity has mainly been used for the study of point-to-point functions. An example of its use in the analysis of multivalued operators can be found in the works of Levin [10, 11], who focused in the study of abstract cyclical monotonicity.

Our first aim is to consider abstract maximal monotone operators and their enlargements. We analyze whether some semicontinuity properties, known to hold in

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the classical setting, can be extended to the abstract setting. We recall the definition of abstract maximal monotonicity given in [10, 11] and based on the works [5, 4, 6, 12] we introduce the definition of enlargement of an abstract monotone map. We prove that the abstract monotone mappings as well as their abstract enlargements are sequentially outer semicontinuous, extending to our general context a well known fact concerning maximal monotone operators and their enlargements.

Some of the classical facts, however, cannot be extended to the abstract setting. For example, it is known that a maximal monotone operator cannot be inner semicontinuous at a point x in the interior of its domain, unless Tx is a singleton (see, e.g., [9, Lemma 1.30] or [3, Theorem 4.6.3]). Hence, a maximal monotone operator cannot be continuous (i.e., outer- and inner- semicontinuous) in the interior of its domain unless it is point-to-point. However, we show in Example 3.6 an abstract maximal monotone map which is (not point-to-point and) continuous at every point in the interior of its domain. This example is particularly relevant because every lower semicontinuous function defined on a compact set is abstract-convex for this choice of H. Therefore, our approach may open the way for using the main ideas of convex analysis for the examination of arbitrary lower semicontinuous functions defined in a bounded set. Another example of a classical fact that cannot be extended to the abstract setting is the maximality of the subdifferential of a proper, lower semicontinuous and convex function. We show this in Example 3.1.

Our second aim is the representation of both the abstract subgradients and the abstract ε-subgradients. This question is relevent because the abstract versions of the subdifferential and the ε-subdifferential might be very large. For obtaining such a representation, we work with the "affine" counterparts of the subdifferential and the ε-subdifferential. We recall the concept of maximal element in this setting and establish conditions under which maximal elements exist.

The paper is organized as follows. Section 2 gives some preliminary material on abstract convexity. In Section 3 we recall the definition of abstract monotonicity and introduce the concept of enlargement of an abstract monotone map. Classical examples of enlargements are also extended to our framework. Under basic assumptions (which hold in the classical setting) we prove that abstract monotone maps and their enlargements are sequentially outer semicontinuous. Approximate abstract subdifferentials is the subject of Section 4, in which we study maximal elements of the "affine" counterparts of the abstract ε-subdifferential (see Proposition 4.2). In Proposition 4.3 we relate specific elements of the support set with maximal elements of the "affine" counterparts of the abstract ε-subdifferential. Finally, we compare our results with those in classical convex analysis and illustrate both the properties and the results with examples.

2. Basic notation and definitions

We use the following notation: $\mathbb{R} = (-\infty, +\infty)$ is the real line and $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$. Let X be an arbitrary set. For given $f, g : X \to \mathbb{R}_{+\infty}$, the inequality $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in X$. Most definitions and known statements related to abstract convexity can be found in [16, 21, 14]. Recall that a lower semicontinuous function defined in a Banach space with values in $\mathbb{R}_{+\infty}$ is convex if and only if it is the *upper envelope* of its affine minorants. In abstract convexity, we keep the idea of upper envelope, and replace the set of linear functions by more general sets.

Namely, we consider a set L of functions $l : X \to \mathbb{R}_{+\infty}$, which will play the role of the linear functions. Our set H is the set of L-*affine* functions, defined as

$$H := \{h_{l,\gamma} : X \to \mathbb{R}_{+\infty} : h_{l,\gamma}(\cdot) := l(\cdot) - \gamma,\ \forall\,(l,\gamma) \in L \times \mathbb{R}\}. \tag{2.1}$$

We may also denote the L-affine function $h_{l,\gamma}$ simply as (l,γ). For this identification to be unique, the set of abstract linear functions should verify the following assumption.

(A_0) Consider the function $\mathbf{1} : X \to \mathbb{R}$ defined by $\mathbf{1}(x) = 1$ for all $x \in X$. For all $l \in L$ and all $0 \neq c \in \mathbb{R}$ we have $l - c\mathbf{1} \notin L$.

We assume from now on that (A_0) holds for L. With the only exception of Example 4.5, we will assume $l(x) \in \mathbb{R}$ for every $(x,l) \in X \times L$.

The *support set of* $f : X \to \mathbb{R}_{+\infty}$ *with respect to* H is defined as

$$\operatorname{supp}(f,H) := \{h \in H : h \leq f\}.$$

Denote by $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$. A function f is called *abstract convex with respect to* H (or H-*convex*, for short) at a point $x \in X$ if there exists a set $U \subset \operatorname{supp}(f,H)$ such that $f(x) = \sup\{h(x) : h \in U\}$. If f is H-convex at each point $x \in Z$, where $Z \subset X$ then f is called H-*convex on* Z.

Note that the set $\operatorname{supp}(f,H)$ might be very large, because it contains all functions h' such that $h' \leq h$ with $h \in \operatorname{supp}(f,H)$. Sets with this property are called *downward sets*. It is therefore convenient to find simple representations of the support set. In order to obtain these representations we need to recall the concepts of *abstract subdifferential* of f and its associated affine set $\mathcal{D}f$.

Let $f : X \to \mathbb{R}_{+\infty}$ and $x_0 \in X$ such that $f(x_0) \in \mathbb{R}$. The *abstract subdifferential* (or L-subdifferential) of f at the point x_0 [16, page 281] is defined as

$$\partial_L f(x_0) := \{l \in L : l(x_0) \in \mathbb{R} \text{ and } l(x) - l(x_0) \leq f(x) - f(x_0) \text{ for all } x \in X\}. \tag{2.2}$$

When $f(x_0) \notin \mathbb{R}$ we define $\partial_L f(x_0) := \emptyset$.

More generally, we will consider in our study the abstract subdifferential with respect to a subset $Z \subset X$ containing the point x_0 such that $f(x_0) \in \mathbb{R}$:

$$\partial_{L,Z} f(x_0) := \{l \in L : l(x_0) \in \mathbb{R} \text{ and } l(x) - l(x_0) \leq f(x) - f(x_0) \text{ for all } x \in Z\}. \tag{2.3}$$

The abstract subdifferential is a subset of L, its counterpart in the set H has been introduced in [16, Section 8.2.3] and it is defined as:

$$\mathcal{D}_{L,Z} f(x_0) := \{h \in H : h(x) = l(x) - l(x_0),\ l \in \partial_{L,Z} f(x_0)\}. \tag{2.4}$$

Since there will be no ambiguity regarding the choice of L and Z, we simply denote $\mathcal{D}_{L,Z} f$ by $\mathcal{D}f$. Let us also recall the definition of the abstract ε-subdifferential $\partial_{L,Z,(\cdot)} f : X \times \mathbb{R}_+ \rightrightarrows L$, given in [16, Definition 7.8]:

$$\begin{aligned}\partial_{L,Z,\varepsilon} f(x_0) \\ := \{l \in L : l(x_0) \in \mathbb{R} \text{ and } l(x) - l(x_0) \leq f(x) - f(x_0) + \varepsilon \text{ for all } x \in Z\},\end{aligned} \tag{2.5}$$

where $x_0 \in Z$ and $\varepsilon \geq 0$. Its affine counterpart is defined as

$$\mathcal{D}_\varepsilon f(x_0) := \{h \in H : h(x) = l(x) - l(x_0),\ l \in \partial_{L,Z,\varepsilon} f(x_0)\}. \tag{2.6}$$

3. Abstract monotonicity

Let X be a Banach space and X^* its dual. Recall that a point-to-set mapping $T : X \rightrightarrows X^*$ is said to be *monotone* when for every pair of elements $(v, x), (u, y)$ such that $v \in Tx$ and $u \in Ty$, we have that

$$0 \leq \langle v - u, x - y\rangle.$$

We extend now this concept to our setting.

Let X be a Haussdorf topological vector space. Assume that the set of elementary functions L is such that $l(x) \in \mathbb{R}$ for all $(x, l) \in X \times L$, a point-to-set mapping $T : X \rightrightarrows L$ is said to be L*-monotone* when for every pair of elements $(l, x), (l', x')$ such that $l \in Tx$ and $l' \in Tx'$, we have that

$$0 \leq l(x) + l'(x') - l(x') - l'(x). \tag{3.1}$$

This definition was introduced in [10, 11], where the stronger concept of L-cyclical monotonicity was studied. Namely, a point-to-set mapping $T : X \rightrightarrows L$ is said to be L*-cyclically monotone* when for every finite collection of elements $(l_1, x_1), \dots, (l_m, x_m)$ such that $l_k \in Tx_k$ for all $k = 1, \dots, m$ we have

$$\sum_{k=1}^{m}[l_k(x_k) - l_k(x_{k+1})] \geq 0,$$

where $x_{m+1} := x_1$. It is immediate to check that the abstract subdifferential is L-cyclically monotone and in particular (take $m = 2$ in the expression above) L-monotone. Call $D(T) := \{x \in X : Tx \neq \emptyset\}$.

Recall that the graph of a point-to-set map $F : A \rightrightarrows B$ is defined as $G(F) := \{(a, b) \in A \times B : b \in F(a)\}$. We give now the definition of L-maximal monotonicity.

As in the classical setting, we say that a point-to-set mapping $T : X \rightrightarrows L$ is L-*maximal* monotone when, for every other L-monotone map T' with $G(T') \supset G(T)$, we must have $T = T'$.

Recall that the subdifferential of a proper, convex and lower semicontinuous function in a Banach space is maximal monotone. Another classical fact is that the subdifferential is not empty at points in the interior of the domain of f. Both properties may no longer be true for abstract subdifferentials, as is shown in the following example.

Example 3.1. Let X be a Banach space and take the set L as in [16, Equation 7.3.13]:

$$l_{v,c}(x) := \begin{cases} c & \text{if } v(x) > 1, \\ 0 & \text{if } v(x) \leq 1, \end{cases}$$

where $v \in X^*$ and $c \geq 0$. So that the set H of L-affine functions becomes

$$j_{v,c,c'}(x) := \begin{cases} c & \text{if } v(x) > 1, \\ c' & \text{if } v(x) \leq 1, \end{cases}$$

where $v \in X^*$ and $c, c' \in \mathbb{R}$ and $c' \leq c$.

Recall that a function $f : X \to \mathbb{R}_{+\infty}$ is *quasiconvex* when for every $a \in \mathbb{R}$ the level sets $\{x \in X : f(x) \leq a\}$ are convex. The following facts were established in [16, Proposition 7.41(1) and Proposition 7.45]. Let $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$.

(a) A function $q : X \to \mathbb{R}_+ \cup \{+\infty\}$ is H-convex if and only if q is lower semicontinuous, quasiconvex and $q(0) = \inf\{q(x) : x \in X\}$.

(b) The L-subdifferential of q at x_0 is empty if x_0 is not a local minimizer of q.

Let $X := \mathbb{R}$ and take $q = |\cdot|$. Then q is H-convex by (a). Part (b) readily implies that $D(\partial_L q) \subset \{0\}$ and therefore $D(\partial_L q)$ does not contain the set $\operatorname{int}\operatorname{dom} q = X$. Let us show now that $\partial_L q$ is not maximal.

By (b) we have that $G(\partial_L q) = \{0\} \times \partial_L q(0)$. It can be proved that

$$\partial_L q(0) = \{(0, c) : c \geq 0\} \cup \{(\alpha, c) : 0 \neq \alpha,\ 0 \leq c \leq \frac{1}{|\alpha|}\}.$$

As we see from the equality above, $(x_0, v) = (0, (2, 2)) \notin G(\partial_L q)$. On the other hand, $(x_0, v) = (0, (2, 2))$ is L-monotonically related with $G(\partial_L q)$. Indeed, take $x = x' = 0$ in inequality (3.1). The definition of $l_{v,c}$ implies that every term in the right hand side of (3.1) is equal to zero, so the inequality holds. Therefore the L-subdifferential is not maximal.

Analysis of monotone maps is closely connected with the specific properties of the scalar product of the space. The role of the scalar product is now taken by the coupling function $\phi : X \times L \to \mathbb{R}$ defined as $\phi(x, l) := l(x)$.

Let H be the set of L-affine functions defined in (2.1). We will consider the following set of assumptions:

(A_1) The topology in H contains the one induced by the pointwise convergence. In other words, we assume that the topology in H is either stronger or it coincides with the one given by the pointwise convergence. We also assume that L is closed in H (with the induced topology).

(A_2) Every $l \in L$ is lower semicontinuous.

(A_3) The coupling function ϕ is upper semicontinuous, i.e., given a directed set I and a net $\{(l_i, x_i)\}_{i \in I} \in L \times X$, we have

$$\limsup_{i \in I} l_i(x_i) \leq l(x),$$

whenever the net (l_i, x_i) converges to (l, x). We denote as $((A_3)_s)$ the sequential statement of (A_3), i.e., when the coupling function ϕ is assumed to be sequentially-upper semicontinuous, or, in other words, when the nets are replaced by sequences.

Remark 3.1. Note that (A_2) and (A_3) imply that all elements of L (and, hence of H) are continuous.

Remark 3.2. Assumptions $(A_1) - (A_2)$ hold for the classical case, i.e., when X is a Banach space and $L = X^*$, the topological dual of X endowed with the weak* topology. Condition (A_3) may not hold in the classical case, this is because weak* convergent nets may not be eventually bounded. However, $(A_3)_s$ (with a weak*-strong convergent sequence $\{(v_n, x_n)\}_{n \in \mathbb{N}}$ instead of a weak*-strong convergent net $\{(l_i, x_i)\} \subset X^* \times X$) holds. When the strong topology is considered both in X and X^*, condition (A_3) (which in this situation is equivalent to $(A_3)_s$) holds. In particular, (A_3) holds when X is finite dimensional. Assumptions $(A_1) - (A_3)$ hold for Examples 4.1 and 4.2 below.

3.1. **Abstract enlargements.** In order to define an enlargement of an L monotone map, we will represent abstract monotone maps by means of abstract convex functions defined in the cartesian product $X \times L$. Note that the linear functions defined in the product $X \times L$ are of the form $(u, y) \in L \times X$ with $^{(u,y)}(x, v) = u(x) + v(y)$. We say that $h : X \times L \to \mathbb{R}_{+\infty}$ is (L, X)-convex (or *dual-abstract convex*) whenever

$$h(x, v) := \sup\{u(x) + v(y) - c : ((u, y), c) \in \operatorname{supp}(h, L \times X)\}.$$

Inspired in [12], we say that an L monotone map T is *represented* by a dual-abstract convex function h when

- (R_1) $h(x, v) \geq v(x)$ for every $(x, v) \in X \times L$,
- (R_2) $h(x, v) = v(x)$ if and only if $(x, v) \in G(T)$.

Representability of monotone maps in the classical sense has been recently studied in [12]. The example below extends well known facts (see [6]) from the classical setting, without any additional assumption on L or X. Given an H-convex function $f : X \to \mathbb{R}_{+\infty}$, its *Fenchel-Moreau conjugate* $f^* : L \to \mathbb{R}_{+\infty}$ is defined as

$$f^*(l) := \sup_{x \in X}\{l(x) - f(x)\}.$$

Example 3.2. Let f be an H-convex function. The abstract subdifferential of f is represented by the dual-abstract convex function $\beta_f(x, v) = f(x) + f^*(v)$, and every L-maximal monotone map is represented by its *Fitzpatrick function* [13]

$$\varphi_T(x, v) := \sup_{(y,u) \in G(T)}\{u(x) + v(y) - u(y)\}. \tag{3.2}$$

The first assertion follows from Equation (7.2.10) and Proposition 7.7 in [16]. We proceed to establish the second one. Note that the Fitzpatrick function is (L, X)-convex by definition. In order to check (R_1), fix $(x, v) \in X \times L$. If $(x, v) \notin G(T)$, then by maximality there exists $(y, u) \in G(T)$ such that

$$0 > v(x) + u(y) - u(x) - v(y).$$

Therefore,

$$\begin{aligned}\varphi_T(x, v) &= \sup_{(y',u') \in G(T)}\{u'(x) + v(y') - u'(y')\}\\ &\geq u(x) + v(y) - u(y) = [u(x) + v(y) - u(y) - v(x)] + v(x) > v(x).\end{aligned}$$

Assume now that $(x, v) \in G(T)$. The argument of the supremum in (3.2) can be expressed as

$$[u(x) + v(y) - u(y) - v(x)] + v(x) \leq v(x),$$

where we used monotonocity of T. So $\varphi_T(x, v) \leq v(x)$. On the other hand, if we take $(y', u') = (x, v) \in G(T)$ in the argument of the supremum we get

$$\varphi_T(x, v) \geq v(x) + v(x) - v(x) = v(x),$$

so $\varphi_T(x, v) = v(x)$ when $(x, v) \in G(T)$. Conversely, assume that $\varphi_T(x, v) = v(x)$. By definition,

$$\varphi_T(x, v) = \sup_{(y',u') \in G(T)}[u'(x) + v(y') - u'(y') - v(x)] + v(x),$$

so the equality $\varphi_T(x,v) = v(x)$ yields $[u'(x) + v(y') - u'(y') - v(x)] \le 0$ for every $(y',u') \in G(T)$. By maximality we must have $(x,v) \in G(T)$. This proves (R_2).

As we gather from the example above, we can study L-maximal monotone maps through their (L,X)-convex representations. Moreover, we can use the (L,X)-convex representations for defining their enlargements.

Given a fixed L monotone map $T : X \rightrightarrows L$, we say that $E : \mathbb{R}_+ \times X \rightrightarrows L$ is an L-*enlargement* of T when it verifies the following property:

(E_1) There exists a dual-abstract convex function h which represents T and such that

$$E(\varepsilon, x) = \{v \in L : h(x,v) \le \varepsilon + v(x)\}. \tag{3.3}$$

This condition can be interpreted as follows. If h represents T, then it represents their enlargements by the rule given in (3.3). By $(R_1) - (R_2)$ we have that $T(x) = E(0,x)$.

Example 3.3. The most important example of enlargement in the literature is the ε-subdifferential, which is an enlargement of the subdifferential defined in [7]. It is a well-known fact that the classical ε-subdifferential verifies (E_1) with $h = \beta_f$ defined in Example 3.2. The fact that these properties can be extended to the abstract ε-subdifferential has been proved in Proposition 7.10 in [16]. In particular, when $\varepsilon = 0$ we have $\partial_L f(x) = \{v \in L : \beta_f(x,v) \le v(x)\}$.

Example 3.4. Fix an L-maximal monotone operator T. Define $B_T : \mathbb{R}_+ \times X \to L$ in the following way.

$$\begin{aligned} B_T(\varepsilon, x) & \\ &:= \{l \in L : l'(y') + l(x) - l(y') - l'(x) \ge -\varepsilon, \ \text{ for all } (y',l') \in G(T)\} \\ &= \{l \in L : \varphi_T(x,l) \le \varepsilon + l(x)\}. \end{aligned} \tag{3.4}$$

This enlargement is an extension to the abstract setting of the one introduced in [4] and further studied in [3, 5, 6]. In the particular case in which $T = \partial_L f$ (see (2.3) for $Z = X$) we have $\partial_{L,\varepsilon} f(x) \subset B_T(\varepsilon, x)$, where the enlargement $\partial_{L,(\cdot)} f(\cdot)$ is given in (2.5). This fact is well-known for the classical case (see [4]) and for our general framework the proof follows the same steps. Indeed, the inclusion $\partial_{L,\varepsilon} f(x) \subset B_{\partial_L f}(\varepsilon, x)$ is a consequence of condition (E_1) and the fact that $\varphi_{\partial_L f} \le \beta_f$. Let us check the latter inequality:

$$\begin{aligned} \varphi_{\partial_L f}(x,v) &= \sup_{y',\, u' \in \partial_L f(y')} \{u'(x) + v(y') - u'(y')\} \\ &\le \sup_{y',\, u' \in \partial_L f(y')} v(y') - f(y') + f(x) \\ &= f^*(v) + f(x) = \beta_f(x,v), \end{aligned}$$

where we used the fact that $f(x) - f(y') \ge u'(x) - u'(y')$ in the inequality and the definition of f^* in the second equality.

Remark 3.3. Under assumptions (A_1) and (A_2), a representation h of T is lower semicontinuous. Indeed, it is enough to check that the (L,X)-linear functions (u,y)

are lower semicontinuous. Let $\{(x_i, v_i)\}_{i\in I}$ be a net converging to (x, v). Then,

$$\begin{aligned}\liminf_i {}^{(u,y)}(x_i, v_i) &= \liminf_i u(x_i) + v_i(y)\\ &\geq \liminf_i u(x_i) + \liminf_i v_i(y)\\ &\geq u(x) + \liminf_i v_i(y)\\ &= u(x) + v(y) = {}^{(u,y)}(x, v),\end{aligned}$$

where we used (A_2) in the second inequality and (A_1) in the second equality.

Example 3.1 suggests that maximality is no longer a property associated with abstract subdifferentials. We establish below a condition under which maximality holds.

Proposition 3.1. *Let f be H-convex and such that both f and f^* are continuous. Assume that the set $D(\partial_L f)$ is dense on* $\operatorname{dom} f$ *and that the set $R(\partial_L f) := \{v \in L : \exists\, x \in X$ such that $v \in \partial_L f(x)\}$ is dense on* $\operatorname{dom} f^*$. *Then $\partial_L f$ is maximal.*

Proof. We mentioned in Example 3.2 that the abstract subdifferential of f is represented by the (L, X)-convex function $\beta_f(x, v) = f(x) + f^*(v)$. Take now a pair (x', v') which is L-monotonically related with $G(\partial_L f)$. So for every $(x, v) \in G(\partial_L f)$ we have

$$v'(x') \geq v'(x) + v(x') - v(x) = v'(x) + v(x') - f(x) - f^*(v),$$

where we used condition (R_2) for β_f in the equality. We can rewrite the inequality above as

$$v'(x') \geq [v'(x) - f(x)] + [x'(v) - f^*(v)].$$

Taking now supremum over the domain and range of $\partial_L f$, and using the density and continuity assumptions, we get

$$v'(x') \geq f^*(v') + f^{**}(x').$$

By Theorem 7.1 in [16] we have that every H-convex function verifies $f = f^{**}$ so we obtain $v'(x') \geq f^*(v') + f(x') = \beta_f(x', v')$. Since we always have $\beta_f(x', v') \geq v'(x')$ we conclude that $v'(x') = \beta_f(x', v')$, which together with (R_2) yields $(x', v') \in G(\partial_L f)$ and hence $\partial_L f$ is maximal. □

Remark 3.4. In connection with Proposition 3.1, we point out that the question of the density of $\operatorname{dom} \partial_L f$ has been intensively studied in the literature (see, e.g. [1, 14, 22]).

3.2. **Outer semicontinuity.** Continuity properties of point-to-set maps are analyzed by studying the topological properties of their graphs. Recall that a point-to-set map is *outer semicontinuous* when its graph is closed with respect to the product topology. Maximal monotone (in the classical sense) maps (and therefore subdifferentials of proper lower semicontinuous convex functions) are *sequentially* outer semicontinuous with respect to the strong topology in X and the weak* topology in X^* (sequentially stands for the fact that the limit is taken over sequences instead of nets). The same property is shared by every enlargement (in the sense of [3–6]) of T. As a consequence, when X is finite dimensional, both the graph of T and the graph of its enlargements are closed with respect to the product topology. We point out that closedness of the graph of T in the product topology (with respect to the strong topology in X and the weak*-topology in X^*) may not hold in general (see [2]).

Our aim is to extend these outer semicontinuity properties to our setting. More precisely, we will prove that our basic assumptions $(A_1)-(A_3)$ yield outer semicontinuity of L-monotone maps and their abstract enlargements.

Theorem 3.1. *Assume that $(A_1)-(A_3)$ hold. Let T be a representable L-monotone mapping and consider E an enlargement of T verifying condition (E_1). Then T and E are outer-semicontinuous. If $(A_3)_s$ holds, then T and E are sequentially outer-semicontinuous. Consequently, L-maximal monotone mappings are outer-semicontinuous.*

Proof. A consequence of condition (E_1) and the properties of h is that $\{0\}\times G(T) = G(E) \cap \{0\} \times (X \times L)$. Hence it is enough to prove that $G(E) = \{(x, l, \varepsilon) \in \mathbb{R}_+ \times L \times X : l \in E(\varepsilon, x))\}$ is closed. Let $\{(x_i, l_i, \varepsilon_i)\}_{i\in I} \subset G(E)$ be a net converging to (x, l, ε). We must prove that $l \in E(\varepsilon, x)$. In other words, we want to show that $h(x, l) \leq \varepsilon + l(x)$, where h represents T and verifies (3.3). From Remark 3.3 we have that h is lower semicontinuous. Hence

$$\begin{aligned} h(x,l) &\leq \liminf_{i\in I} h(x_i, l_i) \leq \liminf_{i\in I} \varepsilon_i + l_i(x_i) \\ &\leq \limsup_{i\in I} \varepsilon_i + l_i(x_i) \\ &\leq \limsup_{i\in I} \varepsilon_i + \limsup_{i\in I} l_i(x_i) \\ &\leq \varepsilon + l(x), \end{aligned}$$

where we used (E_1) in the second inequality and (A_3) in the last one. The statement on sequential outer-semicontinuity is proved in the same way, but using sequences instead of nets. The last statement of the theorem follows from the fact that every L-maximal monotone operator is representable (see Example 3.2). □

Corollary 3.1. *Assume that f is H-convex. Let the point-to-set mappings $\partial_{L,Z} f(\cdot)$ and $\partial_{L,Z,(\cdot)} f(\cdot)$ be given by* (2.3) *and* (2.5)*, respectively. Under assumptions $(A_1)-(A_3)$, their graphs are closed with respect to the product topology. If $(A_3)_s$ holds, then $\partial_{L,Z} f(\cdot)$ and $\partial_{L,Z,(\cdot)} f(\cdot)$ are sequentially outer-semicontinuous.*

Proof. We know that $T := \partial_{L,Z} f(\cdot)$ is L-monotone and representable by $h = \beta_f$. We can now appy Theorem 3.1. □

Remark 3.5. By Remark 3.2, we see that the sequential statement in Theorem 3.1 recovers the classical result on sequential outer semicontinuity for maximal monotone mappings, while Corollary 3.1 recovers the sequential outer semicontinuity of subdifferentials and their enlargements $\partial_{L,Z,(\cdot)} f(\cdot)$ and $B_{\partial_L f}(\cdot,\cdot)$.

The following example uses a set L as in Example 7.5 of [16].

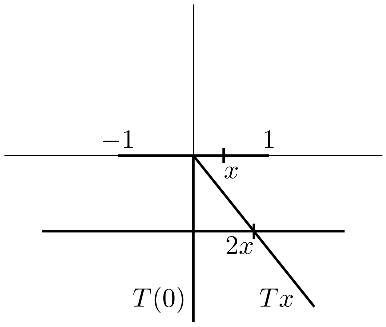
Example 3.5. Let $X = \mathbb{R}$ and take $Z := [-1, 1]$. Consider the set of functions

$$L := \{l(x) = ax^2 + bx : a \leq 0,\, b \in \mathbb{R}\}.$$

We identify the function $l(x) = ax^2 + bx$ with the pair (a, b). We define $T : Z \rightrightarrows L$ as

$$Tx := \{(a, b) \in L : x \in \operatorname{Arg} \max_{t \in [-1,1]} at^2 + bt\}.$$

Is is clear that $D(T) = Z$ because $(0, 0) \in Tx$ for all $x \in [-1, 1]$. The operator T is onto, because for every $(a, b) \in L$ the function $t \mapsto at^2 + bT$ is continuous over the compact set $[-1, 1]$, so there always exists $x \in [-1, 1]$ such that $(a, b) \in Tx$. We claim that T is the subdifferential of any function U which is constant over Z. Indeed, by definition of T we have that $(a, b) = l \in T(t_0)$ if and only if $0 \geq l(x)-l(t_0)$ for every $x \in [-1, 1]$, which is equivalent to $0 = U(x)-U(t_0) \geq l(x)-l(t_0)$ for every $x \in [-1, 1]$, i.e., $l \in \partial_{L,Z}U(t_0)$. Note also that any constant function U on $[-1, 1]$ is H convex, because at every point $t \in [-1, 1]$ we can represent $U(t)$ as the supremum of concave parabolas. The function U^* is continuous, to check this see formula for β_U at the end of this example. Because U and U^* are continuous, $D(T) = Z$ and T is onto, Proposition 3.1 implies that $T = \partial_L U$ is maximal. It is also easy to check that Tx is a closed cone for every $x \in [-1, 1]$. In the figure, we depicted Tx for an $x \in (-1, 1)$ and $T(0)$.



More precisely, we have

$$(3.5) \qquad Tx = \begin{cases} \{\lambda(-1, 2x) : \lambda \geq 0\} & \text{if } x \in (-1, 1), \\ \{\lambda(0, 1) : \lambda \geq 0\} \cup \{\beta(-1, 2) : \beta \geq 0\} & \text{if } x = 1, \\ \{\lambda(0, -1) : \lambda \geq 0\} \cup \{\beta(-1, -2) : \beta \geq 0\} & \text{if } x = -1. \end{cases}$$

Computation of the sets $T(1)$ and $T(-1)$ as well as the inclusions $\{\lambda(-1, 2x) : \lambda \geq 0\} \subset Tx$ for all $x \in (-1, 1)$ are straightforward from the definition of T. Let us check the inclusion $Tx \subset \{\lambda(-1, 2x) : \lambda \geq 0\}$ for $x \in (-1, 1)$. Fix $(a, b) \in Tx$. Because $(0, 0)$ always belongs to the set on the right-hand side, we can assume $(a, b) \neq (0, 0)$. Using the fact that $x \in (-1, 1)$, we must have $a < 0$. By definition of T we must have that $q(t) := at^2 + bt$ attains its unique maximum at the point $x \in (-1, 1)$. This yields $x = \frac{-b}{2a}$. So we must have $(a, b) = -a(-1, 2x)$ as we wanted.

Let us look now at the enlargements of T. Since $T = \partial_{L,Z}U$ we have the enlargement given by Example 3.3, which is the ε-subdifferential of U. Since U is constant we see that $(a, b) \in \partial_{\varepsilon,L,Z}U(x_0)$ if and only if for every $y \in [-1, 1]$

$$0 = U(y) - U(x_0) \geq ay^2 + by - ax_0^2 - bx_0 - \varepsilon.$$

In other words, $(a,b) \in \partial_{\varepsilon,L,Z}U(x_0)$ if and only if x_0 verifies

$$ax_0^2 + bx_0 \geq ay^2 + by - \varepsilon,\ \forall\, y \in [-1,1], \tag{3.6}$$

which means that x_0 is an ε*-maximum of* $q(t) = at^2 + bt$.

The other enlargement is the one induced by the Fitzpatrick function (see Example 3.4). We claim that both enlargements coincide. Indeed, in view Example 3.4, it is enough to check that $B_T(\varepsilon, x_0) \subset \partial_{\varepsilon,L,Z}U(x_0)$. In fact, the latter inclusion is a consequence of the fact that $(0,0) \in Ty'$ for every $y' \in [-1,1]$ in (3.4). Therefore, both representing functions β_U and φ_T given in Examples 3.2 and 3.4 coincide. Indeed, choosing the constant function U to be identically equal to zero we get $\beta_U(x,(a,b)) = U(x) + U^*(a,b) = \sup_{t\in[-1,1]} at^2 + bt$. Using (3.5) it can be verified that the Fitzpatrick function associated with T is

$$\varphi_T(x,(a,b)) = \begin{cases} a-b & \text{if } b \leq 2a,\ a \leq 0, \\ \frac{-b^2}{4a} & \text{if } 2a \leq b \leq -2a,\ a < 0, \\ a+b & \text{if } b \geq -2a,\ a \leq 0, \end{cases}$$

which coincides with $\beta_U(x,(a,b))$.

3.3. **Inner semicontinuity.** Let A, B be topological Haussdorff spaces. A point-to-set map $F : A \rightrightarrows B$ is said to be *inner semicontinuous at* $\bar{x} \in D(F)$ when, for every open set $D \subset B$ such that $D \cap F(\bar{x}) \neq \emptyset$, there exists a neighborhood U of $\bar{x}$ such that $D \cap F(x) \neq \emptyset$ for every $x \in U$. For metric spaces the latter definition can be stated in terms of sequences in the following way: F is *inner semicontinuous at* $\bar{x}$ when for every $y \in F(\bar{x})$ and f or every sequence $\{x_k\} \subset D(F)$ converging to $\bar{x}$ there exists a sequence $\{y_k\}$ such that $y_k \in F(x_k)$ for all k and $\{y_k\}$ converges to y.

We mentioned before that a maximal monotone operator (in the classical sense) $T : X \rightrightarrows X^*$ defined in a separable (i.e., X has a denumerable dense subset) space X is inner semicontinuous at $\bar{x}$ if and only if $T\bar{x}$ is a singleton. Hence *full continuity* (i.e., outer- and inner- semicontinuity) of set-valued maximal monotone operators is a rare event in the classical setting, unless we are dealing with a point-to-point mapping. This fact may no longer be true in the abstract setting, as the next example shows.

Example 3.6. Let us consider again the L-monotone operator T of Example 3.5. This T is maximal and clearly not point-to-point since Tx is a cone for every $x \in (-1,1)$. On the other hand, we can easily check that T is inner-semicontinuous at every point $x \in (-1,1)$. Fix $(a,b) \in Tx$. If $a = 0$, the fact that $x \in (-1,1)$ again forces $b = 0$. Given a sequence $\{x_k\} \subset D(F)$ converging to $\bar{x}$ we can always take $y_k := (0,0) \in Tx_k$ so y_k trivially converges to $(a,b) = (0,0)$. Assume now that $a < 0$. So we must have $x = \frac{-b}{2a}$. Consider a sequence $\{x_k\}$ converging to x. We can assume that $\{x_k\} \subset (-1,1)$. By (3.5) we have that $Tx_k = \{\lambda(-1, 2x_k) : \lambda \geq 0\}$. Let us choose $\lambda_k := -a$. Then $l_k := (-a)(-1, 2x_k) \in Tx_k$ and converges to $(a, -2xa) = (a,b)$. So T is inner-semicontinuous at every point $x \in (-1,1)$. Let us check that T is not inner-semicontinuous at $x = 1$ or $x = -1$. Take $(0,1) \in T(1)$ and the sequence $u_n := 1 - \frac{1}{n} \to 1$. Since $u_n \in (-1,1)$ we have that $Tu_n = \{\lambda(-1, 2u_n) : \lambda \geq 0\}$. If T would be inner-semicontinuous at $x = 1$ we would be able to choose $\lambda_n \geq 0$ such

that
$$\lambda_n(-1, 2u_n) \to (0, 1).$$
Which yields the contradicting facts $\lambda_n \to 0$ and $2\lambda_n(1 - \frac{1}{n}) \to 1$. The lack of semicontinuity at $x = -1$ is proved in a similar manner. Altogether, we have that T is continuous in the interior of its domain. Moreover, we can actually prove that T is Lipschitz continuous in $(-1, 1)$. Here the Lipschitz property is defined using the following distance between the cones K_1, K_2 (see [15]):
$$\delta(K_1, K_2) := \max\{\sup_{z \in K_1 \cap S} dist(z, K_2), \sup_{z \in K_2 \cap S} dist(z, K_1)\},$$
where S denotes the unit sphere. We will use this formula for $K_1 = Tx$ and $K_2 = Tx'$, with $x, x' \in (-1, 1)$, in order to estimate the distance between Tx and Tx'. First note that $K_1 \cap S$ is the unique point $z_1 := \frac{1}{\sqrt{1+4x^2}}(-1, 2x)$. In the same way $K_2 \cap S$ is the unique point $z_2 := \frac{1}{\sqrt{1+4x'^2}}(-1, 2x')$. So the above formula simplifies to
$$\delta(Tx, Tx') = \max\{dist(z_1, K_2), dist(z_2, K_1)\}.$$
Note that
$$dist(z_1, K_2) = \min_{\lambda \geq 0} \sqrt{(\frac{-1}{\sqrt{1+4x^2}} + \lambda)^2 + (\frac{2x}{\sqrt{1+4x^2}} - 2\lambda x')^2}.$$
Direct calculation shows that $dist(z_1, K_2) = \sqrt{1 - \frac{(1+4xx')^2}{(1+4x^2)(1+4x'^2)}}$. Since this expression is symmetric with respect to x and x' we get
$$\delta(Tx, Tx') = \sqrt{1 - \frac{(1+4xx')^2}{(1+4x^2)(1+4x'^2)}} = \frac{2|x - x'|}{\sqrt{(1+4x^2)(1+4x'^2)}} \leq 2|x - x'|,$$
which establishes the desired Lipschitzianity.

4. Approximate subdifferentials

Denote by H the set of all L-affine functions. Fix a subset $Z \subset X$ and $x_0 \in Z$. The set of *L-subdifferentials of f* may be very large, so it is useful to identify special members of it, and relate them with special members of the support set on Z. In [16, Proposition 7.1] it is proved that there exists a bijective correspondence between the set of L-subdifferentials of f at x_0 and the set
$$\mathcal{S}_f(x_0) := \{h \in \operatorname{supp}(f, H, Z) : h(x_0) = f(x_0)\}. \tag{4.1}$$
Remark 4.1. The set $\mathcal{S}_f(x_0)$ is nonempty if and only if f is H-convex at x_0 and the supremum $\sup\{h(x_0) : h \in \operatorname{supp}(f, H, Z)\}$ is attained (see [16, page 279]).

The result in [16, Proposition 7.1] allows to determine whether the abstract subdifferential is or not empty. In some cases, as for instance the one in [16, Example 8.3], the abstract subdifferential at a given point is empty, and hence we are forced to look at the abstract ε-subdifferentials of f at x_0 with respect to the set Z. Note also that every H-convex function on Z has approximate subdifferentials (for a proof of this fact, see page 286 in [16]). Our first result extends [16, Proposition 7.1] to the framework of abstract ε-subdifferentials. The natural replacement for the set defined in (4.1) is

$$\mathcal{S}_\varepsilon f(x_0) := \{h \in \operatorname{supp}(f, H, Z) : h(x_0) = f(x_0) - \varepsilon\}.$$

Note that, when $\varepsilon = 0$, we recover the set $\mathcal{S}_f(x_0)$.

Proposition 4.1. *Let $f : X \to \mathbb{R}_{+\infty}$, $Z \subset X$ and fix $x_0 \in Z$, $l \in L$ and $\varepsilon \geq 0$. Define $c := l(x_0) - f(x_0) + \varepsilon$ and $h(x) := l(x) - c$. Then, the following statements are equivalent;*

(a) *$l \in \partial_{L,Z,\varepsilon} f(x_0)$.*
(b) *$h \in \mathcal{S}_\varepsilon f(x_0)$.*

Items (a) or (b) are true for $\varepsilon = 0$ if and only if f is H-convex at x_0 and the supremum $\sup\{h(x_0) : h \in \operatorname{supp}(f, H, Z)\}$ is attained.

Proof. First note that $h(x_0) = f(x_0) - \varepsilon$. So

$$\begin{aligned} h \in \mathcal{S}_\varepsilon f(x_0) &\Leftrightarrow h(x) \leq f(x) \quad \forall\, x \in Z \\ &\Leftrightarrow l(x) - l(x_0) + f(x_0) - \varepsilon \leq f(x) \quad \forall\, x \in Z \\ &\Leftrightarrow l(x) - l(x_0) \leq f(x) - f(x_0) + \varepsilon \quad \forall\, x \in Z \\ &\Leftrightarrow l \in \partial_{L,Z,\varepsilon} f(x_0). \end{aligned}$$

The last statement of the proposition follows from Remark 4.1. □

Example 4.1. Let $X = \mathbb{R}$ and $Z = [-1, 1]$ and L as in Example 3.5. So $H := \{ax^2 + bx + c : a \leq 0\,,\, b, c \in \mathbb{R}\}$ is the associated set of affine functions. Consider the funcion $f(x) = -|x|$. Note that f is H-convex on Z, and $\partial_{L,Z} f(x) \neq \emptyset$ for every $x \neq 0$. This claim is true because for every fixed $Z \ni x \neq 0$ we can define $h_x = (a_x, b_x, c_x) := (\frac{-1}{2|x|}, 0, \frac{-|x|}{2})$, and it is easy to check that $h_x(t) \leq f(t)$ for all $t \in Z$ as well as the equality $h_x(x) = f(x)$. So for every $Z \ni x \neq 0$ we have $f(x) = \max\{h(x) : h \leq f\}$, the fact that the maximum is attained at h_x implies that $\partial_{L,Z} f(x) \neq \emptyset$ for every $Z \ni x \neq 0$. At $x = 0$ we have that $f(0) = \sup\{h_x(0) : x \in Z,\, x \neq 0\} = \sup\{\frac{-|x|}{2} : x \in Z,\, x \neq 0\} = 0$. So f is H-convex on Z. It can also be seen that $\partial_{L,Z} f(0) = \emptyset$. In fact, $\partial_{L,Z'} f(0) = \emptyset$ for every $Z' \subset \mathbb{R}$ which has 0 as an interior point. Given $\varepsilon > 0$, the abstract ε-subdifferential at 0 is the set of $l \in L$ such that

$$\varepsilon \geq |x| + ax^2 + bx,$$

for all $x \in [-1, 1]$. Call $M(a, b) := \max\{\sup_{x\in(-1,0)} ax^2 + (b - 1)x,\, \sup_{x\in(0,1)} ax^2 + (b + 1)x\}$. Note that $M(a, b) > 0$ for every $(a, b) \in \mathbb{R}_- \times \mathbb{R}$. It is direct from the definition that $(a, b) \in \partial_{L,Z,\varepsilon} f(0)$ if and only if $\varepsilon \geq M(a, b)$. In this case, the "$\Leftrightarrow$" correspondence used in the proof of Proposition 4.1 relates $(a, b) \in \partial_{L,Z,\varepsilon} f(0)$ with $\mathcal{S}_\varepsilon f(0) \ni h(x) = ax^2 + bx - \varepsilon \leq ax^2 + bx - M(a, b)$.

The next example is taken from [16, Example 7.6].

Example 4.2. Let X be a Banach space, Z a subset of X and fix $x_0 \in Z$. Let

$$L := \{a\|x - x_0\| : a \leq 0\}.$$

Given $f : Z \to \mathbb{R}$ define

$$\beta(f, x_0) := \inf_{x \in Z,\, x \neq x_0} \frac{f(x) - f(x_0)}{\|x - x_0\|}.$$

Assume f is Lipschitz on Z with the Lipschitz constant

$$\beta(f) := \sup_{x,y \in Z,\, x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}.$$

It is clear from the above definitions that $\beta(f) \geq \beta(f, x_0) \geq -\beta(f) > -\infty$. Identifying every $l \in L$ with the coefficient a such that $l(x) = a\|x - x_0\|$, it is easy to check that, when f is Lipschitz, the set $\partial_{L,Z} f(x_0)$ is not empty and

$$\partial_{L,Z} f(x_0) = \{a \in \mathbb{R} : a \leq \min\{0, \beta(f, x_0)\}\}.$$

Fix now $\varepsilon \geq 0$. Then $a \in \partial_{L,Z,\varepsilon} f(x_0)$ if and only if $a \leq 0$ and

$$a\|x - x_0\| \leq f(x) - f(x_0) + \varepsilon,$$

for all $x \in Z$. Therefore,

$$a \leq \frac{f(x) - f(x_0)}{\|x - x_0\|} + \frac{\varepsilon}{\|x - x_0\|}.$$

Which yields

$$a \leq \inf_{x \in Z,\, x \neq x_0} \frac{f(x) - f(x_0)}{\|x - x_0\|} + \frac{\varepsilon}{\|x - x_0\|} =: \beta_\varepsilon(f, x_0).$$

Hence we have

$$\partial_{L,Z,\varepsilon} f(x_0) = (-\infty, \min\{0, \beta_\varepsilon(f, x_0)\}] \supset (-\infty, \min\{0, \beta(f, x_0)\}] = \partial_{L,Z} f(x_0).$$

When x_0 is a minimizer of f at Z, then $\beta(f, x_0) \geq 0$ and hence the L-subdifferential at x_0 (as well as every ε-subdifferential) will coincide with the whole set L. Otherwise, there exists $\hat{x} \in Z$ such that $f(\hat{x}) - f(x_0) < 0$ and choosing $\varepsilon > 0$ such that $\varepsilon < f(x_0) - f(\hat{x})$ we will have $\beta(f, x_0) \leq \beta_\varepsilon(f, x_0) < 0$ and in this case

$$\partial_{L,Z,\varepsilon} f(x_0) = (-\infty, \beta_\varepsilon(f, x_0)] \supset (-\infty, \beta(f, x_0)] = \partial_{L,Z} f(x_0).$$

The correspondence defined in Proposition 4.1 sends $a \in \partial_{L,Z,\varepsilon} f(x_0)$ to $h(x) = a\|x - x_0\| + f(x_0) - \varepsilon \leq \beta_\varepsilon(f, x_0)\|x - x_0\| + f(x_0) - \varepsilon$.

4.1. **Maximal Elements.** For $X \supset Z$, let $L \subset \mathbb{R}^X$ be a set of elementary functions defined on X and H be the corresponding set of L-affine functions as in (2.1). Given a subset U of functions defined on Z, we say that $g \in U$ is a *maximal element* of the set U when

$$g' \in U, \quad g'(x) \geq g(x) \text{ for all } x \in Z \implies g' = g.$$

Proposition 8.4 in [16] establishes a bijection between maximal elements of $\operatorname{supp}(f, H, Z)$ and maximal elements of $\mathcal{D}f(x_0)$. Inspired by this result, we will establish a similar connection between maximal elements of $\mathcal{S}_\varepsilon f(x_0)$ and maximal elements of the set $\mathcal{D}_\varepsilon f(x_0)$ given in (2.6).

A careful inspection of Proposition 8.4 in [16] shows that maximal elements in $\mathcal{D}f(x_0)$ are in bijective correspondence with those maximal elements $h \in \operatorname{supp}(f, H, Z)$ which verify $h(x_0) = f(x_0)$. In other words, there is a one-to-one correspondence between maximal elements of the sets $\mathcal{S}_f(x_0)$ and $\mathcal{D}f(x_0)$. With this stronger statement in mind, our next result becomes Proposition 8.4 in [16] when $\varepsilon = 0$.

Proposition 4.2. *Let $f : X \to \mathbb{R}_{+\infty}$, $Z \subset X$ and fix $x_0 \in Z$, $l \in L$ and $\varepsilon \geq 0$. The following statements are equivalent;*

(a) *$h(\cdot) := l(\cdot) - l(x_0)$ is a maximal element of $\mathcal{D}_\varepsilon f(x_0)$.*

(b) *$h'(\cdot) := l(\cdot) - (l(x_0) - f(x_0) + \varepsilon)$ is a maximal element of $\mathcal{S}_\varepsilon f(x_0)$.*

Proof. Note that $h' = h + (f(x_0) - \varepsilon)$, it is immediate to see that $h \in \mathcal{D}_\varepsilon f(x_0) \Leftrightarrow h' \in \mathcal{S}_\varepsilon f(x_0)$. Hence $\mathcal{S}_\varepsilon f(x_0) = \mathcal{D}_\varepsilon f(x_0) + (f(x_0) - \varepsilon)$ and the conclusion follows. $\square$

Let us look now at the maximal elements of the previous examples.

Example 4.3. Consider again Example 4.1 and fix $\varepsilon > 0$. We have that $(a, b) \in \partial_{L,Z,\varepsilon} f(0)$ is maximal if and only if $M(a, b) = \varepsilon$. Indeed, assume first that $(a, b) \in \mathbb{R}_- \times \mathbb{R}$ is such that $M(a, b) = \varepsilon$ and take $(a', b') \in \partial_{L,Z,\varepsilon} f(0)$ such that

$$a'x^2 + b'x \geq ax^2 + bx, \qquad \text{for all } x \in [-1, 1]. \tag{4.2}$$

Since $(a', b') \in \partial_{L,Z,\varepsilon} f(0)$ we must have $M(a', b') \leq \varepsilon = M(a, b)$. On the other hand, (4.2) yields $M(a, b) \leq M(a', b')$. Hence we must have $M(a, b) = M(a', b')$. From this fact and (4.2) for the sequences $\{1/n\}$, $\{-1/n\} \in [-1, 1]$ we see that $b' = b$ and $a' \geq a$. Using the compacity of $[-1, 1]$ and the continuity of the functions involved in the definition of M, it is possible to prove that $a' = a$. Hence (a, b) is maximal. Conversely, assume $(a, b) \in \partial_{L,Z,\varepsilon} f(0)$ is maximal. We know that $\varepsilon \geq M(a, b)$ and we must show that $\varepsilon = M(a, b)$. Assume that $\varepsilon > M(a, b)$. It is possible to find $\delta > 0$ such that $\varepsilon > M(a + \delta, b) > M(a, b)$. The fact that $(a + \delta)x^2 + bx \gneqq ax^2 + bx$ for all $x \in [-1, 1] \setminus \{0\}$ contradicts the maximality of (a, b). For an example, assume that $\varepsilon = 2$. In this case $(0, -1) \in \partial_{L,Z,\varepsilon} f(0)$ and $M(a, b) = 2$. So $(0, -1)$ is a maximal element, and generates the element $h(x) = -x \in \mathcal{D}_\varepsilon f(x_0)$. By Proposition 4.2, h corresponds to the maximal element $h'(x) = -x - \varepsilon \in \mathcal{S}_\varepsilon f(x_0)$.

Example 4.4. Consider now Example 4.2. Let $f(x) = -\|x\|^2$ and take $x_0 = 0$. Note that $\beta(f, 0) = \inf_{x \in Z,\, x \neq 0} -\|x\|$ and $\beta_\varepsilon(f, 0) = \inf_{x \in Z,\, x \neq 0} -\|x\| + \varepsilon/\|x\|$. Hence when Z is unbounded, then both limiting values are $-\infty$ and hence both sets $\partial_{L,Z} f(0)$ and $\partial_{L,Z,\varepsilon} f(0)$ are empty. Let $B(0, 1)$ denote the unit ball of X and take $Z = B(0, 1)$. In this case we have that f is H-convex at 0 (but f is not H-convex at $Z \ni x \neq 0$) and the supremum is attained. We have that $\beta(f, 0) = -1$ and $\beta_\varepsilon(f, 0) = \varepsilon - 1$. So in order to have $\partial_{L,Z,\varepsilon} f(0) \subsetneqq L$ we consider $\varepsilon < 1$. Since $l(0) = 0$ we have that $\mathcal{D}_\varepsilon f(0) = \partial_{L,Z,\varepsilon} f(0)$. Because the set of linear functions is totally ordered the maximal element of $\partial_{L,Z,\varepsilon} f(0)$ is in fact a maximum and it is $l_M(x) = \beta_\varepsilon(f, 0)\|x\| = (\varepsilon - 1)\|x\|$. The corresponding element in $\mathcal{S}_\varepsilon f(0)$ is $h'(x) = (\varepsilon - 1)\|x\| - \varepsilon$.

The following example is inspired in [8].

Example 4.5. Let $Z = C$ be a cone in a vector space X with the order relation $\geq_K$ introduced by a convex cone $K \supset C$. For $y \in C, y \neq 0$ define the function $l_y : C \to \mathbb{R}_{+\infty}$ by $l_y(x) = \sup\{\alpha > 0 : \alpha y \leq_K x\}$ with the convention $\sup \emptyset = 0$. (If $X = \mathbb{R}^n$ and $C = K = \mathbb{R}^n_+$ then $l_y(x) = \min_{i : y_i > 0} \frac{x_i}{y_i}$.) Let $L = \{l_y : y \in C\} \cup \{0\}$. It is known [8] that a function $f : C \to \mathbb{R}_{+\infty}$ is H-convex if and only if f is increasing with respect to $\geq_K$ and the restriction of f on the ray $\{\alpha x : \alpha \geq 0\}$ for each $x \in C$ is a convex and lower semicontinuous function of one variable (these functions are called ICAR: increasing convex along-rays). Let f be an ICAR function and $x_0 \in C$ be a point such that $x_0 \in \operatorname{dom} f$. Then it can be proved (see [8]) that

(i) The function $f_{x_0}(\alpha) := f(\alpha x_0)$ is convex and $1 \in \operatorname{dom} f_{x_0}$.
(ii) The (classical convex) subdifferential $\partial f_{x_0}(1)$ of f_{x_0} is not empty. and it coincides with a closed segment, denote it by $[a, b]$.

A function $f : C \to \mathbb{R}_{+\infty}$ is called *strictly increasing at a point* $x \in C$ if $(y \in C, x <_K y) \implies f(x) < f(y)$. It has been proved in [8] that for the abstract subdifferential it holds

$$\partial_{L,C} f(x_0) \supset \{\tau l_{x_0} : \tau \in \partial f_{x_0}(1)\}. \tag{4.3}$$

If f is strictly increasing at x_0, then equality holds in (4.3). Assume that f is strictly increasing at x_0. Then f_{x_0} is also strictly increasing and hence we must have $b \geq a > 0$ in item (ii). Altogether, we get $\partial_{L,C} f(x_0) = \{\tau l_{x_0} : a \leq \tau \leq b\}$. Since l_{x_0} is nonnegative it follows that $b\, l_{x_0}$ is the maximal element of $\partial_{L,C} f(x_0)$ and each element of $\partial_{L,C} f(x_0)$ is majorized by this maximal element. We have also

$$\begin{aligned}\mathcal{D}f(x_0) &= \{h \in H : h(x) = \tau l_{x_0}(x) - \tau l_{x_0}(x_0) : \tau \in \partial f_{x_0}(1)\} \\ &= \{h \in H : h(x) = \tau(l_{x_0}(x) - 1) : \tau \in \partial f_{x_0}(1)\}.\end{aligned}$$

Note that the function l_{x_0} is positively homogeneous. Then for $x = \lambda x_0$, $\lambda > 0$ and $\tau \in [a, b]$ we have $h(\lambda x_0) = \tau \lambda l_{x_0}(x_0) - \tau = \tau(\lambda - 1)$. Thus $h(\lambda x_0) < 0$ for $\lambda < 1$ and $h(\lambda x_0) > 0$ for $\lambda > 1$, so h has both positive and negative values. This implies that every element of $\mathcal{D}_f(x_0)$ is maximal. Indeed, let $h \in \mathcal{D}f(x_0)$ and let $h' \in \mathcal{D}f(x_0)$ be an element such that $h' \geq h$. Then there exist $\tau \in \partial f_{x_0}(1)$ and $\tau' \in \partial f_{x_0}(1)$ such that $h(x) = \tau(l_{x_0}(x) - 1)$ and $h'(x) = \tau'(l_{x_0}(x) - 1)$. In particular $h(\lambda x_0) = \tau(\lambda - 1)$ and $h'(\lambda x_0) = \tau'(\lambda - 1)$. The inequality $h'(\lambda x_0) \geq h(\lambda x_0)$ is possible for all $\lambda > 0$ if and only if $\tau = \tau'$. Hence $h' = h$, so h is a maximal element of $\mathcal{D}f(x_0)$. Combining this fact with [16, Proposition 8.4] or Proposition 4.2 for the choice $\varepsilon = 0$ we conclude that every element $h(\cdot) = l(\cdot) - l(x_0) \in \mathcal{D}f(x_0)$ is in correspondence with the maximal element $h'(\cdot) = l(\cdot) - l(x_0) + f(x_0) \in \operatorname{supp}(f, H, C)$.

Theorem 8.5 in [16] establishes conditions under which every maximal element in the support set can be associated with a maximal element of $\mathcal{D}f(x_0)$ for some suitable $x_0 \in Z$. We extend this result in the Proposition below.

Proposition 4.3. *Let f be H-convex on Z. Assume that $(l, c) \in \operatorname{supp}(f, H, Z)$ is a maximal element. Then,*

(a) $c = f^*(l)$.
(b) *For every $\varepsilon > 0$ there exists $x_0 \in Z$ and $\hat{\varepsilon} \in [0, \varepsilon)$ such that $\hat{h}(x) := (l, l(x_0))$ is a maximal element of $\mathcal{D}_{\hat{\varepsilon}} f(x_0)$. As a consequence, $\hat{h}$ is a maximal element of $\mathcal{D}f(x_0)$.*
(c) *If there exists $x_0 \in Z$ such that $c = l(x_0) - f(x_0)$ then we can take $\hat{\varepsilon} = 0$ in (b). (this situation holds, for instance, when $\partial_L f : Z \rightrightarrows \operatorname{dom} f^*$ is onto).*

(d) *There exists* $x_0 \in Z$ *such that* $\hat{h}(x) := (l, l(x_0))$ *is a maximal element of* $\mathcal{D}_\varepsilon f(x_0)$ *for the choice* $\varepsilon := f(x_0) - l(x_0) + c$. *As a consequence,* $\hat{h}$ *is a maximal element of* $\mathcal{D} f(x_0)$.

Proof. (a) Since $\mathrm{supp}(f, H, Z) = \mathrm{epi}\, f^*$ we know that $f^*(l) \le c$. Suppose that for some $b \in \mathbb{R}$ we have $f^*(l) < b < c$. Then $(l, b) \in \mathrm{epi}\, f^* = \mathrm{supp}(f, H, Z)$ with (l, b) strictly greater than (l, c), contradicting the maximality of (l, c). This completes the proof of (a).

(b) By definition, we have $f^*(l) = \sup_{x \in Z} l(x) - f(x)$. Hence for every $\varepsilon > 0$, there exists $x_0 \in Z$ such that $c - \varepsilon < l(x_0) - f(x_0) \le c$. Call $\hat{\varepsilon} := c - l(x_0) + f(x_0)$. So we have $\hat{\varepsilon} \in [0, \varepsilon)$. Let us prove now that $\hat{h} = (l, l(x_0))$ is a maximal element of $\mathcal{D}_{\hat{\varepsilon}} f(x_0)$. Assume for this that $h' = (l', l'(x_0))$ verifies $h' \ge \hat{h}$ on Z with $l' \in \partial_{L,Z,\varepsilon} f(x_0)$. Note that $l \in \partial_{L,Z,\varepsilon} f(x_0)$. Indeed,

$$
\begin{aligned}
l(x) - l(x_0) = l(x) - c + c - l(x_0) &= l(x) - c + \hat{\varepsilon} - f(x_0) \\
&\le f(x) - f(x_0) + \hat{\varepsilon},
\end{aligned}
\tag{4.4}
$$

where we used the assumption $(l, c) \in \mathrm{supp}(f, H, Z)$. On the other hand, the inequality $h' \ge \hat{h}$ on Z can be rewritten as $l'(x) - l'(x_0) + f(x_0) \ge l(x) - l(x_0) + f(x_0) = l(x) - c + \hat{\varepsilon}$. Combining this with the fact that $l' \in \partial_{L,Z,\hat{\varepsilon}} f(x_0)$ we get

$$f(x) + \hat{\varepsilon} \ge l'(x) - l'(x_0) + f(x_0) \ge l(x) - c + \hat{\varepsilon},$$

which gives

$$f(x) \ge l'(x) - l'(x_0) + f(x_0) - \hat{\varepsilon} \ge l(x) - c.$$

So the middle function of the expression above belongs to $\mathrm{supp}(f, H, Z)$ and by maximality of (l, c) we must have $l(x) - c = l'(x) - l'(x_0) + f(x_0) - \hat{\varepsilon}$. Using now the definition of $\hat{\varepsilon}$ we can rewrite the last equality as $l(x) - l(x_0) = l'(x) - l'(x_0)$. This proves the maximality of $\hat{h}$ on the set $\mathcal{D}_{\hat{\varepsilon}} f(x_0)$. Since $\mathcal{D} f(x_0) \subset \mathcal{D}_{\hat{\varepsilon}} f(x_0)$ we have in particular the maximality on $\mathcal{D} f(x_0)$.

(c) If there exists $x_0 \in Z$ such that $c = l(x_0) - f(x_0)$, then by item (a) we must have $f^*(l) = l(x_0) - f(x_0) \ge l(x) - f(x)$ for all $x \in Z$. This yields $l \in \partial_{L,Z} f(x_0)$ and $\hat{h} = (l, l(x_0)) \in \mathcal{D} f(x_0)$. The maximality on the set $\mathcal{D} f(x_0)$ is now established in a way similar to item (b).

(d) We know that $\varepsilon = f(x_0) - l(x_0) + c \ge 0$. If $\varepsilon = 0$, we are in case (c), so we get the maximality of $\hat{h}$ on the set $\mathcal{D} f(x_0)$. If $\varepsilon > 0$, then we have

$$f(x) \ge l(x) - c = l(x) - l(x_0) + f(x_0) - \varepsilon,$$

so we get $l \in \partial_{L,Z,\varepsilon} f(x_0)$ and hence $(l, l(x_0)) \in \mathcal{D}_\varepsilon f(x_0)$. Assume $l' \in \partial_{L,Z,\varepsilon} f(x_0)$ is such that $l'(x) - l'(x_0) \ge l(x) - l(x_0)$ for all $x \in Z$. Again we can write $l'(x) - l'(x_0) + f(x_0) \ge l(x) - l(x_0) + f(x_0) = l(x) - c + \varepsilon$. Call $v(x) := l'(x) - l'(x_0) + f(x_0) - \varepsilon$. Since $l' \in \partial_{L,Z,\varepsilon} f(x_0)$ we have

$$f(x) + \varepsilon \ge l'(x) - l'(x_0) + f(x_0) \ge l(x) - c + \varepsilon.$$

So $f(x) \ge v(x) \ge l(x) - c$. Using again the maximality of (l, c) we get

$$v(x) = l'(x) - l'(x_0) + f(x_0) - \varepsilon = l(x) - c,$$

which yields $l(x) - l(x_0) = l'(x) - l'(x_0)$ and therefore the maximality of $\hat{h}$ is established. $\square$

Our next corollary re-writes the above proposition for the classical case.

Corollary 4.1. *Let $f : X \to \mathbb{R}_{+\infty}$ be convex and lower semicontinuous and denote by A the set of affine functions. Fix $(l, c) \in \operatorname{supp}(f, A)$. The following statements are equivalent.*

(a) *(l, c) is maximal in $\operatorname{supp}(f, A)$,*
(b) *$f^*(l) = c$,*
(c) *For every $\varepsilon > 0$ there exists $x_\varepsilon \in Z$ such that $l \in \partial_{Z,\varepsilon} f(x_\varepsilon)$ and $l(\cdot) - l(x_\varepsilon)$ is maximal in $\mathcal{D}_{\hat{\varepsilon}} f(x_\varepsilon)$, where $\hat{\varepsilon} := c - l(x_\varepsilon) + f(x_\varepsilon) \leq \varepsilon$.*

Moreover, the supremum in $f^*(l)$ is attained at some $x_0 \in Z$ if and only if $l \in \partial_Z f(x_0)$. In any case, we always have that $l \in \partial_{Z,\varepsilon} f(x_0)$ for $\varepsilon := f(x_0) - l(x_0) + c \geq 0$.

Proof. Part (a) implies (b) is Proposition 4.3(a). Part (b) implies (a) follows from the fact that

$$l'(x) - l(x) \geq c' - c \text{ for all } x \text{ if and only if } l' = l \text{ and } c' = c. \tag{4.5}$$

Part (b) implies (c) was proved in Proposition 4.3(b). Part (c) implies (a) again follows from (4.5). The first part of the last statement follows from Proposition 4.3(c), while the second part follows from Proposition 4.3(d). □

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Regina Sandra Burachik
University of South Australia, School of Mathematics and Statistics, South Australia, Australia
and
University of Ballarat, Centre for Informatics and Applied Optimization, Victoria, Australia

Alex Rubinov
University of Ballarat, Centre for Informatics and Applied Optimization, Victoria, Australia