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TWIN POSITIVE SOLUTIONS FOR p-LAPLACIAN NONLINEAR NEUMANN PROBLEMS VIA VARIATIONAL AND DEGREE THEORETICAL METHODS

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Abstract. We consider a nonlinear Neumann problem driven by the p-Laplacian and with a nonsmooth potential function (hemivariational inequality). Using a combination of variational and degree theoretic techniques, we show that the problem has two positive smooth solutions. We also show the equivalence of $W_n^{1,p}$ and C_n^1 minimizers for a large class of locally Lipschitz functionals.

1. Introduction

Let $Z \subseteq \mathbb{R}^{\mathbb{N}}$ be a bounded domain with a C^2-boundary ∂Z. In this paper, we investigate the existence of multiple positive solutions for the nonlinear Neumann problem with a nonsmooth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z,x(z)) \text{ a.e. on } Z, \\ \dfrac{\partial x}{\partial n} = 0 \text{ on } \partial Z, \quad 2 \leq p < \infty. \end{array} \right\} \tag{1.1}$$

In this problem the potential function $j(z,x)$ is jointly measurable and $x \to j(z,x)$ is locally Lipschitz and in general nonsmooth. By $\partial j(z,x)$ we denote the generalized subdifferential of $j(z,\cdot)$ (see Section 2).

Recently there have been some results on the existence of multiple nontrivial solutions for the Neumann problems driven by the p-Laplacian. We mention the works of Binding-Drabek-Huang [5], Ricceri [25], Faraci [12], Anello [4], Wu-Tan [28] for problems with a smooth potential (i.e. $j(z,\cdot) \in C^1(\mathbb{R})$) and of Filippakis-Gasinski-Papageorgiou [13], Motreanu-Papageorgiou [24] for problems with a nonsmooth potential (i.e. hemivariational inequalities). In the aforementioned works, the multiplicity results are obtained either by assuming certain symmetry structure on the potential function (see for example Anello [4], Ricceri [25], Filippakis-Gasinski-Papageorgiou [13]) or by requiring that $N < p$ (low dimensional problems) in which case the Sobolev space is embedded compactly in $C(\overline{Z})$ (see Faraci [12], Wu-Tan [28]). In Motreanu-Papageorgiou [24], the multiplicity result is for problems with a potential which is strictly p-sublinear near infinity and strictly p-superlinear near zero and their approach is variational based on a nonsmooth version of the local linking theorem (see Gasinski-Papageorgiou [15]). None of these multiplicity results provides information about the sign of the solutions. Only Binding-Drabek-Huang [5] examine a particular kind of nonlinear eigenvalue problem and for certain values

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of the parameter, they prove the existence of one or two positive solutions. Here the setting and the method of proof are different from those of Binding-Drabek-Huang [5]. Our approach combines variational and degree theoretic arguments. For the degree theoretic methods, we employ the degree map for certain multivalued perturbations of $(S)_+$-operators (see Hu-Papageorgiou [17]). This degree map was also used recently by the authors to prove an existence theorem for p-Laplacian Neumann hemivariational inequalities with an indefinite Euler functional (see Agarwal-Filippakis-O'Regan-Papageorgiou [1]).

2. Mathematical Background

In this section, for the convenience of the reader, we present some of the basic tools that are used in the analysis of problem (1.1).

Let X be a reflexive Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X, X^*). Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, the generalized directional derivative $\varphi^0(x; h)$ of φ at $x \in X$ in the direction h, is defined by

$$\varphi^0(x; h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $\varphi^0(x, \cdot)$ is sublinear, continuous and so by the Hahn-Banach theorem it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x) \subseteq X^*$, defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \le \varphi^0(x; h) \text{ for all } h \in X\}.$$

The multifunction $\partial\varphi : X \to 2^{X^*} \backslash \{\emptyset\}$ is called the "generalized subdifferential" of φ. If $\varphi : X \to \mathbb{R}$ is continuous convex, then it is well-known that φ is locally Lipschitz and the generalized subdifferential coincides with the subdifferential in the sense of convex analysis $\partial_c\varphi(x)$, defined by

$$\partial_c\varphi(x) = \{x^* \in X^* : \langle x^*, h \rangle \le \varphi(x + h) - \varphi(x) \text{ for all } h \in X\}.$$

Also, if $\varphi \in C^1(X)$, then φ is locally Lipschitz and

$$\partial\varphi(x) = \{\varphi'(x)\}.$$

If $\varphi, \psi : X \to \mathbb{R}$ are both locally Lipschitz and $\lambda \in \mathbb{R}$, then

$$\partial(\varphi + \psi)(x) \subseteq \partial\varphi(x) + \partial\psi(x) \text{ and } \partial(\lambda\varphi)(x) = \lambda\partial\varphi(x) \text{ for all } x \in X.$$

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, a point $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. It is easy to see that, if $x \in X$ is a local extremum of φ (i.e. a local minimum or a local maximum of φ), then $x \in X$ is a critical point of φ. For further details on these and related issues, we refer to Clarke [11].

If Y, V are Hausdorff topological spaces, a multifunction (set-valued function) $G : Y \to 2^V \setminus \{\emptyset\}$ is said to be upper semicontinuous (usc for short), if for every $C \subseteq V$ closed, the set

$$G^-(C) = \{y \in Y : G(y) \cap C \ne 0\}$$

is closed in Y. The generalized subdifferential $\partial\varphi : X \to 2^{X^*} \backslash \{\emptyset\}$ is usc from X with the norm topology into X^* with the weak topology (denoted by X^*_w). We say that the multifunction $G : X \to 2^{X^*}$ belongs in the "class (P)", if it has nonempty, closed and convex values, it is usc and for every bounded set $B \subseteq X$, we have that

$$G(B) = \bigcup_{x\in B} G(x)$$

is relatively compact in X^*.

Let $G : D \subseteq X \to 2^{X^*}\backslash\{\emptyset\}$ be an usc multifunction with closed and convex values. By virtue of a result of Cellina [9] (see also Hu-Papageorgiou [18], p.105), for every $\varepsilon > 0$, we can find a continuous map $g_\varepsilon : D \to X^*$ such that

$$\begin{aligned} g_\varepsilon(x) &\in G((x + B_\varepsilon) \cap D) + B^*_\varepsilon \text{ for all } x \in D \\ \text{and } g_\varepsilon(D) &\subseteq \overline{conv}G(D), \end{aligned}$$

where $B_\varepsilon = \{x \in X : \|x\| < \varepsilon\}$ and $B^*_\varepsilon = \{x^* \in X^* : \|x^*\| < \varepsilon\}$. Note that, if G belongs in the class (P), then g_ε is compact.

Recall that a map $A : X \to X^*$ is said to be of type $(S)_+$, if $x_n \xrightarrow{w} x$ in X and $\limsup_{n\to\infty}\langle A(x_n), x_n - x\rangle \leq 0$, imply $x_n \to x$ in X.

From Troyanski's renorming theorem (see for example Gasinski-Papageorgiou [16], p.911), we know that we can equivalently renorm X so that both X and X^* are locally uniformly convex with Frechet differentiable norms. So, in what follows, we assume that both spaces X and X^* are locally uniformly convex.

Let U be a bounded open set in X and $A : \overline{U} \to X^*$ is a demicontinuous operator of type $(S)_+$. Let $\{X_a\}_{a\in J}$ be the collection of all finite dimensional subspaces of X and by A_a we denote the Galerkin approximation of A with respect to X_a, that is

$$\langle A_a(x), y\rangle_{X_a} = \langle A(x), y\rangle \text{ for all } x \in \overline{U} \cap X_a \text{ and all } y \in X_a.$$

Here by $\langle\cdot,\cdot\rangle_{X_a}$ we denote the duality brackets for the pair (X^*_a, X_a).

If $0 \notin A(\partial U)$, $d_{(S)_+}(A, U, 0)$ is defined by

$$d_{(S)_+}(A, U, 0) = d_B(A, U \cap X_a, 0)$$

for X_a large enough (in the sense of inclusion). Here d_B stands for the classical finite dimensional Brouwer's degree. For details on the degree map $d_{(S)_+}$ we refer to Browder [7] and Skrypnik [26].

Note that, if $A : \overline{U} \to X^*$ is of type $(S)_+$ and $g : \overline{U} \to X^*$ is compact, then $A + g : \overline{U} \to X^*$ is still an $(S)_+$-map. Suppose $G : \overline{U} \to 2^{X^*}\backslash\{\emptyset\}$ is a (P)-multifunction and $0 \notin (A + G)(\partial U)$. Then $\widehat{d}(A + G, U, 0)$, is defined by

$$\widehat{d}(A + G, U, 0) = d_{(S)_+}(A + g_\varepsilon, U, 0)$$

for $\varepsilon > 0$ small, where g_ε is a continuous ε-approximate selector of the multifunction G described earlier.

This degree map has all the usual properties such as, normalization, domain additivity, homotopy invariance, excision property and solution property. We need to elaborate further on the normalization and homotopy invariance properties.

Let $\mathcal{F} : X \to X^*$ be the duality map of X, i.e.

$$\mathcal{F}(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

Since we have assumed that both X and X^* are uniformly convex, the duality map $\mathcal{F}$ is a homeomorphism and it is also bounded (i.e. it maps bounded sets to bounded ones), maximal monotone and of type $(S)_+$ (see Gasinski-Papageorgiou [16], p.316). The normalization property of the degree map $\widehat{d}$, has the form

$$\widehat{d}(\mathcal{F}, U, x^*) = d_{(S)_+}(\mathcal{F}, U, x^*) = 1 \text{ provided } x^* \in \mathcal{F}(U).$$

To formulate the homotopy invariance property, we need to specify the admissible homotopies for the maps A and G.

Definition 2.1. (a) A one-parameter family $\{A_t\}_{t\in[0,1]}$ of maps $A_t : \overline{U} \to X^*$ is said to be an "$(S)_+$-homotopy", if for any $\{x_n\}_{n\geq 1} \subseteq \overline{U}$ such that $x_n \xrightarrow{w} x$ in X and for any $\{t_n\}_{n\geq 1} \subseteq [0,1]$ with $t_n \to t$, for which

$$\limsup_{n\to\infty} \langle A_{t_n}(x_n), x_n - x\rangle \leq 0,$$

one has that $x_n \to x$ in X and $A_{t_n}(x_n) \xrightarrow{w} A_t(x)$ in X^*.

(b) A one-parameter family $\{G_t\}_{t\in[0,1]}$ of multifunctions $G_t : \overline{U} \to 2^{X^*}\backslash\{\emptyset\}$ is said to be a "homotopy of class (P)", if $(t,x) \to G(t,x)$ is usc from $[0,1]\times\overline{U}$ into $2^{X^*}\backslash\{\emptyset\}$ with closed convex values and

$$\overline{\bigcup\{G_t(x) : t\in[0,1],\ x\in\overline{U}\}} \text{ is compact in } X^*.$$

With these as admissible homotopies, the homotopy invariance property for the degree map $\widehat{d}$, reads as follows:

"If $\{A_t\}_{t\in[0,1]}$ is an $(S)_+$-homotopy with A_t bounded for every $t\in[0,1]$, $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) and $x^* : [0,1] \to X^*$ is a continuous map such that $x_t^* \notin (A_t + G_t)(\partial U)$ for all $t \in [0,1]$, then $\widehat{d}(A_t + G_t, U, x_t^*)$ is independent of $t\in[0,1]$".

Finally let us recall some basic facts about the spectrum of the negative p-Laplacian with Neumann boundary conditions. Let $m \in L^\infty(Z)_+$, $m \neq 0$ and consider the following nonlinear weighted (with weight m) eigenvalue problem:

$$\left\{\begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = \widehat{\lambda} m(z)|x(z)|^{p-2}x(z) \text{ a.e. on } Z, \\ \dfrac{\partial x}{\partial n} = 0 \text{ on } \partial Z,\ 1<p<\infty,\ \widehat{\lambda}\in\mathbb{R}. \end{array}\right\} \tag{2.1}$$

A $\widehat{\lambda} \in \mathbb{R}$ for which problem (2.1) has a nontrivial solution, is said to be an eigenvalue of $(-\triangle_p, W^{1,p}(Z), m)$ and the nontrivial solution is an eigenfunction corresponding to the eigenvalue $\widehat{\lambda}$. It is easy to see that a necessary condition for $\widehat{\lambda}$ to be an eigenvalue, is that $\widehat{\lambda} \geq 0$. Moreover, zero is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (the space of constant functions). The eigenvalue $\widehat{\lambda}_0(m) = 0$, is isolated and admits the following variational characterization

$$0 = \widehat{\lambda}_0(m) = \inf\left[\frac{\|Dx\|_p^p}{\int_Z m|x|^p dz} : x\in W^{1,p}(Z),\ x\neq 0\right]. \tag{2.2}$$

Clearly constant functions realize the infimum in (2.2). In addition to $\widehat{\lambda}_0(m) = 0$, the Lusternik-Schnirelmann theory, gives a whole strictly increasing sequence $\{\widehat{\lambda}_k = \widehat{\lambda}_k(m)\}_{k\geq 1} \subseteq \mathbb{R}_+$ of eigenvalues such that $\widehat{\lambda}_k \to +\infty$ as $k\to\infty$. These are the so-called "LS-eigenvalues". If $p = 2$ (linear eigenvalue problem), then these are all the eigenvalues. If $p \neq 2$ (nonlinear eigenvalue problem), we do not know if this is the case.

Nevertheless since $\widehat{\lambda}_0(m) = 0$ is isolated and the set of eigenvalues is closed, we can define

$$\widehat{\lambda}_1^* = \inf[\widehat{\lambda} : \widehat{\lambda} \text{ is an eigenvalue, } \widehat{\lambda} > 0] > 0.$$

This is the second eigenvalue of $(-\triangle_p, W^{1,p}(Z), m)$ and $\widehat{\lambda}_1^* = \widehat{\lambda}_1$. For details see Le [22] and Gasinski-Papageorgiou [16]. If $m \equiv 1$, we write $\widehat{\lambda}_k = \lambda_k$ for all $k \geq 0$.

3. Auxiliary Results

It is well-known (see Amann [3]), that if H is a Hilbert space, $\varphi \in C^1(H)$, $\nabla\varphi$ is a compact vector field and $x_0 \in H$ is an isolated local minimizer of φ, then we can find $r > 0$ small such that $d_{LS}(\nabla\varphi, B_r(x_0), 0) = 1$. Here d_{LS} denotes the Leray-Schauder degree. This result was extended to the $\widehat{d}$-degree map by Aizicovici-Papageorgiou-Staicu [2].

Let X be a reflexive Banach space which is embedded compactly and densely into $L^p(Z)$. Then $L^{p'}(Z) = L^p(Z)^*$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ is embedded compactly and densely into X^*. Suppose that $\theta \in C^1(X)$ and $A = \theta' : X \to X^*$ is a bounded $(S)_+$-map.

Also we consider a function $j_0 : Z \times \mathbb{R} \to \mathbb{R}$ satisfying:

$\underline{(H_0)}$
(i) for all $x \in \mathbb{R}$, $z \to j_0(z, x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to j_0(z, x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$, we have

$$|u| \leq a(z) + c|x|^{p-1} \text{ with } a \in L^\infty(Z)_+, c > 0.$$

We define the integral functional $\widehat{J}_0 : L^p(Z) \to \mathbb{R}$ by

$$\widehat{J}_0(x) = \int_Z j_0(z, x(z))dz \text{ for all } x \in L^p(Z). \tag{3.1}$$

Hypotheses (H_0) imply that $\widehat{J}_0$ is Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [11], p.83). A fortiori then, $J_0 = \widehat{J}_0|_X$ is locally Lipschitz too. Moreover, we have

$$\partial J_0(x) = \partial\widehat{J}_0(x) \subseteq L^{p'}(Z)$$

and $\partial J_0(x) = N_0(x) = \{u \in L^{p'}(Z) : u(z) \in \partial j_0(z, x(z)) \text{ a.e. on } Z\}$ for all $x \in X$ (see Clarke [11], pp.47 and 83). Exploiting the compact embedding of $L^{p'}(Z)$ into X^*, we can verify that $x \to N_0(x)$ is a multifunction of type (P). Hence we can define the $\widehat{d}$-degree of the map $x \to A(x) + N_0(x)$. The next theorem extends the above mentioned result of Amann [3] to the degree map $\widehat{d}$.

Theorem 3.1. *If X is a reflexive Banach space which is embedded compactly and densely in $L^p(Z)$ $(1 < p < \infty)$, $U \subseteq X$ is a nonempty open set, $\varphi = \theta + J_0 : X \to \mathbb{R}$ is locally Lipschitz with $\theta \in C^1(X)$, $A = \theta' : X \to X^*$ is bounded and of type $(S)_+$-type, $J_0 = \widehat{J}_0|_X$ with $\widehat{J}_0$ as in (3.1), $x_0 \in U$, $\xi, \mu, r \in \mathbb{R}$, $\xi < \mu$ and $r > 0$ satisfy*

(i) *$x_0 \in V = \{\varphi \leq \mu\} \cap U$ and $\{\varphi \leq \mu\} \cap \overline{U}$ is bounded subset of U;*

(ii) *If* $x \in \{\varphi \leq \xi\} \cap U$, *then* $tx_0 + (1-t)x \in V$ *for all* $t \in [0,1]$;

(iii) $0 \notin \partial\varphi(x)$ *for all* $x \in \{\xi \leq \varphi \leq \mu\} \cap \overline{U}$,

then $\widehat{d}(\partial\varphi, V, 0) = \widehat{d}(A + N_0, V, 0) = 1$.

In our analysis of problem (1.1) we will use the following two spaces:

$$W_n^{1,p}(Z) = \{x \in W^{1,p}(Z) : x = \lim_{k\to\infty} x_k \text{ in } W^{1,p}(Z),$$
$$x_k \in C^\infty(\overline{Z}), \frac{\partial x_k}{\partial n} = 0 \text{ on } \partial Z\}$$
$$\text{and } C_n^1(\overline{Z}) = \{x \in C^1(\overline{Z}) : \frac{\partial x}{\partial n} = 0 \text{ on } \partial Z\}.$$

Both are ordered Banach spaces with order cones given by

$$W_+ = \{x \in W_n^{1,p}(Z) : x(z) \geq 0 \text{ a.e. on } Z\}$$
$$\text{and } C_+ = \{x \in C_n^1(\overline{Z}) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\}.$$

Note that $\operatorname{int} C_+ \neq \emptyset$ and in fact

$$\operatorname{int} C_+ = \{x \in C_+ : x(z) > 0 \text{ for all } z \in \overline{Z}\}.$$

We introduce the operator $A : W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ defined by

$$\langle A(x), y\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } x, y \in W_n^{1,p}(Z).$$

The next three propositions were proved in [1]. For easy reference, we have included the results here.

Proposition 3.2. $A : W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ *is bounded demicontinuous monotone and of type* $(S)_+$.

Remark 3.3. Since A is demicontinuous monotone, it is maximal monotone (see Gasinski-Papageorgiou [16], p.310).

Proposition 3.4. *If* $m, m' \in L^\infty(Z)_+$, $m \neq 0$ *and* $m(z) < m'(z)$ *for a.a.* $z \in Z$, *then* $\widehat{\lambda}_1(m') < \widehat{\lambda}_1(m)$.

Proposition 3.5. *If* $\theta \in L^\infty(Z)$, $\theta(z) \leq 0$ *a.e. on* Z *and* $\theta \neq 0$, *then there exists* $\xi_0 > 0$ *such that*

$$\|Dx\|_p^p - \int_Z \theta(z)|x(z)|^p dz \geq \xi_0\|x\|^p \text{ for all } x \in W^{1,p}(Z).$$

The next result is of independent interest and is related to earlier results obtained by Brezis-Nirenberg [6], Garcia Azorero-Manfredi-Peral Alonso [14] and Kyritsi-Papageorgiou [20]. In Brezis-Nirenberg [6] $p = 2$ (semilinear case) and in Garcia Azorero-Manfredi-Peral Alonso [14] $p \neq 2$ (nonlinear case). In both works the potential is smooth and the boundary condition is Dirichlet. In Kyritsi-Papageorgiou [20], $p \geq 2$, the potential is nonsmooth and the boundary condition is Dirichlet (see also Gasinski-Papageorgiou [15], p.655). We introduce the following hypotheses:

$\underline{(H_1)}$ $\widehat{j} : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \to \widehat{j}(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to \widehat{j}(z,x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \le \widehat{a}(z) + \widehat{c}|x|^{r-1},$$

with $\widehat{a} \in L^\infty(Z)_+$, $\widehat{c} > 0$ and $1 < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \ge N \end{cases}$.

We consider the locally Lipschitz functional

$$\widehat{\varphi}(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z \widehat{j}(z, x(z))dz \text{ for all } x \in W_n^{1,p}(Z).$$

Proposition 3.6. *If $x_0 \in W_n^{1,p}(Z)$ is a local $C_n^1(\overline{Z})$-minimizer of $\widehat{\varphi}$, i.e. there exists $r > 0$ such that*

$$\widehat{\varphi}(x_0) \le \widehat{\varphi}(x_0 + h) \textit{ for all } h \in C_n^1(\overline{Z}),\ \|h\|_{C_n^1(\overline{Z})} \le r,$$

then $x_0 \in C_n^1(\overline{Z})$ and it is a local $W_n^{1,p}(Z)$-minimizer of $\widehat{\varphi}$, i.e. there exists $r' > 0$ such that

$$\widehat{\varphi}(x_0) \le \widehat{\varphi}(x_0 + h) \textit{ for all } h \in W_n^{1,p}(Z),\ \|h\| \le r'.$$

Proof. Take $h \in C_n^1(\overline{Z})$. Then for $\lambda > 0$ small, we have

$$\begin{aligned} &\widehat{\varphi}(x_0) \le \widehat{\varphi}(x_0 + \lambda h) \\ \Rightarrow &0 \le \widehat{\varphi}^0(x_0; h). \end{aligned} \tag{3.2}$$

Since $C_n^1(\overline{Z})$ is dense in $W_n^{1,p}(Z)$ and $\widehat{\varphi}^0(x_0; \cdot)$ is continuous on $W_n^{1,p}(Z)$, from (3.2) we infer that

$$\begin{aligned} &0 \le \widehat{\varphi}^0(x_0; h) \text{ for all } h \in W_n^{1,p}(Z), \\ \Rightarrow &0 \in \partial\widehat{\varphi}(x_0). \end{aligned} \tag{3.3}$$

From (3.3) it follows that

$$A(x_0) = u_0, \tag{3.4}$$

with $u_0 \in L^{r'}(Z)$ $(\frac{1}{r} + \frac{1}{r'} = 1)$, $u_0(z) \in \partial\widehat{j}(z, x_0(z))$ a.e. on Z. From the representation theorem for the elements of $W^{-1,p'}(Z) = W_0^{1,p}(Z)^*$ (see for example Gasinski-Papageorgiou [16], p.212), we know that

$$-\operatorname{div}(\|Dx_0\|^{p-2}Dx_0) \in W^{-1,p'}(Z). \tag{3.5}$$

We act on (3.4) with $v \in C_c^1(Z)$. Then

$$\langle A(x_0), v\rangle = \int_Z \|Dx_0\|^{p-2}(Dx_0, Dv)_{\mathbb{R}^N}dz = \int_Z u_0 v dz. \tag{3.6}$$

If by $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$ $(\frac{1}{p}+\frac{1}{p'}=1)$, then from the definition of the weak (distributional) derivative (3.5) and (3.6), we have

$$\langle -\operatorname{div}(\|Dx_0\|^{p-2}Dx_0), v\rangle_0 = \int_Z u_0 v dz = \langle u_0, v\rangle. \tag{3.7}$$

Since $v \in C_c^1(Z)$ is arbitrary and $C_c^1(Z)$ is dense in $W_0^{1,p}(Z)$, from (3.7) we infer that

$$-\operatorname{div}(\|Dx_0(z)\|^{p-2}Dx_0(z)) = u_0(z) \text{ a.e. on } Z.$$

Moreover, as in [1] (see the proof of Proposition 3.8), using the nonlinear Green's identity (see Kenmochi [19] and Casas-Fernandez [8]), we also obtain

$$\frac{\partial x_0}{\partial n} = 0 \text{ in } W^{-\frac{1}{p'},p'}(\partial Z). \tag{3.8}$$

Invoking Theorem 7.1 p.286, of Ladyzhenskaya-Uraltseva [21], we have $x_0 \in L^\infty(Z)$ and then Theorem 2 of Lieberman [23] can be used to conclude that $x_0 \in C_n^{1,\beta}(\overline{Z})$ for some $0<\beta<1$. So (3.8) can be interpreted in the pointwise sense. Hence $x_0 \in C_n^1(\overline{Z})$.

Now suppose that x_0 is not a local $W_n^{1,p}(Z)$ minimizer of $\widehat{\varphi}$. Because $\widehat{\varphi}$ is weakly lower semicontinuous on $W_n^{1,p}(Z)$ and the closed ε-ball $\overline{B}_\varepsilon = \{h \in W_n^{1,p}(Z) : \|h\| \leq \varepsilon\}$ is w-compact, by the Weierstrass theorem, for any $\varepsilon>0$ we can find $h_\varepsilon \in \overline{B}_\varepsilon$ such that

$$\widehat{\varphi}(x_0+h_\varepsilon) = \min[\widehat{\varphi}(x_0+h) : h \in \overline{B}_\varepsilon] < \widehat{\varphi}(x_0). \tag{3.9}$$

Applying the nonsmooth Lagrange multiplier rule of Clarke [10], we can find $\lambda_\varepsilon < 0$ such that

$$\lambda_\varepsilon \eta_\varepsilon'(h_\varepsilon) \in \partial\widehat{\varphi}(x_0+h_\varepsilon),$$

where $\eta_\varepsilon(h) = \frac{1}{p}(\|h\|^p - \varepsilon^p)$ (the constraint function). So

$$A(x_0+h_\varepsilon) - u_\varepsilon = \lambda_\varepsilon A(h_\varepsilon) + \lambda_\varepsilon K_p(h_\varepsilon), \tag{3.10}$$

with $u_\varepsilon \in L^{r'}(Z)$, $u_\varepsilon(z) \in \partial\widehat{j}(z,(x_0+h_\varepsilon)(z))$ a.e. on Z and $K_p : L^p(Z) \to L^{p'}(Z)$ is the bounded, continuous map defined by $K_p(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot)$ for all $x \in L^p(Z)$.

From (3.10) and (3.4), we have

$$\begin{aligned} & A(x_0+h_\varepsilon) - A(x_0) - \lambda_\varepsilon A(h_\varepsilon) = u_\varepsilon - u_0 + \lambda_\varepsilon K_p(h_\varepsilon) \\ & \quad \Rightarrow -\triangle_p(x_0+h_\varepsilon)(z) + \triangle_p x_0(z) + \lambda_p \triangle_p h_\varepsilon(z) \\ & \quad\quad = u_\varepsilon(z) - u_0(z) + \lambda_\varepsilon |h_\varepsilon(z)|^{p-2}h_\varepsilon(z) \text{ a.e. on } Z \text{ (as before).} \end{aligned} \tag{3.11}$$

We introduce the map $H : Z \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$H(z,\xi) = \|Dx_0(z)+\xi\|^{p-2}(Dx_0(z)+\xi) - \|Dx_0(z)\|^{p-2}Dx_0(z) - \lambda_\varepsilon \|\xi\|^{p-2}\xi.$$

Clearly $H(z,\xi)$ is a Caratheodory function (i.e. measurable in $z \in Z$ and continuous in $\xi \in \mathbb{R}^{\mathbb{N}}$) and it has a $(p-1)$-polynomial growth in $\xi \in \mathbb{R}^{\mathbb{N}}$. We rewrite (3.11) as

$$-\operatorname{div} H(z, Dh_\varepsilon(z)) = u_\varepsilon(z) - u_0(z) + \lambda_\varepsilon |h_\varepsilon(z)|^{p-2}h_\varepsilon(z) \text{ a.e. on } Z. \tag{3.12}$$

As before, using the nonlinear Green's identity and (3.10), (3.12), we obtain

$$\frac{\partial h_\varepsilon}{\partial n}(z) = 0 \text{ for all } z \in \partial Z.$$

Since $p \geq 2$ and $\lambda_\varepsilon \leq 0$

$$(H(z,\xi),\xi)_{\mathbb{R}^{\mathbb{N}}} \geq c_1\|\xi\|^p \text{ for a.a. } z \in Z, \text{ all } \xi \in \mathbb{R}^{\mathbb{N}} \text{ and some } c_1 > 0. \tag{3.13}$$

Then because of (3.12), (3.13), Theorem 7.1 p.286 of Ladyzhenskaya-Uraltseva [21] and Theorem 2 of Lieberman [23], we can find $\beta_0 \in (0,1)$ and $M_0 > 0$, both independent of $\varepsilon \in (0,1]$ and λ_ε such that

$$h_\varepsilon \in C_n^{1,\beta_0}(\overline{Z}) \text{ and } \|h_\varepsilon\|_{C_n^{1,\beta_0}(\overline{Z})} \leq M_0 \text{ for all } \varepsilon \in (0,1].$$

Let $\varepsilon \downarrow 0$ and set $h_n = h_{\varepsilon_n}$. Recalling that $C_n^{1,\beta_0}(\overline{Z})$ is embedded compactly in $C_n^1(\overline{Z})$, we may assume that

$$h_n \to \widehat{h} \text{ in } C_n^1(\overline{Z}) \text{ as } n \to \infty.$$

On the other hand

$$h_n \to 0 \text{ in } C_n^1(\overline{Z}) \text{ as } n \to \infty.$$

So $\widehat{h} = 0$. Then for $n \geq 1$ large, we have

$$\begin{aligned}&\|h_n\|_{C_n^1(\overline{Z})} \leq r\\ \Rightarrow &\widehat{\varphi}(x_0) \leq \widehat{\varphi}(x_0 + h_n),\end{aligned}$$

which contradicts (3.9). This completes the proof of the proposition □

4. Multiple Positive Solutions

The hypotheses on the nonsmooth potential function $j(z,x)$ are the following:

<u>(Hj)</u> $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(z,0) = 0$, $\partial j(z,0) \subseteq \mathbb{R}_+$ a.e. on Z and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, all $x \to j(z,x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $a_r \in L^\infty(Z)_+$ such that for a.a. $z \in Z$, all $|x| \leq r$ and all $u \in \partial j(z,x)$, we have $|u| \leq a_r(z)$;

(iv) there exists $\theta \in L^\infty(Z)$, $\theta(z) \leq 0$ a.e. on Z, $\theta \neq 0$ such that
$$\limsup_{x \to +\infty} \frac{u}{x^{p-1}} \leq \theta(z)$$
uniformly for a.a. $z \in Z$, all $u \in \partial j(z,x)$;

(v) there exist functions $\eta_1, \eta_2 \in L^\infty(Z)_+$ such that $\eta_1 \neq 0$, $\eta_1(z) \leq \eta_2(z) < \lambda_1$ a.e. on Z
$$\eta_1(z) \leq \liminf_{x \to 0^+} \frac{u}{x^{p-1}} \leq \limsup_{x \to 0^+} \frac{u}{x^{p-1}} \leq \eta_2(z)$$
uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$;

(vi) there exist $\overline{v}, \overline{M}, \overline{c} > 0$ such that

$$\int_Z j(z, \overline{v})dz > 0$$

$$\partial j(z,x) \subseteq \mathbb{R}_+ \text{ for a.a. } z \in Z, \text{ all } x \geq \overline{M}$$

$$\text{and } -\overline{c}x^{p-1} \leq u \text{ for a.a. } z \in Z, \text{ all } x \geq 0 \text{ and all } u \in \partial j(z,x).$$

Remark 4.1. Note that hypotheses $H(j)(iv), (v), (vi)$ all concern $j(z,x)$ for $x \geq 0$, so we may as well assume without any loss of generality that $j(z,x) = 0$ for a.a. $z \in Z$, all $x \leq 0$. Evidently hypotheses $H(j)(iv)$ and (v) are nonresonance conditions at $+\infty$ and at 0^+ with respect to the first two eigenvalues $\lambda_0 = 0 < \lambda_1$. More precisely, near $+\infty$ we allow partial interaction (nonuniform nonresonance) with $\lambda_0 = 0$ from the left, of the generalized slopes $\{\frac{u}{x^{p-1}} : u \in \partial j(z,x)\}$, while near 0^+, the generalized slopes remain in the spectral $[\lambda_0 = 0, \lambda_1]$, allowing partial interaction (nonuniform nonresonance) with $\lambda_0 = 0$, while we avoid completely $\lambda_1 > 0$ (uniform nonresonance). Note that as the variable $x \in \mathbb{R}_+$ moves from 0^+ to $+\infty$, the generalized slopes cross the principal eigenvalue (crossing nonlinearity).

In what follows $A : W_n^{1,p}(Z) \to W_n^{1,p}(Z)^*$ is the nonlinear, maximal monotone, $(S)_+$-operator introduced in Section 3 and $N : W_n^{1,p}(Z) \to 2^{L^{p'}(Z)} \backslash \{\emptyset\}$ is the multifunction defined by

$$N(x) = \{u \in L^{p'}(Z) : u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z\}.$$

As in [1] (see Proposition 3.2 and Corollary 3.3), we can prove the following result.

Proposition 4.2. *If hypotheses $H(j)(i), (ii), (iii)$ hold, then $N : W_n^{1,p}(Z) \to 2^{W_n^{1,p}(Z)^*} \backslash \{\emptyset\}$ is a multifunction of type (P).*

Also let $\varphi : W_n^{1,p}(Z) \to \mathbb{R}$ be the Euler functional for problem (1.1) defined by

$$\varphi(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j(z, x(z))dz \text{ for all } x \in W_n^{1,p}(Z).$$

We know that φ is locally Lipschitz.

In the next proposition, using a variational argument, we establish the existence of a positive solution for problem (1.1).

Proposition 4.3. *If hypotheses $H(j)$ hold, then problem (1.1) has a solution $x_0 \in \operatorname{int} C_+$ which is a local minimizer of φ.*

Proof. By virtue of hypothesis $H(j)(iv)$, given $\varepsilon > 0$, we can find $M_1 = M_1(\varepsilon) > 0$ such that for almost all $z \in Z$, all $x \geq M_1$ and all $u \in \partial j(z,x)$, we have

$$u \leq (\theta(z) + \varepsilon)x^{p-1}. \tag{4.1}$$

Also because of hypothesis $H(j)(iii)$ and since $j(z,x) = 0$ for a.a. $z \in Z$ and all $x \leq 0$, we have

$$|u| \leq a_{M_1}(z) \text{ for a.a. } z \in Z, \text{ all } x \leq M_1 \text{ and all } u \in \partial j(z,x). \tag{4.2}$$

From (4.1) and (4.2), we see that, if $x^+ = \max\{x, 0\}$, then

$$u \leq (\theta(z) + \varepsilon)(x^+)^{p-1} + \widehat{a}_\varepsilon(z) \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R}, \text{ all } u \in \partial j(z,x) \tag{4.3}$$

and with $\widehat{a}_\varepsilon \in L^\infty(Z)_+$. Let $N_0 \subseteq Z$ be the Lebesgue-null set such that for all $z \in Z \backslash N_0$, the function $x \to j(z,x)$ is locally Lipschitz (see hypothesis $H(j)(ii)$). By Rademacher's theorem $x \to j(z,x)$ is differentiable almost everywhere on $\mathbb{R}$. Moreover, if $z \in Z\backslash N_0$ and $r \in \mathbb{R}$ is a point of differentiability of $j(z,\cdot)$, we have

$$\frac{d}{dr} j(z,r) \in \partial j(z,r) \text{ (see Clarke [11],p.32).}$$

Integrating over $[0,x]$, $x > 0$ and using (4.3), we obtain

$$j(z,x) \leq \frac{1}{p}(\theta(z)+\varepsilon)(x^+)^p + a_\varepsilon(z)x^+ \text{ for all } z \in Z\backslash N_0, \text{ all } x \in \mathbb{R}. \tag{4.4}$$

Let $x \in W_+$. Then

$$\begin{aligned} \varphi(x) &= \frac{1}{p}\|Dx\|_p^p - \int_Z j(z,x(z))dz \\ &\geq \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}\int_Z \theta |x|^p dz - \frac{\varepsilon}{p}\|x\|_p^p - c_2\|x\| \\ &\quad \text{for some } c_2 > 0 \text{ (see (4.4))} \\ &\geq \frac{\xi_0 - \varepsilon}{p}\|x\|^p - c_2\|x\|. \end{aligned} \tag{4.5}$$

We choose $0 < \varepsilon < \xi_0$. Then from (4.5) it follows that $\varphi|_{W_+}$ is coercive. In addition, we can easily check that φ is weakly lower semicontinuous. So by virtue of the Weierstrass theorem, we can find $x_0 \in W_+$ such that

$$-\infty < m_+ = \inf_{W_+} \varphi = \varphi(x_0).$$

Let $\overline{c} > 0$ be as in hypothesis $H(j)(vi)$. Then

$$\varphi(\overline{c}) = -\int_Z j(z,\overline{c})dz < 0, \tag{4.6}$$

$$\Rightarrow m_+ = \varphi(x_0) < 0, \text{ i.e. } x_0 \neq 0. \tag{4.7}$$

Also from the optimality condition of Clarke [11], we have

$$0 \in \partial\varphi_+(x_0) + N_{W_+}(x_0), \tag{4.8}$$

where $N_{W_+}(x_0)$ is the normal cone to W_+ at x_0. Recall that

$$N_{W_+}(x_0) = \{x^* \in W_n^{1,p}(Z)^* : \langle x^*, w - x_0\rangle \leq 0 \text{ for all } w \in W_+\}. \tag{4.9}$$

From (4.8), we see that we can find $x^* \in \partial\varphi(x_0)$ such that

$$-x^* \in N_{W_+}(x_0).$$

We know that $x^* = A(x_0) - u_0$ with $u_0 \in N(x_0)$ and so

$$-A(x_0) + u_0 \in N_{W_+}(x_0),$$

$$\Rightarrow 0 \leq \langle A(x_0) - u_0, w - x_0\rangle \text{ for all } w \in W_+ \text{ (see (4.9)).} \tag{4.10}$$

Fix $\varepsilon > 0$ and $v \in W_n^{1,p}(Z)$, otherwise arbitrary and let

$$w = (x_0 + \varepsilon v)^+ = (x_0 + \varepsilon v) + (x_0 + \varepsilon v)^- \in W_+.$$

Using this as a test function in (4.10), we obtain

$$0 \leq \langle A(x_0) - u_0, \varepsilon v + (x_0 + \varepsilon v)^- \rangle$$

$$\Rightarrow - \langle A(x_0) - u_0, (x_0 + \varepsilon v)^- \rangle \leq \varepsilon \langle A(x_0) - u_0, v \rangle. \tag{4.11}$$

We set $Z_\varepsilon^- = \{z \in Z : (x_0 + \varepsilon v)(z) < 0\}$. We know that

$$D(x_0 + \varepsilon v)^-(z) = \begin{cases} -D(x_0 + \varepsilon v)(z) & \text{for a.a. } z \in Z_\varepsilon^- \\ 0 & \text{for a.a. } z \in Z \backslash Z_\varepsilon^- \end{cases}. \tag{4.12}$$

Then

$$- \langle A(x_0), (x_0 + \varepsilon v)^- \rangle + \int_Z u_0 (x_0 + \varepsilon v)^- dz \tag{4.13}$$

$$= - \int_Z \|Dx_0\|^{p-2} (Dx_0, D(x_0 + \varepsilon v)^-)_{\mathbb{R}^{\mathbb{N}}} dz + \int_Z u_0 (x_0 + \varepsilon v)^- dz.$$

Using (4.12), we have

$$\begin{aligned} & - \int_Z \|Dx_0\|^{p-2} (Dx_0, D(x_0 + \varepsilon v)^-)_{\mathbb{R}^{\mathbb{N}}} dz \\ & = \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2} (Dx_0, D(x_0 + \varepsilon v))_{\mathbb{R}^{\mathbb{N}}} dz \\ & \geq \varepsilon \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2} (Dx_0, Dv)_{\mathbb{R}^{\mathbb{N}}} dz. \end{aligned} \tag{4.14}$$

Moreover,

$$\begin{aligned} \int_Z u_0 (x_0 + \varepsilon v)^- dz &= - \int_{Z_\varepsilon^-} u_0 (x_0 + \varepsilon v) dz \\ &= - \int_{Z_\varepsilon^- \cap \{x_0 = 0\}} u_0 \varepsilon v dz - \int_{Z_\varepsilon^- \cap \{x_0 > 0\}} u_0 (x_0 + \varepsilon v) dz \end{aligned} \tag{4.15}$$

(recall $x_0 \in W_+$).

Recalling the definition of the set Z_ε^-, we see that $v(z) < 0$ a.e. on $Z_\varepsilon^- \cap \{x_0 = 0\}$. Also by hypothesis $\partial j(z, 0) \subseteq \mathbb{R}_+$ for a.a. $z \in Z$. Therefore $u_0(z) \geq 0$ a.e. on $Z_\varepsilon^- \cap \{x_0 = 0\}$. It follows then that

$$- \int_{Z_\varepsilon^- \cap \{x_0 = 0\}} u_0 \varepsilon v_0 dz \geq 0. \tag{4.16}$$

Also

$$\begin{aligned} & - \int_{Z_\varepsilon^- \cap \{x_0 > 0\}} u_0 (x_0 + \varepsilon v) dz \\ & = - \int_{Z_\varepsilon^- \cap \{0 < x_0 < \overline{M}\}} u_0 (x_0 + \varepsilon v) dz - \int_{Z_\varepsilon^- \cap \{x_0 \geq \overline{M}\}} u_0 (x_0 + \varepsilon v) dz \\ & \geq - \int_{Z_\varepsilon^- \cap \{0 < x_0 < \overline{M}\}} u_0 (x_0 + \varepsilon v) dz \text{ (see hypothesis } H(j)(vi)) \\ & \geq \int_{Z_\varepsilon^- \cap \{0 < x_0 < \overline{M}\}} a_{\overline{M}}(z) (x_0 + \varepsilon v) dz \text{ (see hypothesis } H(j)(iii)) \end{aligned} \tag{4.17}$$

$$\geq \varepsilon \int_{Z_\varepsilon^- \cap \{0<x_0<\overline{M}\}} a_{\overline{M}} v dz.$$

We use (4.16) and (4.17) in (4.15). Then

$$\int_Z u_0(x_0+\varepsilon v)^- dz \geq \varepsilon \int_{Z_\varepsilon^- \cap \{0<x_0<\overline{M}\}} a_{M_0} v dz. \tag{4.18}$$

We return to (4.13) and we use (4.14) and (4.18). We obtain

$$\begin{aligned} &- \langle A(x_0), (x_0+\varepsilon v)^- \rangle + \int_Z u_0(x_0+\varepsilon v)^- dz \\ &\qquad \geq \varepsilon \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2}(Dx_0, Dv)_{\mathbb{R}^{\mathbb{N}}} dz + \varepsilon \int_{Z_\varepsilon^- \cap \{0<x_0<\overline{M}\}} a_{M_0} v dz \\ &\Rightarrow \langle A(x_0) - u_0, v \rangle \geq \int_{Z_\varepsilon^-} \|Dx_0\|^{p-2}(Dx_0, Dv)_{\mathbb{R}^{\mathbb{N}}} dz + \int_{Z_\varepsilon^- \cap \{0<x_0<\overline{M}\}} a_{M_0} v dz \\ &\qquad\qquad \text{(see (4.11))}. \end{aligned}$$

From Stampacchia's theorem (see for example Gasinksi-Papageorgiou [16], pp. 195–196), we know that

$$Dx_0(z) = 0 \text{ a.e. on } \{x_0 = 0\}.$$

Hence

$$\begin{aligned} &\langle A(x_0) - u_0, v \rangle \\ &\qquad \geq \int_{Z_\varepsilon^- \cap \{0<x_0\}} \|Dx_0\|^{p-2}(Dx_0, Dv)_{\mathbb{R}^{\mathbb{N}}} dz + \int_{Z_\varepsilon^- \cap \{0<x_0<\overline{M}\}} a_M v dz. \end{aligned} \tag{4.19}$$

Note that, if by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^{\mathbb{N}}$, then

$$|Z_\varepsilon^- \cap \{0 < x_0 < \overline{M}\}|_N \leq |Z_\varepsilon^- \cap \{0 < x_0\}|_N \to 0 \text{ as } \varepsilon \downarrow 0.$$

So, if in (4.19) we pass to the limit as $\varepsilon \downarrow 0$, then

$$\langle A(x_0) - u_0, v \rangle \geq 0.$$

Recall that $v \in W_n^{1,p}(Z)$ was arbitrary. It follows that

$$A(x_0) = u_0, \ \ u_0 \in N(x_0). \tag{4.20}$$

Then as in [1] using the nonlinear Green's identity, from (4.20) we obtain

$$-\operatorname{div}(\|Dx_0(z)\|^{p-2}Dx_0(z)) = u_0(z) \text{ a.e. on } Z, \ \frac{\partial x_0}{\partial n} = 0 \text{ on } \partial Z. \tag{4.21}$$

As before nonlinear regularity theory implies that $x_0 \in C_+$. From (4.21) and hypothesis $H(j)(vi)$, we have

$$\operatorname{div}(\|Dx_0(z)\|^{p-2}Dx_0(z)) \leq \overline{c} x_0(z)^{p-1} \text{ a.e. on } Z.$$

Invoking the nonlinear strong maximum principle of Vazquez [27], we infer that $x_0 \in \operatorname{int} C_+$. So $x_0 \in \operatorname{int} C_+$ is a local $C_n^1(\overline{Z})$-minimizer of φ. By Proposition 3.6 it is also a local $W_n^{1,p}(Z)$-minimizer of φ. $\square$

Let $\varepsilon \in (0,1)$ and consider the functional $\varphi_\varepsilon : W_n^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_\varepsilon(x) = \frac{1}{p}\|Dx\|_p^p + \frac{\varepsilon}{p}\|x\|_p^p - \int_Z j(z, x(z))dz - \frac{\varepsilon}{p}\|x^+\|_p^p$$

for all $x \in W_n^{1,p}(Z)$.

Recall that $x^+ = \max\{x, 0\} \in W_n^{1,p}(Z)$. Since $x_0 \in \operatorname{int} C_+$, we can find $r_1 > 0$ such that

$$\varphi_\varepsilon|_{B_{r_1}^{C_n^1(\overline{Z})}(x_0)} = \varphi|_{B_{r_1}^{C_n^1(\overline{Z})}(x_0)},$$

where $B_{r_1}^{C_n^1(\overline{Z})}(x_0) = \{x \in C_n^1(\overline{Z}) : \|x - x_0\|_{C_n^1(\overline{Z})} < r_1\}$. This means that

$$\begin{aligned}
&x_0 \text{ is a local } C_n^1(\overline{Z})\text{-minimizer of } \varphi_\varepsilon \text{ (see Proposition 4.2)},\\
\Rightarrow &x_0 \text{ is a local } W_n^{1,p}(Z)\text{-minimizer of } \varphi_\varepsilon \text{ (see Proposition 3.6)},\\
\Rightarrow &0 \in \partial\varphi_\varepsilon(x_0).
\end{aligned}$$

Without loss of generality, we may assume that x_0 is an isolated local minimizer (and critical point) of the functional φ_ε. Indeed, if this is not the case, we can find $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ distinct from x_0 such that

$$0 \in \partial\varphi_\varepsilon(x_n) \text{ for all } n \geq 1 \text{ and } x_n \to x_0 \text{ in } W_n^{1,p}(Z) \text{ as } n \to \infty. \tag{4.22}$$

Let $K_p, K_p^+ : L^p(Z) \to L^{p'}(Z)$ be the bounded continuous maps defined by

$$K_p(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot) \text{ and } K_p^+(x)(\cdot) = (x^+(\cdot))^{p-1}.$$

Then from the inclusion in (4.22)

$$A(x_n) + \varepsilon K_p(x_n) = u_n + \varepsilon K_p^+(x_n) \text{ with } u_n \in N(x_n), \ \ n \geq 1. \tag{4.23}$$

On (4.23) we act with the test function $-x_n^- \in W_n^{1,p}(Z)$ and obtain

$$\begin{aligned}
&\|Dx_n^-\|_p^p + \varepsilon\|x_n^-\|_p^p = 0,\\
\Rightarrow &\varepsilon\|x_n^-\|^p = 0, \text{ i.e. } x_n \geq 0 \text{ for all } n \geq 1.
\end{aligned}$$

So (4.23) becomes

$$A(x_n) = u_n, \ \ n \geq 1.$$

From this as in the proof of Proposition 4.2, we infer that $x_n \in \operatorname{int} C_+$, $n \geq 1$. So

$$\begin{aligned}
&\varphi_\varepsilon(x_n) = \varphi(x_n) \text{ and } \partial\varphi_\varepsilon(x_n) = \partial\varphi(x_n) \text{ for all } n \geq 1,\\
\Rightarrow &\{x_n\}_{n\geq 1} \text{ is a sequence of nontrivial distinct critical points of } \varphi,\\
\Rightarrow &\{x_n\}_{n\geq 1} \text{ is a sequence of distinct positive solutions of (1.1) and so we are done.}
\end{aligned}$$

Therefore, without any loss of generality, we may assume that $x_0 \in \operatorname{int} C_+$ is an isolated local minimizer (and critical point) of φ_ε.

We know that

$$\partial\varphi_\varepsilon(x) = A(x) + \varepsilon K_p(x) - N(x) - \varepsilon K_p^+(x).$$

Note that due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we see that $K_p|_{W_0^{1,p}(Z)}$ and $K_p^+|_{W_0^{1,p}(Z)}$ are both completely continuous (hence compact too, see Gasinski-Papageorgiou [16], p.268) and so $x \to A(x) + \varepsilon K_p(x) - \varepsilon K_p^+$ is an

$(S)_+$-map. From Proposition 4.2, we know that N is a multifunction of type (P). Therefore we can speak about the $\widehat{d}$-degree of $\partial\varphi_\varepsilon$.

Proposition 4.4. *If hypotheses $H(j)$ hold and $x_0 \in \operatorname{int} C_+$ is as in Proposition 4.3, then we can find $r > 0$ such that*

$$\widehat{d}(\partial\varphi_\varepsilon, B_r(x_0), 0) = 1.$$

Proof. As we already noted above, we may assume that $x_0 \in \operatorname{int} C_+$ is an isolated local minimizer (and critical point) of φ_ε. Therefore there exists $r_0 > 0$ such that

$$\varphi_\varepsilon(x_0) < \varphi_\varepsilon(y) \text{ and } 0 \notin \partial\varphi_\varepsilon(y) \text{ for all } y \in \overline{B}_{r_0}(x_0)\backslash\{y_0\}, \tag{4.24}$$

where $\overline{B}_{r_0}(x_0) = \{x \in W_n^{1,p}(Z) : \|x - x_0\| \le r_0\}$.

Claim. *For all $0 < r < r_0$, we have*

$$\inf[\varphi_\varepsilon(x) : x \in \overline{B}_{r_0}(x_0)\backslash B_r(x_0)] > \varphi_\varepsilon(x_0). \tag{4.25}$$

Suppose that the Claim is not true. Then there exists $r \in (0, r_0)$ and a sequence $\{x_n\}_{n\ge1} \subseteq \overline{B}_{r_0}(x_0)\backslash B_r(x_0)$ such that

$$\varphi_\varepsilon(x_n) \downarrow \varphi_\varepsilon(x_0) \text{ as } n \to \infty. \tag{4.26}$$

Clearly $\{x_n\}_{n\ge1} \subseteq W_n^{1,p}(Z)$ is bounded. So we may assume that

$$x_n \xrightarrow{w} y \text{ in } W_n^{1,p}(Z),\ x_n \to y \text{ in } L^p(Z),\ x_n(z) \to y(z) \text{ a.e. on } Z,$$
$$\text{and } |x_n(z)| \le k(z) \text{ for a.a. } z \in Z, \text{ all } n \ge 1, \text{ with } k \in L^p(Z)_+.$$

The functional φ_ε is weakly lower semicontinuous. Hence

$$\varphi_\varepsilon(y) \le \lim_{n\to\infty} \varphi_\varepsilon(x_n) = \varphi_\varepsilon(x_0) \text{ (see (4.26))}.$$

Since $y \in \overline{B}_r(x_0)$, from (4.24) we infer that $y = x_0$.

Using the nonsmooth mean value theorem (see Clarke [11], p.41), we can find

$$w_n^* \in \partial\varphi_\varepsilon(t_n x_n + (1 - t_n)\frac{x_n + x_0}{2}),\ \ t_n \in (0, 1)\ \ n \ge 1$$

such that

$$\varphi_\varepsilon(x_n) - \varphi_\varepsilon(\frac{x_n + x_0}{2}) = \langle w_n^*, \frac{x_n - x_0}{2}\rangle\ \ n \ge 1.$$

We know that

$$\begin{aligned} w_n^* =& A(t_n x_n + (1 - t_n)\frac{x_n + x_0}{2}) + \varepsilon K_p(t_n x_n + (1 - t_n)\frac{x_n + x_0}{2}) - u_n \\ &- \varepsilon K_p^+(t_n x_n + (1 - t_n)\frac{x_n + x_0}{2}) \\ &\text{with } u_n \in N(t_n x_n + (1 - t_n)\frac{x_n + x_0}{2}). \end{aligned}$$

Therefore

$$\begin{aligned} \varphi_\varepsilon(x_n) - \varphi_\varepsilon(\frac{x_n + x_0}{2}) &= \frac{1}{2}\langle A(t_n x_n + (1 - t_n)\frac{x_n + x_0}{2}), x_n - x_0\rangle \\ + \frac{\varepsilon}{2}\int_Z |\lambda_n x_n + (1 - \lambda_n)&\frac{x_n + x_0}{2}|^{p-2}(\lambda_n x_n + (1 - \lambda_n)\frac{x_n + x_0}{2})(x_n - x_0)dz \end{aligned} \tag{4.27}$$

$$-\frac{1}{2}\int_Z u_n(x_n-x_0)dz$$
$$-\frac{\varepsilon}{2}\int_Z |(t_nx_n+(1-t_n)\frac{x_n+x_0}{2})^+|^{p-2}(t_nx_n+(1-t_n)\frac{x_n+x_0}{2})^+(x_n-x_0)dz.$$

Recall that $\varphi_\varepsilon(x_n)\to\varphi_\varepsilon(x_0)$ (see (4.26)) and because $\frac{x_n+x_0}{2}\xrightarrow{w} x_0$ in $W_n^{1,p}(Z)$ and φ_ε is weakly lower semicontinuous, we have

$$\varphi_\varepsilon(x_0)\le\liminf_{n\to\infty}\varphi_\varepsilon(\frac{x_n+x_0}{2}).$$

So, if in (4.27) we pass to the limit as $n\to\infty$, then

$$\limsup_{n\to\infty}\langle A(t_nx_n+(1-t_n)\frac{x_n+x_0}{2}),x_n-x_0\rangle\le 0 \tag{4.28}$$

We may assume that $t_n\to t\in[0,1]$ and so

$$t_nx_n+(1-t_n)\frac{x_n+x_0}{2}\xrightarrow{w} x_0 \text{ in } W_n^{1,p}(Z).$$

From (4.28), we have

$$\limsup_{n\to\infty}\langle A(t_nx_n+(1-t_n)\frac{x_n+x_0}{2}),t_nx_n+(1-t_n)\frac{x_n+x_0}{2}-x_0\rangle\le 0.$$

Since A is of type $(S)_+$ (see Proposition 3.2), it follows that

$$t_nx_n+(1-t_n)\frac{x_n+x_0}{2}\to x_0 \text{ in } W_n^{1,p}(Z). \tag{4.29}$$

However, note that

$$\|t_nx_n+(1-t_n)\frac{x_n+x_0}{2}-x_0\|=(1+t_n)\|\frac{x_n-x_0}{2}\|\ge\frac{r}{2}, \tag{4.30}$$

which of course contradicts (4.29). Therefore the Claim is true and (4.25) holds.

Set

$$\mu=\inf[\varphi_\varepsilon(x):x\in\overline{B}_{r_0}(x_0)\backslash B_{\frac{r_0}{2}}(x_0)]-\varphi_\varepsilon(x_0). \tag{4.31}$$

Because of (4.25), $\mu>0$. Also we set

$$V=\{x\in B_{\frac{r_0}{2}}(x_0):\varphi_\varepsilon(x)-\varphi_\varepsilon(x_0)<\mu\}. \tag{4.32}$$

Clearly the set V is open and $x_0\in V$. Let $r\in(0,\frac{r_0}{2})$ be such that $\overline{B}_r(x_0)\subseteq V$. Then we can apply Theorem 3.1 with the following data

$$U=B_{r_0}(x_0),\ \ \varphi=\varphi_\varepsilon-\varphi_\varepsilon(x_0),\ \ x_0,\ \ \mu>0 \text{ as above}$$
$$\text{and } 0<\xi<\inf[\varphi_\varepsilon(x):x\in B_{r_0}(x_0)-B_r(x_0)]-\varphi_\varepsilon(x_0) \text{ (see (4.25))}.$$

Indeed, note that because $r<\frac{r_0}{2}$, from (4.31) and (4.32), we have

$$\{x\in B_{r_0}(x_0):\varphi_\varepsilon(x)-\varphi_\varepsilon(x_0)\le\xi\}\subseteq B_r(x_0)\subseteq\overline{B}_r(x_0)\subseteq V.$$

Also, because of (4.24)

$$0\notin\partial\varphi_\varepsilon(x) \text{ for all } x\in\overline{B}_{r_0}(x_0) \text{ satisfying } \xi\le\varphi_\varepsilon(x)-\varphi_\varepsilon(x_0)\le\mu.$$

Therefore Theorem 3.1 can be applied and we have

$$\widehat{d}(\partial\varphi_\varepsilon,V,0)=1.$$

From the previous considerations, we have

$$0 \notin \partial\varphi_\varepsilon(\overline{V}\backslash B_r(x_0)).$$

Then, the excision property of the $\widehat{d}$-degree map, implies

$$\widehat{d}(\partial\varphi_\varepsilon, B_r(x_0), 0) = 1. \qquad \square$$

Next we compute the $\widehat{d}$-degree of $\partial\varphi_\varepsilon$ for large balls.

Proposition 4.5. *If hypotheses $H(j)$ hold, then there exists $R_0 > 0$ such that for all $R \geq R_0$*

$$\widehat{d}(\partial\varphi_\varepsilon, B_R, 0) = 1 \;\; (B_R = \{x \in W_n^{1,p}(Z) : \|x\| < R\}).$$

Proof. We consider the admissible homotopy $h_1 : [0,1] \times W_n^{1,p}(Z) \to 2^{W_n^{1,p}(Z)^*}\backslash\{\emptyset\}$ defined by

$$h_1(t,x) = A(x) + \varepsilon K_p(x) - tN(x) - t\varepsilon K_p^+(x).$$

Claim. *We can find $R_0 > 0$ such that $0 \notin h_1(t,x)$ for all $t \in [0,1]$ and all $\|x\| \geq R_0$.*

We proceed by a contradiction argument. So suppose that the Claim is not true. We can find $\{t_n\}_{n\geq1} \subseteq [0,1]$ and $\{x_n\}_{n\geq1} \subseteq W_n^{1,p}(Z)$ such that

$$t_n \to t \text{ in } [0,1], \;\|x_n\| \to \infty \text{ and } 0 \in h_1(t_n, x_n) \text{ for all } n \geq 1. \tag{4.33}$$

From the inclusion in (4.33), we have

$$A(x_n) + \varepsilon K_p(x_n) = t_n u_n + t_n\varepsilon K_p^+(x_n) \text{ with } u_n \in N(x_n),\; n \geq 1. \tag{4.34}$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_n^{1,p}(Z),\; y_n \to y \text{ in } L^p(Z),\;\; y_n(z) \to y(z) \text{ a.e. on } Z$$
$$\text{and } |y_n(z)| \leq k(z) \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \text{ with } k \in L^p(Z)_+.$$

We divide (4.34) by $\|x_n\|^{p-1}$. Then

$$A(y_n) + \varepsilon K_p(y_n) = t_n\frac{u_n}{\|x_n\|^{p-1}} + t_n\varepsilon K_p^+(y_n)\;\; n \geq 1. \tag{4.35}$$

Note that hypotheses $H(j)(iii), (iv), (vi)$, imply

$$|u| \leq \widetilde{a}(z) + \widetilde{c}|x|^{p-1} \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and all } u \in \partial j(z,x),$$

with $\widetilde{a} \in L^\infty(Z)_+$, $\widetilde{c} > 0$. So it follows that

$$\{h_n = \frac{u_n}{\|x_n\|^{p-1}}\}_{n\geq1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

Hence we may assume that

$$h_n \xrightarrow{w} h \text{ in } L^{p'}(Z).$$

Arguing as in the proof of Proposition 3.7 in [1], we show that

$$h(z) = g(z)y^+(z)^{p-1}$$

with $g \in L^\infty(Z)$, $-\overline{c} \leq g(z) \leq \theta(z)$ a.e. on Z, $\overline{c} > 0$ as in hypothesis $H(j)(vi)$.

Moreover, acting on (4.35) with $y_n - y$ and passing to the limit, we have

$$\lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0,$$

$$\Rightarrow y_n \to y \text{ in } W_n^{1,p}(Z) \text{ (see Proposition 3.2), hence } \|y\| = 1.$$

So, if we pass to the limit as $n \to \infty$ in (4.35), then

$$A(y) + \varepsilon K_p(y) = t(g + \varepsilon) K_p^+(y). \tag{4.36}$$

On (4.36), we act with the test function $-y^- \in W_n^{1,p}(Z)$. Then

$$\varepsilon \|y^-\|^p = 0, \text{ i.e. } y^- = 0 \text{ and so } y \geq 0, \quad y \neq 0.$$

Hence (4.36) becomes

$$A(y) + \varepsilon K_p(y) = t(g + \varepsilon) K_p(y). \tag{4.37}$$

We act with $y \in W_n^{1,p}(Z)$ and so

$$\begin{aligned} & \|Dy\|_p^p + \varepsilon \|y\|_p^p \leq t\varepsilon \|y\|_p^p \leq \varepsilon \|y\|_p^p \text{ (since } g \leq 0, t \in [0,1]), \quad (4.38) \\ \Rightarrow & \|Dy\|_p = 0, \\ \Rightarrow & y = \xi \in \mathbb{R}, \xi > 0 \text{ (since } y \geq 0, \ y \neq 0). \end{aligned}$$

If $t = 0$, then from (4.38) we have

$$\varepsilon \|y\|^p = 0, \text{ i.e. } y = 0, \text{ a contradiction.}$$

If $0 < t \leq 1$, then from (4.37) we have

$$0 \leq t\xi^p \Big(\int_Z g(z) dz + \varepsilon |Z|_N \Big).$$

Choosing $\varepsilon > 0$ small, we have $\int_Z g(z)dz + \varepsilon |Z|_N < 0$, a contradiction. This proves the Claim.

The Claim permits the use of the homotopy invariance property. Hence

$$\widehat{d}(\partial \varphi_\varepsilon, B_R, 0) = d_{(S)_+}(A + \varepsilon K_p, B_R, 0) \text{ for all } R \geq R_0. \tag{4.39}$$

But from the proof of Proposition 3.7 in [1], we have

$$d_{(S)_+}(A + \varepsilon K_p, B_R, 0) = 1 \text{ for all } R > 0. \tag{4.40}$$

From (4.39) and (4.40), we conclude that

$$\widehat{d}(\partial \varphi_\varepsilon, B_R, 0) = 1 \text{ for all } R \geq R_0. \qquad \square$$

Next we perform a similar computation for small balls.

Proposition 4.6. *If hypotheses $H(j)$ hold, then there exists $\rho_0 > 0$ such that for all $0 < \rho \leq \rho_0$*

$$\widehat{d}(\partial \varphi_\varepsilon, B_\rho, 0) = 1 \ \ (B_\rho = \{x \in W_n^{1,p}(Z) : \|x\| < \rho\})$$

Proof. We fix $\eta \in L^\infty(Z)_+$ such that

$$\eta_1(z) \leq \eta(z) \leq \eta_2(z) \text{ a.e. on and } \underset{Z}{essinf}\, \eta \geq \gamma > 0.$$

We consider the admissible homotopy $h_2 : [0,1] \times W_n^{1,p}(z) \to 2^{W_n^{1,p}(Z)} \setminus \{\emptyset\}$ defined by

$$h_2(t,x) = A(x) + \varepsilon K_p(x) - (1-t)\eta K_p^+(x) - tN(x) - t\varepsilon K_p^+(x).$$

Claim. *We can find $\rho_0 > 0$ such that $0 \notin h_2(t,x)$ for all $t \in [0,1]$ and all $0 < \|x\| \leq \rho_0$.*

We argue indirectly. So suppose that the Claim is not true. We can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W_n^{1,p}(Z)$ such that

$$t_n \to t \in [0,1], \ \ \|x_n\| \to 0 \text{ and } 0 \in h_2(t_n, x_n) \text{ for all } n \geq 1. \tag{4.41}$$

The inclusion in (4.41) implies that

$$A(x_n) + \varepsilon K_p(x_n) = (1-t_n)\eta K_p^+(x_n) + t_n u_n + t_n \varepsilon K_p^+(x_n) \text{ with } u_n \in N(x_n). \tag{4.42}$$

We set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. Since $\|y_n\| = 1$ for all $n \geq 1$, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_n^{1,p}(Z), \ y_n \to y \text{ in } L^p(Z), \ \ y_n(z) \to y(z) \text{ a.e. on } Z$$
$$\text{and } |y_n(z)| \leq k(z) \text{ for a.a. } z \in Z, \text{ all } n \geq 1, \text{ with } k \in L^p(Z)_+.$$

From (4.42), we obtain

$$A(y_n) + \varepsilon K_p(y_n) = (1-t_n)\eta K_p^+(y_n) + t_n \frac{u_n}{\|x_n\|^{p-1}} + t_n \varepsilon K_p^+(y_n). \tag{4.43}$$

Note that by virtue of hypothesis $H(j)(v)$, we can find $\widetilde{\eta} \in L^\infty(Z)_+ \backslash \{\emptyset\}$ such that

$$|u| \leq \widetilde{\eta}(z)|x|^{p-1} \text{ for a.a. } z \in Z, \text{ all } x \leq \delta \text{ and all } u \in \partial j(z,x). \tag{4.44}$$

On the other hand from the proof of Proposition 4.5, we have

$$|u| \leq \widetilde{a}(z) + \widetilde{c}|x|^{p-1} \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and all } u \in \partial j(z,x),$$

with $\widetilde{a} \in L^\infty(Z)_+$, $\widetilde{c} > 0$. Hence

$$|u| \leq (\frac{\widetilde{a}(z)}{\delta^{p-1}} + \widetilde{c})|x|^{p-1} \text{ for a.a. } z \in Z, \text{ all } x \geq \delta \text{ and all } u \in \partial j(z,x). \tag{4.45}$$

Combining (4.44) and (4.45), we infer that

$$|u| \leq \overline{\eta}(z)|x|^{p-1} \text{ for a.a. } z \in Z, \text{ all } x \in \mathbb{R} \text{ and all } u \in \partial j(z,x),$$

with $\overline{\eta} \in L^\infty(Z)_+$. Therefore

$$|u_n(z)| \leq \overline{\eta}(z)|x_n(z)|^{p-1} \text{ a.e. on } Z,$$
$$\Rightarrow \{\frac{u_n}{\|x_n\|^{p-1}}\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

So, we may assume that

$$\widehat{h}_n = \frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} \widehat{h} \text{ in } L^{p'}(Z).$$

Arguing as in the proof of Proposition 3.8 in [1], we show that

$$\widehat{h}(z) = \widehat{g}(z) y^+(z)^{p-1} \text{ a.e. on } Z,$$

with $\widehat{g} \in L^\infty(Z)_+$, $\eta_1(z) \leq \widehat{g}(z) \leq \eta_2(z)$ a.e. on Z. Moreover, as before, acting on (4.43) with $y_n - y$, passing to the limit as $n \to \infty$ and using the $(S)_+$-property of A, we obtain

$$y_n \to y \text{ in } W_n^{1,p}(Z), \ \ \|y\| = 1.$$

From (4.43), in the limit as $n \to \infty$, we have

$$A(y) + \varepsilon K_p(y) = (1-t)\eta K_p^+(y) + t(\widehat{g} + \varepsilon)K_p^+(y),$$
$$\Rightarrow A(y) + \varepsilon K_p(y) = (\xi + t\varepsilon)K_p^+(y) \text{ with } \xi = (1-t)\eta + t\widehat{g} \in L^\infty(Z)_+. \tag{4.46}$$

We act with the test function $-y^- \in W^{1,p}_n(Z)$. Then

$$\begin{aligned}&\varepsilon\|y^-\|^p = 0\\ \Rightarrow &y^- = 0 \text{ and so } y \geq 0,\ y \neq 0.\end{aligned}$$

So (4.46) becomes

$$A(y) = (\xi - (1-t)\varepsilon)K_p(y).$$

As before, using the nonlinear Green's identity, we have

$$\left\{\begin{array}{l} -\operatorname{div}(\|Dy(z)\|^{p-2}Dy(z)) = (\xi(z) - (1-t)\varepsilon)|y(z)|^{p-2}y(z) \text{ a.e. on } Z, \\ \dfrac{\partial y}{\partial n} = 0 \text{ on } \partial Z. \end{array}\right\} \tag{4.47}$$

Nonlinear regularity theory implies $y \in C_+$. We choose $\varepsilon < \gamma \leq essinf\, \eta$. Then $\xi - (1-t)\varepsilon \geq 0$, $\xi - (1-t)\varepsilon \neq 0$ and so $\widehat{\lambda}_0(\xi - (1-t)\varepsilon) = 0$. Moreover, Proposition 3.4 implies

$$\widehat{\lambda}_1(\xi - (1-t)\varepsilon) \geq \widehat{\lambda}_1(\xi) > \widehat{\lambda}_1(\lambda_1) = 1. \tag{4.48}$$

From (4.47) and (4.48), we infer that $y = 0$, a contradiction to the fact that $\|y\| = 1$. This proves the Claim.

By homotopy invariance, we have

$$\widehat{d}(\partial\varphi_\varepsilon, B_\rho, 0) = \widehat{d}(A + \varepsilon K_p - \eta K_p^+, B_\rho, 0) \text{ for all } 0 < \rho \leq \rho_0. \tag{4.49}$$

We need to compute $\widehat{d}(A + \varepsilon K_p - \eta K_p^+, B_\rho, 0)$. To this end we consider the $(S)_+$-homotopy $h_3 : [0,1] \times W^{1,p}_n(Z) \to W^{1,p}_n(Z)^*$

$$h_3(t,x) = A(x) + \varepsilon K_p(x) - t\eta K_p^+(x).$$

Suppose that for $t \in [0,1]$ and $x \neq 0$, we have

$$h_3(t,x) = 0,$$

$$\Rightarrow A(x) + \varepsilon K_p(x) = t\eta K_p^+(x). \tag{4.50}$$

We act on (4.50) with $-x^- \in W^{1,p}_n(Z)$. Then

$$\begin{aligned}&\varepsilon\|x^-\|^p = 0\\ \Rightarrow &x \geq 0,\quad x \neq 0.\end{aligned}$$

So from (4.50) we have

$$A(x) + \varepsilon K_p(x) = t\eta K_p(x).$$

If $t = 0$, then

$$\begin{aligned}&A(x) + \varepsilon K_p(x) = 0,\\ \Rightarrow &x = 0, \text{ a contradiction.}\end{aligned}$$

If $0 < t \leq 1$, then

$$A(x) = (t\eta - \varepsilon)K_p(x),$$

$$\Rightarrow \left\{\begin{array}{l} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) = (t\eta(z) - \varepsilon)|x(z)|^{p-2}x(z) \quad \text{a.e. on } Z, \\ \dfrac{\partial y}{\partial n} = 0. \end{array}\right\} \tag{4.51}$$

Choose $0 < \varepsilon < t\gamma$. Then $t\eta(\cdot) - \varepsilon \in L^\infty(Z)_+$, $t\eta - \varepsilon \neq 0$ and $\widehat{\lambda}_0(t\eta - \varepsilon) = 0$. Also Proposition 3.4 implies

$$\widehat{\lambda}_1(t\eta - \varepsilon) > \widehat{\lambda}_1(\lambda_1) = 1.$$

Because of (4.51) we infer $y = 0$, a contradiction. Therefore by homotopy invariance

$$d_{(S)_+}(A + \varepsilon K_p - \eta K_p^+, B_\rho, 0) = d_{(S)_+}(A + \varepsilon K_p, B_\rho, 0) = 1 \text{ for all } \rho > 0, \tag{4.52}$$
$$\Rightarrow \widehat{d}(\partial\varphi_\varepsilon, B_\rho, 0) = 1 \text{ for all } 0 < \rho \leq \rho_0 \text{ (see (4.49) and (4.52)).}$$ □

Now we are ready to prove the multiplicity result for the positive solutions of (1.1).

Theorem 4.7. *If hypotheses $H(j)$ hold, then problem (1.1) has at least two solutions $x_0, \widehat{x} \in \operatorname{int} C_+$.*

Proof. From Proposition 4.3 we already have one solution $x_0 \in \operatorname{int} C_+$.

Let $0 < \rho \leq \rho_0$, $R \geq R_0$ and $r > 0$ be such that

$$B_r(x_0) \cap B_\rho = \emptyset \text{ and } \overline{B}_r(x_0) \subseteq B_R.$$

Then from Propositions 4.4, 4.5, 4.6 and the domain additivity and excision properties of the degree map, we have

$$\widehat{d}(\partial\varphi_\varepsilon, B_R, 0) = \widehat{d}(\partial\varphi_\varepsilon, B_\rho, 0) + \widehat{d}(\partial\varphi_\varepsilon, B_r(x_0), 0) + \widehat{d}(\partial\varphi_\varepsilon, B_R \backslash (\overline{B_r(x_0) \cup B_\rho}), 0),$$
$$\Rightarrow \widehat{d}(\partial\varphi_\varepsilon, B_R \backslash (\overline{B_r(x_0) \cap B_\rho}), 0) = -1.$$

By virtue of the solution property, we can find $\widehat{x} \in B_R \backslash (\overline{B_r(x_0) \cap B_\rho}), 0)$, hence $\widehat{x} \neq x_0$, $\widehat{x} \neq 0$, such that

$$A(\widehat{x}) + \varepsilon K_p(\widehat{x}) = \widehat{u} + \varepsilon K_p^+(\widehat{x}) \text{ with } \widehat{u} \in N(\widehat{x}). \tag{4.53}$$

We act with the test function $-\widehat{x}^- \in W_n^{1,p}(Z)$. Recalling that $j(z,x) = 0$ for a.a. $z \in Z$ and all $x \leq 0$, we obtain

$$\varepsilon \|\widehat{x}\|^p = 0, \text{ i.e. } \widehat{x} \geq 0, \ \widehat{x} \neq 0.$$

So (4.53) becomes

$$A(\widehat{x}) = \widehat{u} \text{ with } \widehat{u} \in N(\widehat{x}),$$
$$\Rightarrow \left\{ \begin{array}{l} -\operatorname{div}(\|D\widehat{x}(z)\|^{p-2} D\widehat{x}(z)) = \widehat{u}(z) \text{ a.e. on } Z, \\ \dfrac{\partial \widehat{x}}{\partial n} = 0 \text{ on } \partial Z. \end{array} \right\}$$

From nonlinear regularity theory and the nonlinear strong maximum principle of Vazquez [27], we conclude that $\widehat{x} \in \operatorname{int} C_+$. □

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