Journal of Nonlinear and Convex Analysis
Volume 8, Number 3, 2007, 471–489

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON A STRONGLY NONEXPANSIVE SEQUENCE IN HILBERT SPACES

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Abstract. In order to discuss some weak convergence theorems for nonexpansive mappings, we introduce a notion of a sequence of nonexpansive mappings, which is called a strongly nonexpansive sequence. We investigate some properties of such sequences and prove weak convergence theorems. Then we deal with some applications for monotone operators.

1. Introduction

The purpose of this paper is to introduce a notion of a sequence of nonexpansive mappings, which is called a *strongly nonexpansive sequence*, and to discuss its properties and applications.

The notion of strong nonexpansiveness for a single mapping was introduced and studied by Bruck and Reich [7]; see also Browder [3]. Every firmly nonexpansive mapping [6], which is an important mapping in nonlinear analysis, is strongly nonexpansive. Strongly nonexpansive mappings have several nice properties, for example, the composition of two such mappings is also strongly nonexpansive; see [7] for more details.

On the other hand, in the study of approximating fixed points of nonexpansive mappings, we often treat some kind of sequence of mappings. The property of such sequences, as well as that of each mapping, is important in the study; see [1, 2].

This paper is constructed as follows: In §3, motivated by results mentioned above, we discuss a strong nonexpansiveness for a sequence of nonexpansive mappings. Roughly speaking, we see that the strong nonexpansiveness is preserved under the composition, and that such a sequence can be constructed for an arbitrary sequence of nonexpansive mappings. Then, in §4, we prove some weak convergence theorems for a strongly nonexpansive sequence. Finally, in §5, we apply our results obtained in §3 and §4 to some problems for monotone operators.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of positive integers, H a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, $F(T)$ the set of fixed points of a mapping $T\colon C \to H$, where C is a nonempty subset of H. Strong convergence of a sequence $\{x_n\}$ to x is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let C be a nonempty subset of H and T a mapping of C into H. A mapping T is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. It is known

2000 *Mathematics Subject Classification.* Primary 47H09, 47H10, 41A65.

Key words and phrases. Strongly nonexpansive sequence, nonexpansive mapping, weak convergence theorem, monotone operator.

that if C is a closed convex subset of H and T is nonexpansive, then $F(T)$ is closed and convex, and moreover, $I - T$ is demiclosed, that is, $u \in F(T)$ whenever $\|x_n - Tx_n\| \to 0$ and $x_n \rightharpoonup u$. A mapping T is said to be strongly nonexpansive [7] if T is nonexpansive and

$$\lim_{n\to\infty} \|x_n - y_n - (Tx_n - Ty_n)\| = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$. A mapping T is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|x - y - (Tx - Ty)\|^2$$

for all $x, y \in C$; see, for example, [9]. It is obvious that every firmly nonexpansive mapping is strongly nonexpansive.

Let C be a nonempty closed convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive and

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \tag{2.1}$$

holds for all $x \in H$ and $y \in C$; see [9, 15].

Let $\alpha > 0$ be a given constant. A mapping $A\colon C \to H$ is said to be α-inverse-strongly-monotone if $\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. It is known that

$$\|Ax - Ay\| \leq \frac{1}{\alpha} \|x - y\| \tag{2.2}$$

for all $x, y \in C$ if A is α-inverse-strongly-monotone; see, for example, [5, 17].

Let B be a mapping of H into 2^H, where 2^H denotes the set of all subsets of H. Such a mapping B is said to be a multi-valued mapping on H. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator B' on H. It is known that, for a maximal monotone operator B on H and $r > 0$, we may define a single-valued mapping $(I + rB)^{-1}\colon H \to \mathrm{dom}(B)$; see [10], [4] and [14]. Such a mapping $(I + rB)^{-1}$ is called the resolvent of B for r.

Let B be a maximal monotone operator on H and $B^{-1}0 = \{x \in H : Bx \ni 0\}$. It is known that the resolvent $(I + rB)^{-1}$ is firmly nonexpansive and $B^{-1}0 = F((I + rB)^{-1})$ for all $r > 0$. It is also known that

$$\frac{x - J_\lambda x}{\lambda} \in AJ_\lambda x \tag{2.3}$$

and

$$\|J_\lambda x - J_\mu x\| \leq \frac{|\lambda - \mu|}{\lambda} \|x - J_\lambda x\| \tag{2.4}$$

hold for all $\lambda, \mu > 0$ and $x \in H$, where $J_\lambda = (I + \lambda B)^{-1}$; see [9, 15, 8] for more details.

We use the following lemmas:

Lemma 2.1 ([18]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers. If $\sum_{n=1}^{\infty} b_n < \infty$ and $a_{n+1} \le a_n + b_n$ for every $n \in \mathbb{N}$, then $\{a_n\}$ is convergent.*

Lemma 2.2 ([17]). *Let F be a nonempty closed convex subset of a Hilbert space H and P_F the metric projection of H onto F. Let $\{x_n\}$ be a sequence in H such that*

$$\|x_{n+1} - u\| \le \|x_n - u\|$$

for all $u \in F$ and $n \in \mathbb{N}$. Then $\{P_F x_n\}$ converges strongly.

It is known that all Hilbert spaces satisfy Opial's condition [11], that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\|$$

if $x_n \rightharpoonup u$ and $u \neq v$. It is also known that

$$\lambda(1-\lambda)\|x-y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \|\lambda x + (1-\lambda)y\|^2 \tag{2.5}$$

holds for all $x, y \in H$ and $\lambda \in \mathbb{R}$; see, for instance, [16].

3. Strongly nonexpansive sequences

In this section, we introduce the definition of a strongly nonexpansive sequence. Then we show several examples and investigate some properties of such sequences.

Let C be a nonempty subset of a Hilbert space H. A sequence $\{T_n\}$ of mappings of C into H is said to be a strongly nonexpansive sequence if each T_n is nonexpansive and

$$\lim_{n\to\infty} \|x_n - y_n - (T_n x_n - T_n y_n)\| = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are two sequences in C such that $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0$.

Example 3.1. Let $T\colon C \to H$ be a strongly nonexpansive mapping. Put $T_n = T$ for $n \in \mathbb{N}$. Then it is obvious that $\{T_n\}$ is a strongly nonexpansive sequence.

Example 3.2. Let $\{T_n\}$ be a sequence of firmly nonexpansive mappings of C into H. Then it is clear that $\{T_n\}$ is a strongly nonexpansive sequence. In particular, if B is a maximal monotone operator on H, then $\{(I + r_n B)^{-1}\}$ is a strongly nonexpansive sequence, where $\{r_n\}$ is a sequence of positive real numbers. Furthermore, a sequence $\{P_{C_n}\}$ of metric projections is a strongly nonexpansive sequence, where $\{C_n\}$ is a sequence of nonempty closed convex subset of H.

Example 3.3. Let A be an α-inverse-strongly-monotone mapping of C into H. It is known that

$$\begin{aligned} &\|(I-\lambda A)x - (I-\lambda A)y\|^2 \\ &\quad \le \|x-y\|^2 - \lambda(2\alpha - \lambda)\|Ax - Ay\|^2 \\ &\quad = \|x-y\|^2 - \frac{2\alpha-\lambda}{\lambda}\|x - y - ((I-\lambda A)x - (I-\lambda A)y)\|^2 \end{aligned} \tag{3.1}$$

holds for all $x, y \in C$ and $\lambda > 0$. This shows that a mapping $I - \lambda A$ is nonexpansive for all $\lambda \in (0, 2\alpha]$. Let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < \inf_{n\in\mathbb{N}} \lambda_n \le \sup_{n\in\mathbb{N}} \lambda_n < 2\alpha$. Then the inequality (3.1) also implies that $\{I - \lambda_n A\}$ is a strongly nonexpansive sequence.

Let $\{T_n\}$ be a sequence of mappings of C into H. A sequence $\{z_n\}$ in C is said to be an approximate fixed point sequence of $\{T_n\}$ if $\|z_n - T_n z_n\| \to 0$. The set of all bounded approximate fixed point sequences of $\{T_n\}$ is denoted by $\tilde{F}(\{T_n\})$, that is,

$$\tilde{F}(\{T_n\}) = \left\{ \{z_n\} : \sup_{n \in \mathbb{N}} \|z_n\| < \infty,\ \|z_n - T_n z_n\| \to 0,\ z_n \in C \text{ for all } n \in \mathbb{N} \right\}.$$

Clearly, if $\{T_n\}$ has a common fixed point, then all bounded sequences in the common fixed point set are approximate fixed point sequences of $\{T_n\}$.

The composition of two strongly nonexpansive sequences is a strongly nonexpansive sequence as follows:

Theorem 3.4. *Let C and D be nonempty subsets of a Hilbert space H. Let $\{S_n\}$ be a sequence of mappings of D into H and $\{T_n\}$ a sequence of mappings of C into H. Suppose that both $\{S_n\}$ and $\{T_n\}$ are strongly nonexpansive sequences and $T_n(C) \subset D$ for each $n \in \mathbb{N}$. Then $\{S_n T_n\}$ is a strongly nonexpansive sequence.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\|x_n - y_n\| - \|S_n T_n x_n - S_n T_n y_n\| \to 0. \tag{3.2}$$

From the assumption that each S_n and T_n are nonexpansive, we have

$$0 \le \|x_n - y_n\| - \|T_n x_n - T_n y_n\| \le \|x_n - y_n\| - \|S_n T_n x_n - S_n T_n y_n\|$$

for every $n \in \mathbb{N}$. Therefore, from (3.2), we see that

$$\|x_n - y_n\| - \|T_n x_n - T_n y_n\| \to 0.$$

Since $\{T_n\}$ is a strongly nonexpansive sequence, we obtain

$$\|x_n - y_n - (T_n x_n - T_n y_n)\| \to 0. \tag{3.3}$$

Clearly we have

$$\begin{aligned} 0 &\le \|T_n x_n - T_n y_n\| - \|S_n T_n x_n - S_n T_n y_n\| \\ &\le \|x_n - y_n\| - \|S_n T_n x_n - S_n T_n y_n\|. \end{aligned}$$

It follows from (3.2) that

$$\|T_n x_n - T_n y_n\| - \|S_n T_n x_n - S_n T_n y_n\| \to 0.$$

Since $\{T_n x_n - T_n y_n\}$ is bounded and $\{S_n\}$ is a strongly nonexpansive sequence, we obtain

$$\|T_n x_n - T_n y_n - (S_n T_n x_n - S_n T_n y_n)\| \to 0. \tag{3.4}$$

Therefore, from (3.3) and (3.4), we have

$$\begin{aligned} &\|x_n - y_n - (S_n T_n x_n - S_n T_n y_n)\| \\ &\qquad \le \|x_n - y_n - (T_n x_n - T_n y_n)\| + \|T_n x_n - T_n y_n - (S_n T_n x_n - S_n T_n y_n)\| \to 0 \end{aligned}$$

as $n \to \infty$. This completes the proof. □

Remark 3.5. Bruck and Reich [7] showed that the composition of two strongly nonexpansive mappings is strongly nonexpansive; see [7, Proposition 1.1].

In order to examine a property of approximate fixed point sequences under composition, we need the following lemma:

Lemma 3.6. *Let C and D be nonempty subsets of a Hilbert space H. Let $\{S_n\}$ be a sequence of nonexpansive mappings of C into H and $\{T_n\}$ a sequence of mappings of D into H. If $T_n(D) \subset C$ for every $n \in \mathbb{N}$, then $\tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \subset \tilde{F}(\{S_nT_n\})$.*

Proof. Suppose that $\tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\})$ is nonempty. Let $\{w_n\} \in \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\})$ be fixed. Note that $w_n \in C \cap D$ for every $n \in \mathbb{N}$. Since each S_n is nonexpansive, it follows that

$$\begin{aligned}\|w_n - S_nT_nw_n\| &\leq \|w_n - S_nw_n\| + \|S_nw_n - S_nT_nw_n\| \\ &\leq \|w_n - S_nw_n\| + \|w_n - T_nw_n\| \to 0.\end{aligned}$$

Therefore $\{w_n\} \in \tilde{F}(\{S_nT_n\})$. □

Theorem 3.7. *Let C and D be nonempty subsets of a Hilbert space H. Let $\{S_n\}$ be a strongly nonexpansive sequence of mappings of C into H and $\{T_n\}$ a sequence of nonexpansive mappings of D into H. Suppose that*

$$\tilde{F} = \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset$$

and $T_n(D) \subset C$ for every $n \in \mathbb{N}$. Then $\tilde{F} = \tilde{F}(\{S_nT_n\})$.

Proof. By assumption and Lemma 3.6, we see that $\emptyset \neq \tilde{F} \subset \tilde{F}(\{S_nT_n\})$. Thus we have only to show $\tilde{F} \supset \tilde{F}(\{S_nT_n\})$. Let $\{z_n\} \in \tilde{F}(\{S_nT_n\})$ and $\{w_n\} \in \tilde{F}$ be fixed. Since both S_n and T_n are nonexpansive, it follows that

$$\begin{aligned}\|T_nz_n - T_nw_n\| &\leq \|z_n - w_n\| \\ &\leq \|z_n - S_nT_nz_n\| + \|S_nT_nz_n - S_nT_nw_n\| \\ &\qquad + \|S_nT_nw_n - S_nw_n\| + \|S_nw_n - w_n\| \\ &\leq \|S_nT_nz_n - S_nT_nw_n\| + \beta_n,\end{aligned}$$

where $\beta_n = \|z_n - S_nT_nz_n\| + \|T_nw_n - w_n\| + \|S_nw_n - w_n\|$. This implies that

$$0 \leq \|T_nz_n - T_nw_n\| - \|S_nT_nz_n - S_nT_nw_n\| \leq \beta_n \to 0.$$

Since $\{T_nz_n - T_nw_n\}$ is bounded in C and $\{S_n\}$ is a strongly nonexpansive sequence, it follows that

$$\|T_nz_n - T_nw_n - (S_nT_nz_n - S_nT_nw_n)\| \to 0.$$

Therefore we obtain that

$$\begin{aligned}\|z_n - T_nz_n\| &= \|z_n - S_nT_nz_n + S_nT_nz_n - S_nT_nw_n - (T_nz_n - T_nw_n) \\ &\qquad + S_nT_nw_n - S_nw_n + S_nw_n - w_n + w_n - T_nw_n\| \\ &\leq \|z_n - S_nT_nz_n\| + \|S_nT_nz_n - S_nT_nw_n - (T_nz_n - T_nw_n)\| \\ &\qquad + \|S_nT_nw_n - S_nw_n\| + \|S_nw_n - w_n\| + \|w_n - T_nw_n\| \\ &\leq \|z_n - S_nT_nz_n\| + \|S_nT_nz_n - S_nT_nw_n - (T_nz_n - T_nw_n)\| \\ &\qquad + \|T_nw_n - w_n\| + \|S_nw_n - w_n\| + \|w_n - T_nw_n\| \to 0\end{aligned}$$

as $n \to \infty$. Thus $\{z_n\} \in \tilde{F}(\{T_n\})$. From this fact, we also obtain

$$\|z_n - S_nz_n\| \leq \|z_n - S_nT_nz_n\| + \|S_nT_nz_n - S_nz_n\|$$

$$\leq \|z_n - S_nT_nz_n\| + \|T_nz_n - z_n\| \to 0$$

as $n \to \infty$. Thus $\{z_n\} \in \tilde{F}(\{S_n\})$ and hence $\tilde{F}(\{S_nT_n\}) \subset \tilde{F}$. □

Using Theorem 3.7, we directly obtain the following:

Corollary 3.8. *Let C and D be nonempty subsets of a Hilbert space H. Let $\{S_n\}$ be a strongly nonexpansive sequence of mappings of D into H and $\{T_n\}$ a sequence of nonexpansive mappings of C into H. Suppose that*

$$F = \bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$$

and $T_n(D) \subset C$ for every $n \in \mathbb{N}$. Then $\tilde{F}(\{S_nT_n\}) = \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset$.

Proof. Let $\{w_n\}$ be a bounded sequence in F. Then it is clear that $\{w_n\} \in \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset$. Therefore we obtain the conclusion from Theorem 3.7. □

As a special case of Corollary 3.8, we get an improvement of [7, Lemma 2.1].

Corollary 3.9. *Let C and D be nonempty subsets of a Hilbert space H. Let S be a strongly nonexpansive mapping of C into H and T a nonexpansive mapping of D into H. Suppose that $F(S) \cap F(T) \neq \emptyset$ and $T(D) \subset C$. Then $F(S) \cap F(T) = F(ST)$.*

As in the proof of Theorem 3.7, we obtain the following:

Theorem 3.10. *Let C and D be nonempty subsets of a Hilbert space H. Let $\{S_n\}$ be a strongly nonexpansive sequence of mappings of C into H and $\{T_n\}$ a sequence of nonexpansive mappings of D into H. Suppose that*

$$\tilde{F} = \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset$$

and $S_n(C) \subset D$ for every $n \in \mathbb{N}$. Then $\tilde{F} = \tilde{F}(\{T_nS_n\})$.

For the remainder of this section we observe some properties concerning the convex combination of sequences of mappings.

Theorem 3.11. *Let C be a nonempty subset of a Hilbert space H. Let $\{S_n\}$ be a strongly nonexpansive sequence of mappings of C into H and $\{T_n\}$ a sequence of nonexpansive mappings of C into H. Let $\{\lambda_n\}$ be a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \lambda_n > 0$. Let $\{U_n\}$ be a sequence of mappings of C into H defined by $U_n = \lambda_nS_n + (1-\lambda_n)T_n$ for $n \in \mathbb{N}$. Then $\{U_n\}$ is a strongly nonexpansive sequence.*

Proof. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in C such that $\{x_n - y_n\}$ is bounded and

$$\|x_n - y_n\| - \|U_nx_n - U_ny_n\| \to 0. \tag{3.5}$$

Since each S_n and T_n are nonexpansive, we have

$$\begin{aligned}
\|U_nx_n - U_ny_n\| &= \|\lambda_n(S_nx_n - S_ny_n) + (1-\lambda_n)(T_nx_n - T_ny_n)\| \\
&\leq \lambda_n \|S_nx_n - S_ny_n\| + (1-\lambda_n)\|T_nx_n - T_ny_n\| \\
&\leq \lambda_n \|S_nx_n - S_ny_n\| + (1-\lambda_n)\|x_n - y_n\| \\
&\leq \|x_n - y_n\|.
\end{aligned}$$

This yields that

$$0 \leq \lambda_n(\|x_n - y_n\| - \|S_n x_n - S_n y_n\|) \leq \|x_n - y_n\| - \|U_n x_n - U_n y_n\|$$

for every $n \in \mathbb{N}$. Therefore it follows from (3.5) that

$$\|x_n - y_n\| - \|S_n x_n - S_n y_n\| \to 0. \tag{3.6}$$

Since $\{S_n\}$ is a strongly nonexpansive sequence, we have

$$\|x_n - y_n - (S_n x_n - S_n y_n)\| \to 0. \tag{3.7}$$

On the other hand, from (2.5), we have

$$\begin{aligned}
&\lambda_n(1-\lambda_n)\|S_n x_n - S_n y_n - (T_n x_n - T_n y_n)\|^2 \\
&\quad = \lambda_n \|S_n x_n - S_n y_n\|^2 + (1-\lambda_n)\|T_n x_n - T_n y_n\|^2 \\
&\qquad - \|\lambda_n(S_n x_n - S_n y_n) + (1-\lambda_n)(T_n x_n - T_n y_n)\|^2 \\
&\quad = \lambda_n \|S_n x_n - S_n y_n\|^2 + (1-\lambda_n)\|T_n x_n - T_n y_n\|^2 - \|U_n x_n - U_n y_n\|^2 \\
&\quad \leq \lambda_n \|S_n x_n - S_n y_n\|^2 + (1-\lambda_n)\|x_n - y_n\|^2 - \|U_n x_n - U_n y_n\|^2 \\
&\quad = \lambda_n(\|S_n x_n - S_n y_n\|^2 - \|x_n - y_n\|^2) + \|x_n - y_n\|^2 - \|U_n x_n - U_n y_n\|^2.
\end{aligned}$$

Since both $\{S_n x_n - S_n y_n\}$ and $\{U_n x_n - U_n y_n\}$ are bounded, it follows from (3.5) and (3.6) that

$$\lambda_n(1-\lambda_n)\|S_n x_n - S_n y_n - (T_n x_n - T_n y_n)\|^2 \to 0.$$

Thus, by $\liminf_{n\to\infty} \lambda_n > 0$, we get

$$(1-\lambda_n)\|S_n x_n - S_n y_n - (T_n x_n - T_n y_n)\| \to 0.$$

From this fact combined with (3.7), it follows that

$$\begin{aligned}
&\|x_n - y_n - (U_n x_n - U_n y_n)\| \\
&\quad = \|x_n - y_n - (S_n x_n - S_n y_n) + (1-\lambda_n)(S_n x_n - S_n y_n - (T_n x_n - T_n y_n))\| \\
&\quad \leq \|x_n - y_n - (S_n x_n - S_n y_n)\| \\
&\qquad + (1-\lambda_n)\|S_n x_n - S_n y_n - (T_n x_n - T_n y_n)\| \to 0.
\end{aligned}$$

This completes the proof. □

Remark 3.12. It is known that the convex combination of a nonexpansive mapping and a strongly nonexpansive mapping is strongly nonexpansive in a uniformly convex Banach space; see [7, Theorem 1.3].

It is obvious that the identity mapping I is strongly nonexpansive. Therefore we immediately deduce from Theorem 3.11 the following result, which provides us to construct a strongly nonexpansive sequence from a given sequence of nonexpansive mappings.

Corollary 3.13. *Let C be a nonempty subset of a Hilbert space H. Let $\{T_n\}$ be a sequences of nonexpansive mappings of C into H. Let $\{\lambda_n\}$ be a sequence in $[0,1]$ such that $\liminf_{n\to\infty} \lambda_n > 0$. Let $\{U_n\}$ be a sequence of mappings of C into H defined by $U_n = \lambda_n I + (1-\lambda_n)T_n$ for $n \in \mathbb{N}$, where I is the identity mapping on C. Then $\{U_n\}$ is a strongly nonexpansive sequence.*

We examine some properties of approximate fixed point sequences under convex combinations.

Lemma 3.14. *Let C be a nonempty subset of a Hilbert space H. Let $\{S_n\}$ and $\{T_n\}$ be two sequences of mappings of C into H. Let $\{\lambda_n\}$ be a bounded sequence of real numbers and $\{U_n\}$ a sequence of mappings of C into H defined by $U_n = \lambda_n S_n + (1-\lambda_n)T_n$ for $n \in \mathbb{N}$. Suppose that*

$$\tilde{F} = \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset.$$

Then $\tilde{F} \subset \tilde{F}(\{U_n\})$.

Proof. Let $\{w_n\} \in \tilde{F}$ be given. Then it is easily shown that

$$\begin{aligned}\|w_n - U_n w_n\| &= \|\lambda_n(w_n - S_n w_n) + (1-\lambda_n)(w_n - T_n w_n)\| \\ &\leq |\lambda_n|\,\|w_n - S_n w_n\| + |1-\lambda_n|\,\|w_n - T_n w_n\| \to 0\end{aligned}$$

as $n \to \infty$. Thus $\{w_n\} \in \tilde{F}(\{U_n\})$. □

Theorem 3.15. *Let C be a nonempty subset of a Hilbert space H. Let $\{S_n\}$ be a strongly nonexpansive sequence of mappings of C into H and $\{T_n\}$ a sequence of nonexpansive mappings of C into H. Let $\{\lambda_n\}$ be a sequence in $[0,1]$ such that $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 1$. Let $\{U_n\}$ be a sequence of mappings of C into H defined by $U_n = \lambda_n S_n + (1-\lambda_n)T_n$ for $n \in \mathbb{N}$. Suppose that*

$$\tilde{F} = \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset.$$

Then $\tilde{F} = \tilde{F}(\{U_n\})$.

Proof. From the consequence of Lemma 3.14, we have $\tilde{F} \subset \tilde{F}(\{U_n\})$ and hence $\tilde{F}(\{U_n\}) \neq \emptyset$. We have only to show that $\tilde{F} \supset \tilde{F}(\{U_n\})$. Let $\{z_n\} \in \tilde{F}(\{U_n\})$ and $\{w_n\} \in \tilde{F}$ be fixed. Since both S_n and T_n are nonexpansive, we have

$$\begin{aligned}\|z_n - w_n\| &= \|z_n - U_n z_n + U_n z_n - U_n w_n + U_n w_n - w_n\| \\ &\leq \|z_n - U_n z_n\| + \lambda_n \|S_n z_n - S_n w_n\| \\ &\qquad + (1-\lambda_n)\|T_n z_n - T_n w_n\| + \|U_n w_n - w_n\| \\ &\leq \|z_n - U_n z_n\| + \lambda_n \|S_n z_n - S_n w_n\| \\ &\qquad + (1-\lambda_n)\|z_n - w_n\| + \|U_n w_n - w_n\|.\end{aligned}$$

Therefore it follows that

$$0 \leq \lambda_n(\|z_n - w_n\| - \|S_n z_n - S_n w_n\|) \leq \|z_n - U_n z_n\| + \|U_n w_n - w_n\| \to 0.$$

Taking into account $\liminf_{n\to\infty} \lambda_n > 0$, we have

$$\|z_n - w_n\| - \|S_n z_n - S_n w_n\| \to 0.$$

Since $\{z_n - w_n\}$ is bounded and $\{S_n\}$ is a strongly nonexpansive sequence, we conclude that

$$\begin{aligned}\|z_n - S_n z_n\| &= \|z_n - w_n - (S_n z_n - S_n w_n) + w_n - S_n w_n\| \\ &\leq \|z_n - w_n - (S_n z_n - S_n w_n)\| + \|w_n - S_n w_n\| \to 0\end{aligned}$$

as $n \to \infty$. This means that $\{z_n\} \in \tilde{F}(\{S_n\})$, and moreover,

$$\begin{aligned}(1-\lambda_n)\|z_n - T_n z_n\| &= \|\lambda_n(S_n z_n - z_n) + z_n - U_n z_n\| \\ &\leq \lambda_n \|S_n z_n - z_n\| + \|z_n - U_n z_n\| \to 0.\end{aligned}$$

Taking into account $\liminf_{n\to\infty}(1-\lambda_n) > 0$, we have $\|z_n - T_n z_n\| \to 0$. Thus $\{z_n\} \in \tilde{F}(\{T_n\})$. This completes the proof. □

Theorem 3.15 yields the following:

Corollary 3.16. *Suppose that C, $\{S_n\}$, $\{T_n\}$, $\{\lambda_n\}$, and $\{U_n\}$ are the same as in Theorem 3.15. Suppose that*

$$F = \bigcap_{n=1}^{\infty} F(S_n) \cap \bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset.$$

Then $\tilde{F}(\{U_n\}) = \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset$.

Proof. By assumption, we may choose a point $w \in F$. Let $\{w_n\}$ be a sequence in C defined by $w_n = w$ for $n \in \mathbb{N}$. Then we see that $\{w_n\} \in \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\}) \neq \emptyset$. Theorem 3.15 implies that $\tilde{F}(\{U_n\}) = \tilde{F}(\{S_n\}) \cap \tilde{F}(\{T_n\})$. □

4. Weak Convergence Theorems

In this section, we discuss weak convergence theorems for a strongly nonexpansive sequence. In the following lemma, we do not need to assume the existence of a common fixed point of a given sequence of mappings.

Lemma 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_n\}$ be a strongly nonexpansive sequence of self mappings of C such that $F(T_n)$ is nonempty for every $n \in \mathbb{N}$. Suppose that there exists a nonempty closed convex subset C_0 of C such that the following two conditions hold:*

1. *$\sum_{n=1}^{\infty} \|T_n z - z\| < \infty$ for any $z \in C_0$;*
2. *if for a weakly convergent sequence $\{u_i\}$ in C with $u_i \rightharpoonup u$ there is a subsequence $\{T_{n_i}\}$ of $\{T_n\}$ such that $\{u_i\} \in \tilde{F}(\{T_{n_i}\})$, then $u \in C_0$.*

Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and $x_{n+1} = T_n x_n$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to some point in C_0.

Proof. Let $z \in C_0$ be fixed. Since each T_n is nonexpansive, it follows that

$$\begin{aligned}\|x_{n+1} - z\| &\leq \|T_n x_n - T_n z\| + \|T_n z - z\| \\ &\leq \|x_n - z\| + \|T_n z - z\|\end{aligned} \tag{4.1}$$

for every $n \in \mathbb{N}$. Thus the condition (1) and Lemma 2.1 imply that $\{\|x_n - z\|\}$ is convergent and hence both $\{x_n\}$ and $\{x_n - z\}$ are bounded. It also follows from (4.1) that

$$0 \leq \|x_n - z\| - \|T_n x_n - T_n z\| \leq \|x_n - z\| - \|x_{n+1} - z\| + \|T_n z - z\|.$$

Therefore we obtain $\|x_n - z\| - \|T_n x_n - T_n z\| \to 0$. Since $\{T_n\}$ is a strongly nonexpansive sequence, we have

$$\lim_{n\to\infty} \|x_n - T_n x_n\| = 0. \tag{4.2}$$

Since $\{x_n\}$ is bounded, we see that $\{x_n\}$ has a weakly convergent subsequence. We show that the set of weak subsequential limits of $\{x_n\}$ is a singleton. Suppose that $x_{n_i} \rightharpoonup v$, $x_{m_i} \rightharpoonup w$ and $v \neq w$. Note that $v, w \in C_0$ by the condition (2) and (4.2). Also note that both $\{\|x_n - v\|\}$ and $\{\|x_n - w\|\}$ are convergent. Thus, by Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - v\| &= \liminf_{i\to\infty} \|x_{n_i} - v\| \\ &< \liminf_{i\to\infty} \|x_{n_i} - w\| = \lim_{n\to\infty} \|x_n - w\| = \liminf_{i\to\infty} \|x_{m_i} - w\| \\ &< \liminf_{i\to\infty} \|x_{m_i} - v\| = \lim_{n\to\infty} \|x_n - v\| .\end{aligned}$$

This is a contradiction. Hence $v = w$. This means that $\{x_n\}$ converges weakly to some point in C_0. □

A sufficient condition of the conditions (1) and (2) in Lemma 4.1 is stated as follows:

Lemma 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_n\}$ be a sequence of mappings of C into H such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that for any nonempty bounded closed convex subset D of C and for any increasing sequence $\{n_i\}$ in $\mathbb{N}$ there exist a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ and a nonexpansive mapping T of C into H such that*

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n) \text{ and } \lim_{j\to\infty} \sup_{y\in D} \left\| Ty - T_{n_{i_j}} y \right\| = 0. \tag{4.3}$$

Define $C_0 = \bigcap_{n=1}^{\infty} F(T_n)$. Then the conditions (1) and (2) in Lemma 4.1 hold.

Proof. It is clear that C_0 is nonempty closed convex subset of C and $\|T_n z - z\| = \|z - z\| = 0$ for all $z \in C_0$. Therefore the condition (1) holds. Let $\{u_i\}$ be a sequence in C such that $u_i \rightharpoonup u$. Suppose that there is a subsequence $\{T_{n_i}\}$ of $\{T_n\}$ such that

$$\lim_{i\to\infty} \|u_i - T_{n_i} u_i\| = 0. \tag{4.4}$$

Since $\{u_i\}$ is bounded, there is a bounded closed convex subset D of C such that $u_i \in D$ for all $i \in \mathbb{N}$. By assumption, for D and $\{n_i\}$, there exist a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ and a nonexpansive mapping T of C into H such that (4.3) is satisfied. It is obvious that

$$\begin{aligned}\left\| u_{n_{i_j}} - T u_{n_{i_j}} \right\| &\le \left\| u_{n_{i_j}} - T_{n_{i_j}} u_{n_{i_j}} \right\| + \left\| T_{n_{i_j}} u_{n_{i_j}} - T u_{n_{i_j}} \right\| \\ &\le \left\| u_{n_{i_j}} - T_{n_{i_j}} u_{n_{i_j}} \right\| + \sup_{y\in D} \left\| T_{n_{i_j}} y - Ty \right\|\end{aligned}$$

for every $j \in \mathbb{N}$. Thus it follows from (4.3) and (4.4) that $\left\| u_{n_{i_j}} - T u_{n_{i_j}} \right\| \to 0$. Since $I - T$ is demiclosed and $u_{n_{i_j}} \rightharpoonup u$, we conclude that $u \in F(T) = \bigcap_{n=1}^{\infty} F(T_n) = C_0$. This completes the proof. □

Using Lemma 4.1 combined with Lemma 4.2, we obtain the following:

Theorem 4.3. *Let C be a nonempty closed convex subset of a Hilbert space H. Let $\{T_n\}$ be a strongly nonexpansive sequence of self mappings of C such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Suppose that, for any nonempty bounded closed convex subset D of C and for any increasing sequence $\{n_i\}$ of $\mathbb{N}$, there exist a subsequence $\{T_{n_{i_j}}\}$ of $\{T_{n_i}\}$ and a nonexpansive mapping T of C into H such that*

$$F(T) = \bigcap_{n=1}^{\infty} F(T_n) \text{ and } \lim_{j\to\infty} \sup_{y\in D} \left\| Ty - T_{n_{i_j}} y \right\| = 0.$$

Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and $x_{n+1} = T_n x_n$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to some point $v \in \bigcap_{n=1}^{\infty} F(T_n)$, and moreover, $v = \lim_{n\to\infty} Px_n$, where P is the metric projection of H onto $\bigcap_{n=1}^{\infty} F(T_n)$.

Proof. From Lemmas 4.1 and 4.2, we deduce that $\{x_n\}$ converges weakly to some point $v \in \bigcap_{n=1}^{\infty} F(T_n)$. We have only to show that $v = \lim_{n\to\infty} Px_n$. Since each T_n is nonexpansive, it is clear that

$$\|x_{n+1} - u\| = \|T_n x_n - T_n u\| \le \|x_n - u\|$$

for all $u \in \bigcap_{n=1}^{\infty} F(T_n)$ and $n \in \mathbb{N}$. Therefore Lemma 2.2 implies that $\{Px_n\}$ converges strongly to some point $w \in \bigcap_{n=1}^{\infty} F(T_n)$. On the other hand, since $v \in \bigcap_{n=1}^{\infty} F(T_n)$ and P is the metric projection onto $\bigcap_{n=1}^{\infty} F(T_n)$, it follows from (2.1) that

$$\langle x_n - Px_n, v - Px_n\rangle \le 0$$

for every $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, we obtain $\langle v - w, v - w\rangle \le 0$. Consequently, we conclude that $w = v$. This completes our proof. □

5. Applications to problems for monotone operators

In this section, we apply our results obtained in §3 and §4 to some problems for monotone operators. We discuss the following three problems:

(1) The problem of finding a common zero of a finite family of monotone operators.
(2) The problem of finding a solution of a variational inequality problem for a monotone operator.
(3) The problem of finding a zero of the sum of two monotone operators.

We first consider an iterative scheme for a finite number of maximal monotone operators, which converges weakly to a common zero point. The following lemma is a well-known result; see [13].

Lemma 5.1. *Let B be a maximal monotone operator on a Hilbert space H, $\{u_n\}$ a weakly convergent sequence of H with a limit $u \in H$, and $\{r_n\}$ a positive real sequence such that $\inf_{n\in\mathbb{N}} r_n > 0$. Suppose that $\{u_n\} \in \tilde{F}(\{(I + r_n B)^{-1}\})$, that is,*

$$\lim_{n\to\infty} \|u_n - (I + r_n B)^{-1} u_n\| = 0.$$

Then, $u \in B^{-1}0$.

Proof. Put $J_{r_n} = (I + r_n B)^{-1}$ for $n \in \mathbb{N}$. It follows from (2.3) that $(u_n - J_{r_n} u_n)/r_n \in BJ_{r_n} u_n$ for every $n \in \mathbb{N}$. Since B is a monotone operator,

$$\left\langle x - J_{r_n} u_n, y - \frac{u_n - J_{r_n} u_n}{r_n} \right\rangle \geq 0$$

for all $x \in \operatorname{dom}(B)$, $y \in Bx$, and $n \in \mathbb{N}$. By assumption, we see that $J_{r_n} u_n \rightharpoonup u$ and $\|(u_n - J_{r_n} u_n)/r_n\| \to 0$. Therefore $\langle x - u, y - 0\rangle \geq 0$ for all $x \in \operatorname{dom}(B)$ and $y \in Bx$. Thus the maximality of B implies that $0 \in Bu$. □

For our result, we need the following lemma.

Lemma 5.2. *Let $\{x_n\}$ be a sequence of a Hausdorff topological space X and $m \in \mathbb{N}$. Suppose that, for each $k \in \{0, 1, 2, \dots, m-1\}$, a subsequence $\{x_{mi-k}\}_{i=1}^{\infty}$ of $\{x_n\}$ is a convergent sequence. If, in addition, a subsequence $\{x_{(m+1)i}\}_{i=1}^{\infty}$ is convergent, then $\{x_n\}$ is also a convergent sequence.*

Proof. Fix $m \in \mathbb{N}$ and suppose that

$$\lim_{i\to\infty} x_{mi-0} = z_0,\ \lim_{i\to\infty} x_{mi-1} = z_1,\ \lim_{i\to\infty} x_{mi-2} = z_2, \dots,\ \lim_{i\to\infty} x_{mi-(m-1)} = z_{m-1},$$

and $\lim_{i\to\infty} x_{(m+1)i} = z$. Then, it is sufficient to show that $z = z_k$ for all $k \in \{0, 1, 2, \dots, m-1\}$. Fix $k \in \{0, 1, 2, \dots, m-1\}$ and let $\{y_i\}$ be a sequence defined by

$$y_i = x_{(m+1)(mi-k)} \text{ for all } i \in \mathbb{N}.$$

Then, obviously $\{y_i\}$ is a subsequence of $\{x_{(m+1)i}\}_{i=1}^{\infty}$ and hence $\lim_{i\to\infty} y_i = z$. Further, since

$$y_i = x_{(m+1)(mi-k)} = x_{m(mi+i-k)-k}$$

for all $i \in \mathbb{N}$, $\{y_i\}$ is also a subsequence of $\{x_{mi-k}\}_{i=1}^{\infty}$. Thus we have that $\lim_{i\to\infty} y_i = z_k$. Since X is Hausdorff, a limit of a sequence is unique. Therefore we obtain that $z = z_k$ for each $k \in \{0, 1, 2, \dots, m-1\}$ and we conclude the desired result. □

In particular, we obtain the following corollary, which is the case that $m = 3$ in the previous lemma.

Corollary 5.3. *Let $\{x_n\}$ be a sequence of a Hausdorff topological space. Suppose that $\{x_{3i-2}\}$, $\{x_{3i-1}\}$, and $\{x_{3i}\}$ are convergent subsequences of $\{x_n\}$. If, in addition, a subsequence $\{x_{4i}\}$ is convergent, then $\{x_n\}$ is also a convergent sequence.*

Now we obtain a weak convergence theorem for a finite family of maximal monotone operators as follows.

Theorem 5.4. *Let $\{B_0, B_1, \dots, B_{N-1}\}$ be a finite family of maximal monotone operators on a Hilbert space H such that $C_0 = \bigcap_{k=0}^{N-1} B_k^{-1} 0$ is nonempty, where N is a positive integer. Let $\{r_n\}$ be a positive real sequence such that $\inf_{n\in\mathbb{N}} r_n > 0$. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in H$ and*

$$x_{n+1} = \left(I + r_n B_{c(n)}\right)^{-1} x_n$$

for every $n \in \mathbb{N}$, where $c(n) = n \bmod N$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to some point in C_0.

Proof. For the sake of simplicity, we will prove only the case that $N = 3$, that is, $\{x_n\}$ is generated by three maximal monotone operators $\{B_0, B_1, B_2\}$. For more general cases, the proof is left for the readers since the method of the proof is similar to this spacial case.

Let $Q_n = (I + r_{3n-2}B_1)^{-1}$, $R_n = (I + r_{3n-1}B_2)^{-1}$ and $S_n = (I + r_{3n}B_0)^{-1}$ for $n \in \mathbb{N}$. First, we show subsequences $\{x_{3n-2}\}$, $\{x_{3n-1}\}$, and $\{x_{3n}\}$ converges weakly to $z_2 \in C_0$, $z_1 \in C_0$, and $z_0 \in C_0$, respectively. Let $v_n = x_{3n-2}$ and $T_n = S_n R_n Q_n$ for every $n \in \mathbb{N}$. Then, from the definition of the iterative sequence $\{x_n\}$, we have $v_1 = x_1 \in H$ and

$$\begin{aligned} v_{n+1} &= x_{3(n+1)-2} = x_{3n+1} \\ &= S_n x_{3n} = S_n R_n x_{3n-1} = S_n R_n Q_n x_{3n-2} \\ &= S_n R_n Q_n v_n = T_n v_n \end{aligned}$$

for $n \in \mathbb{N}$. Since Q_n, R_n, and S_n are all firmly nonexpansive, Example 3.2 shows that $\{Q_n\}$, $\{R_n\}$, and $\{S_n\}$ are strongly nonexpansive sequences. Thus, by Theorem 3.4, $\{T_n\}$ is also a strongly nonexpansive sequence. Since $F(Q_n) \cap F(R_n) \cap F(S_n) = B_1^{-1}0 \cap B_2^{-1}0 \cap B_0^{-1}0 \neq \emptyset$, by Corollary 3.9 we have

$$\begin{aligned} F(T_n) = F(S_n R_n Q_n) &= F(S_n) \cap F(R_n Q_n) = F(S_n) \cap F(R_n) \cap F(Q_n) \\ &= B_1^{-1}0 \cap B_2^{-1}0 \cap B_0^{-1}0 = C_0 \end{aligned}$$

for every $n \in \mathbb{N}$. It is obvious that $\{T_n\}$ and C_0 satisfy the condition (1) in Lemma 4.1. For (2), let $\{u_i\}$ be a weakly convergent sequence of H with a limit $u \in H$ and $\{T_{n_i}\}$ a subsequence of $\{T_n\}$ such that $\{u_i\} \in \tilde{F}(\{T_{n_i}\})$. Then, since

$$\bigcap_{i=1}^{\infty} F(S_{n_i}) \cap \bigcap_{i=1}^{\infty} F(R_{n_i}) \cap \bigcap_{i=1}^{\infty} F(Q_{n_i}) = B_1^{-1}0 \cap B_2^{-1}0 \cap B_0^{-1}0 = C_0 \neq \emptyset,$$

by Corollary 3.8 we have

$$\begin{aligned} \{u_i\} \in \tilde{F}(\{T_{n_i}\}) &= \tilde{F}(\{S_{n_i} R_{n_i} Q_{n_i}\}) \\ &= \tilde{F}(\{S_{n_i}\}) \cap \tilde{F}(\{R_{n_i} Q_{n_i}\}) \\ &= \tilde{F}(\{S_{n_i}\}) \cap \tilde{F}(\{R_{n_i}\}) \cap \tilde{F}(\{Q_{n_i}\}). \end{aligned}$$

It follows from Lemma 5.1 that $u \in B_1^{-1}0 \cap B_2^{-1}0 \cap B_0^{-1}0 = C_0$, and therefore the condition (2) in Lemma 4.1 is satisfied. Hence we have $\{v_n\} = \{x_{3n-2}\}$ converges weakly to $z_2 \in C_0$. In the same way, we also have that $\{x_{3n-1}\}$ and $\{x_{3n}\}$ converge weakly to $z_1 \in C_0$ and $z_0 \in C_0$, respectively.

Let $y_n = x_{4n}$ for every $n \in \mathbb{N}$ and let us show that $\{y_n\}$ also converges weakly. From the definition of $\{x_n\}$, we have $y_1 = x_4 = S_1 R_1 Q_1 x_1 \in H$ and

$$\begin{aligned} &y_2 = Q_3 S_2 R_2 Q_2 y_1, \ y_3 = R_4 Q_4 S_3 R_3 y_2, \ y_4 = S_5 R_5 Q_5 S_4 y_3, \\ &y_5 = Q_7 S_6 R_6 Q_6 y_4, \ y_6 = R_8 Q_8 S_7 R_7 y_5, \ y_7 = S_9 R_9 Q_9 S_8 y_6, \dots. \end{aligned}$$

Thus, letting

$$U_n = \begin{cases} Q_{4(n+2)/3-1} S_{4(n+2)/3-2} R_{4(n+2)/3-2} Q_{4(n+2)/3-2}, & (n \bmod 3 = 1) \\ R_{4(n+1)/3-0} Q_{4(n+1)/3-0} S_{4(n+1)/3-1} R_{4(n+1)/3-1}, & (n \bmod 3 = 2) \\ S_{4(n+0)/3+1} R_{4(n+0)/3+1} Q_{4(n+0)/3+1} S_{4(n+0)/3+0}, & (n \bmod 3 = 0) \end{cases}$$

for every $n \in \mathbb{N}$, we have $y_{n+1} = U_n y_n$ for $n \in \mathbb{N}$. By a similar way to that shown above, we can easily verify that $\{U_n\}$ is a strongly nonexpansive sequence. Also, using Corollary 3.9 repeatedly, we have

$$F(U_n) = B_1^{-1}0 \cap B_2^{-1}0 \cap B_0^{-1}0 = C_0$$

for $n \in \mathbb{N}$. Thus we obtain that $\{U_n\}$ and C_0 satisfy the condition (1) in Lemma 4.1. For (2), let $\{u_i\}$ be a weakly convergent sequence of H with a limit $u \in H$ and $\{U_{n_i}\}$ a subsequence of $\{U_n\}$ such that $\{u_i\} \in \tilde{F}(\{U_{n_i}\})$. Then, the subsequence $\{n_i\}$ of $\mathbb{N}$ must contain infinitely many elements of at least one of subsequences $\{3n-2\}$, $\{3n-1\}$, and $\{3n\}$ of $\mathbb{N}$. Suppose that a sequence $\{n_i\}$ contains infinitely many elements of $\{3n-2\}$. Then, we may take a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ which consists of only elements of $\{3n-2\}$. Since $n_{i_j} \bmod 3 = 1$ for every $j \in \mathbb{N}$, it follows that

$$\begin{aligned} U_{n_{i_j}} &= Q_{4(n_{i_j}+2)/3-1} S_{4(n_{i_j}+2)/3-2} R_{4(n_{i_j}+2)/3-2} Q_{4(n_{i_j}+2)/3-2} \\ &= Q_{m_j-1} S_{m_j-2} R_{m_j-2} Q_{m_j-2} \end{aligned}$$

where $m_j = 4(n_{i_j}+2)/3$ for $j \in \mathbb{N}$. Obviously, the subsequence $\{u_{i_j}\}$ of $\{u_i\}$ converges weakly to u. It is also clear that $\{u_{i_j}\} \in \tilde{F}(\{U_{n_{i_j}}\})$ and

$$\begin{aligned} &\bigcap_{j=1}^{\infty} F(Q_{m_j-1}) \cap \bigcap_{j=1}^{\infty} F(S_{m_j-2}) \cap \bigcap_{j=1}^{\infty} F(R_{m_j-2}) \cap \bigcap_{j=1}^{\infty} F(Q_{m_j-2}) \\ &= B_1^{-1}0 \cap B_0^{-1}0 \cap B_2^{-1}0 \cap B_1^{-1}0 = C_0 \neq \emptyset. \end{aligned}$$

Then, Corollary 3.8 implies that

$$\begin{aligned} \{u_{i_j}\} &\in \tilde{F}(\{U_{n_{i_j}}\}) \\ &= \tilde{F}(\{Q_{m_j-1} S_{m_j-2} R_{m_j-2} Q_{m_j-2}\}) \\ &= \tilde{F}(\{Q_{m_j-1}\}) \cap \tilde{F}(\{S_{m_j-2}\}) \cap \tilde{F}(\{R_{m_j-2}\}) \cap \tilde{F}(\{Q_{m_j-2}\}). \end{aligned}$$

It follows from Lemma 5.1 that $u \in B_1^{-1}0 \cap B_2^{-1}0 \cap B_0^{-1}0 = C_0$. In other cases that $\{n_i\}$ contains infinitely many elements of $\{3n-1\}$ or $\{3n\}$, using a similar method to that mentioned above, we can also obtain that $u \in C_0$. Therefore the condition (2) in Lemma 4.1 is satisfied and hence we obtain $\{y_n\} = \{x_{4n}\}$ converges weakly to $z \in C_0$. Then, using Corollary 5.3, we have that $\{x_n\}$ converges weakly to $z \in C_0$, which completes the proof. □

The following corollary is a generalization of the result obtained by Xu and Ori [19].

Corollary 5.5. *Let C be a nonempty closed convex subset of a Hilbert space H and let $\{T_0, T_1, \ldots, T_{N-1}\}$ be a finite family of nonexpansive self mapping of C such that $C_0 = \bigcap_{k=0}^{N-1} F(T_k) \neq \emptyset$, where N is a positive integer. Let $\{x_n\}$ be a sequence defined by $x_1 = x \in C$ and*

$$x_{n+1} = t_n x_n + (1-t_n) T_{c(n)} x_{n+1}$$

for every $n \in \mathbb{N}$, where $c(n) = n \bmod N$ and $0 < t_n \leq \sup_{k\in\mathbb{N}} t_k < 1$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to some point in C_0.

Proof. Let $B_k = I - T_k P_C$ for $k = 0, 1, \dots, N-1$, where P_C is the metric projection onto C. Then the iterative scheme given in this theorem is equivalent that $x_1 = x \in C$ and

$$x_{n+1} = \left(I + \left(\frac{1}{t_n} - 1\right) B_{c(n)}\right)^{-1} x_n$$

for $n \in \mathbb{N}$. Since T_k and P_C are nonexpansive, so is $T_k P_C$. Thus B_k is a monotone operator on H whose zero is a fixed point of T_k for $k = 0, 1, \dots, N-1$. So we have

$$\bigcap_{k=0}^{N-1} B_k^{-1} 0 = \bigcap_{k=0}^{N-1} F(T_k) = C_0.$$

Moreover, since B_k is a single-valued continuous monotone operator defined on the whole space H, it is maximal monotone for each $k = 0, 1, \dots, N-1$. On the other hand, the condition $0 < t_n \leq \sup_{k \in \mathbb{N}} t_k < 1$ guarantees that

$$\inf_{n \in \mathbb{N}} \left(\frac{1}{t_n} - 1\right) > 0.$$

Therefore, using Theorem 5.4, we obtain that $\{x_n\}$ converges weakly to a point of C_0, which completes the proof. □

Remark 5.6. We can replace the function c appearing in Theorem 5.4 and Corollary 5.5 to a more general one which is called an admissible function. The exact definition is as follows: Let M be a subset of $\mathbb{N}$. We say that a function c of $\mathbb{N}$ into M is admissible [3] if for each $k \in M$, there exists $m(k) \in \mathbb{N}$ such that $k \in \{c(n), c(n+1), c(n+2), \dots, c(n+m(k))\}$ for every $n \in \mathbb{N}$. If M is a finite subset of $\mathbb{N}$, then c is admissible if and only if there exists $m_0 \in \mathbb{N}$ such that

$$\{c(n), c(n+1), c(n+2), \dots, c(n+m_0)\} = M$$

for every $n \in \mathbb{N}$.

Next we consider variational inequality problems for monotone operators. In particular, we treat the case that its feasible set is approximated by a sequence of closed convex sets. Such a problem, for example, has been discussed in [20]. Let C be a nonempty closed convex subset of a Hilbert space H and A a mapping of C into H. Then the variational inequality problem for A is formulated as follows: Find $x \in C$ such that $\langle y - x, Ax \rangle \geq 0$ for all $y \in C$. In this case such a point x is a solution of this problem and the solution set is denoted by $\mathrm{VI}(C, A)$, that is, $\mathrm{VI}(C, A) = \{x \in C : \langle y - x, Ax \rangle \geq 0 \text{ for all } y \in C\}$. By (2.1), it is known that

$$\mathrm{VI}(C, A) = F(P_C(I - \lambda A)) \tag{5.1}$$

for all $\lambda > 0$, where P_C is the metric projection of H onto C and I is the identity mapping on C.

Using Lemma 4.1, we obtain the following:

Theorem 5.7. *Let A be an α-inverse-strongly-monotone mapping of a Hilbert space H into itself, where $\alpha > 0$. For every $n \in \mathbb{N}$, let C_n be a closed convex subset of H such that $\mathrm{VI}(C_n, A) \neq \emptyset$. Let C be a closed convex subset of H such that $\mathrm{VI}(C, A) \neq \emptyset$. Assume that $\sum_{n=1}^{\infty} \|P_{C_n} y - P_C y\| < \infty$ for all $y \in H$ and $\lim_{n \to \infty} \sup\{\|P_{C_n} z - P_C z\| : \|z\| \leq r\} = 0$ for all $r > 0$. Let $\{\lambda_n\}$ be a sequence of* *positive numbers such that* $0 < \inf_{n\in\mathbb{N}} \lambda_n \le \sup_{n\in\mathbb{N}} \lambda_n < 2\alpha$ *and* $\sum_{n=1}^{\infty} |\lambda_n - \lambda| < \infty$ *for some* $\lambda > 0$. *Let* $\{x_n\}$ *be a sequence in* H *defined by* $x_1 = x \in H$ *and* $x_{n+1} = P_{C_n}(x_n - \lambda_n A x_n)$ *for every* $n \in \mathbb{N}$. *Then* $\{x_n\}$ *converges weakly to some point* $v \in \mathrm{VI}(C, A)$.

Proof. Put $T_n = P_{C_n}(I - \lambda_n A)$ for $n \in \mathbb{N}$ and $T = P_C(I - \lambda A)$, where I is the identity mapping on C. We obtain that $\{I - \lambda_n A\}$ and $\{P_{C_n}\}$ are strongly nonexpansive sequences; see Examples 3.2 and 3.3. Thus, by Theorem 3.4, $\{T_n\}$ is a strongly nonexpansive sequence. We also know that T is a nonexpansive mapping by the nonexpansiveness of P_C and $I - \lambda A$. By (5.1), $F(T_n) = \mathrm{VI}(C_n, A)$ for all $n \in \mathbb{N}$ and $F(T) = \mathrm{VI}(C, A)$.

Put $C_0 = F(T)$. We show that C_0 satisfies (1) in Lemma 4.1. Let $z \in F(T)$. Since P_{C_n} is nonexpansive, it follows that

$$\begin{aligned} \|T_n z - z\| &\le \|P_{C_n}(I - \lambda_n A)z - P_{C_n}(I - \lambda A)z\| + \|P_{C_n}(I - \lambda A)z - P_C(I - \lambda A)z\| \\ &\le \|(I - \lambda_n A)z - (I - \lambda A)z\| + \|P_{C_n} y - P_C y\| \\ &= |\lambda_n - \lambda|\, \|Az\| + \|P_{C_n} y - P_C y\| \end{aligned}$$

for every $n \in \mathbb{N}$, where $y = (I - \lambda A)z$. Therefore we have

$$\sum_{n=1}^{\infty} \|T_n z - z\| \le \|Az\| \sum_{n=1}^{\infty} |\lambda_n - \lambda| + \sum_{n=1}^{\infty} \|P_{C_n} y - P_C y\| < \infty.$$

Thus the condition (1) in Lemma 4.1 is satisfied.

Next we show that C_0 satisfies (2) in Lemma 4.1. Suppose that $\{u_i\}$ is a sequence in C such that $\{u_i\}$ converges weakly to some point $u \in C$ and $u_i - T_{n_i} u_i \to 0$. It is not hard to verify that $\{(I - \lambda A)u_i\}$ is bounded, so that there is $M > 0$ such that $\|(I - \lambda A)u_i\| \le M$ for every $i \in \mathbb{N}$. Since each $P_{C_{n_i}}$ is nonexpansive, we have

$$\begin{aligned} \|u_i - Tu_i\| &\le \|u_i - T_{n_i} u_i\| + \|T_{n_i} u_i - Tu_i\| \\ &\le \|u_i - T_{n_i} u_i\| + \left\| P_{C_{n_i}}(I - \lambda_{n_i} A)u_i - P_{C_{n_i}}(I - \lambda A)u_i \right\| \\ &\qquad + \left\| P_{C_{n_i}}(I - \lambda A)u_i - P_C(I - \lambda A)u_i \right\| \\ &\le \|u_i - T_{n_i} u_i\| + \|(I - \lambda_{n_i} A)u_i - (I - \lambda A)u_i\| \\ &\qquad + \sup\{ \left\| P_{C_{n_i}} y - P_C y \right\| : \|y\| \le M\} \\ &= \|u_i - T_{n_i} u_i\| + |\lambda_{n_i} - \lambda|\, \|Au_i\| + \sup\{ \left\| P_{C_{n_i}} y - P_C y \right\| : \|y\| \le M\}. \end{aligned}$$

Since $u_i - T_{n_i} u_i \to 0$, $\lambda_n \to \lambda$, $\{Au_i\}$ is bounded, and $\sup\{ \left\| P_{C_{n_i}} y - P_C y \right\| : \|y\| \le M\} \to 0$, it holds that $u_i - Tu_i \to 0$. Since $I - T$ is demiclosed, we conclude that $u \in F(T)$. Therefore, by Lemma 4.1, $x_n \rightharpoonup v \in F(T) = \mathrm{VI}(C, A)$. □

Lastly, we consider the problem of finding a zero of the sum of two monotone operators. Such a problem, for instance, was discussed by Passty [12]. To obtain the next theorem, we need the following:

Lemma 5.8. *Let C be a nonempty subset of a Hilbert space H. Let A be a mapping of C into H. Let B be a maximal monotone operator on H and $J_r = (I + rB)^{-1}$ the resolvent of B for $r > 0$. Then $F(J_r(I - rA)) = (A + B)^{-1}0$ for all $r > 0$.*

Proof. Let $r > 0$ be fixed. Then we have the following:

$$\begin{aligned} u \in F(J_r(I - rA)) &\Leftrightarrow u = J_r(I - rA)u = (I + rB)^{-1}(I - rA)u \\ &\Leftrightarrow (I + rB)u \ni (I - rA)u \\ &\Leftrightarrow rBu \ni -rAu \\ &\Leftrightarrow Bu \ni -Au \\ &\Leftrightarrow (A + B)u \ni 0 \\ &\Leftrightarrow u \in (A + B)^{-1}0. \end{aligned}$$

This completes the proof. □

Using Theorem 4.3 with Lemma 5.8, we obtain the following:

Theorem 5.9. *Let C be a nonempty closed convex subset of a Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H, where $\alpha > 0$ is a constant. Let B be a maximal monotone operator on H and $J_r = (I + rB)^{-1}$ the resolvent of B for $r > 0$. Suppose that $(A + B)^{-1}0$ is nonempty and $\operatorname{dom}(B) \subset C$. Let $\{r_n\}$ be a sequence of positive real numbers such that $0 < \inf_{n\in\mathbb{N}} r_n \leq \sup_{n\in\mathbb{N}} r_n < 2\alpha$. Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and $x_{n+1} = J_{r_n}(x_n - r_n A x_n)$ for $n \in \mathbb{N}$. Then $\{x_n\}$ converges weakly to some point $v \in (A + B)^{-1}0$, and moreover, $v = \lim_{n\to\infty} P x_n$, where P is the metric projection of H onto $(A + B)^{-1}0$.*

Proof. Put $T_n = J_{r_n}(I - r_n A)$ for $n \in \mathbb{N}$. Then it follows from Lemma 5.8 that $F(T_n) = (A + B)^{-1}0$ for every $n \in \mathbb{N}$ and hence

$$\bigcap_{n=1}^{\infty} F(T_n) = (A + B)^{-1}0 \neq \emptyset. \tag{5.2}$$

Since each J_{r_n} is firmly nonexpansive, $\{J_{r_n}\}$ is a strongly nonexpansive sequence. Example 3.3 shows that $\{I - r_n A\}$ is also a strongly nonexpansive sequence. Thus Theorem 3.4 implies that $\{T_n\}$ is a strongly nonexpansive sequence. Let D be a nonempty bounded closed convex subset of C and $\{n_i\}$ a increasing sequence in $\mathbb{N}$. Since $0 < \inf_{n\in\mathbb{N}} r_n \leq r_{n_i} \leq \sup_{n\in\mathbb{N}} r_n < 2\alpha$ for every i, there exists a subsequence $\{r_{n_{i_j}}\}$ of $\{r_{n_i}\}$ such that $r_{n_{i_j}} \to r \in [\inf_{n\in\mathbb{N}} r_n, \sup_{n\in\mathbb{N}} r_n]$. Let T be a self mapping of C defined by $T = J_r(I - rA)$. It is clear from Lemma 5.8 and (5.2) that $F(T) = (A + B)^{-1}0 = \bigcap_{n=1}^{\infty} F(T_n)$. From (2.4), it holds that

$$\begin{aligned} &\|J_s(I - sA)y - J_t(I - tA)y\| \\ &\quad \leq \|J_s(I - sA)y - J_s(I - tA)y\| + \|J_s(I - tA)y - J_t(I - tA)y\| \\ &\quad \leq \|(I - sA)y - (I - tA)y\| + \frac{|t - s|}{t} \|(I - tA)y - J_t(I - tA)y\| \\ &\quad = |t - s| \left(\|Ay\| + \frac{1}{t} \|(I - tA)y - J_t(I - tA)y\| \right) \end{aligned}$$

for all $s, t > 0$ and $y \in C$. Thus it follows that

$$\left\| Ty - T_{n_{i_j}} y \right\| \leq \left| r - r_{n_{i_j}} \right| \left(\|Ay\| + \frac{1}{r} \|(I - rA)y - J_r(I - rA)y\| \right) \tag{5.3}$$

for all $y \in C$ and $j \in \mathbb{N}$. Since A is $1/\alpha$-Lipschitz continuous and $J_r(I - rA)$ is nonexpansive, we conclude that

$$\sup_{y \in D} \left(\|Ay\| + \frac{1}{r} \|(I - rA)y - J_r(I - rA)y\| \right) < \infty. \tag{5.4}$$

Consequently, from (5.3) and (5.4) we obtain

$$\lim_{j \to \infty} \sup_{y \in D} \left\| Ty - T_{n_{i_j}} y \right\| = 0.$$

Therefore Theorem 4.3 implies that $\{x_n\}$ converges weakly to some point $v \in (A + B)^{-1}0$ and $v = \lim_{n \to \infty} Px_n$. □

In the preceding theorem, it is noticed that we do not need to assume either the existence of a common zero of two monotone operators or that of a zero of each monotone operator.

We apply Theorem 5.9 to a variational inequality problem for a monotone mapping. The direct consequence of Theorem 5.9 is the following corollary.

Corollary 5.10. *Let C be a nonempty closed convex subset of a Hilbert space H. Let A be an α-inverse-strongly-monotone mapping of C into H, where $\alpha > 0$ is a constant. Suppose that $\mathrm{VI}(C, A)$ is nonempty. Let $\{r_n\}$ be a sequence of positive real numbers such that $0 < \inf_{n \in \mathbb{N}} r_n \leq \sup_{n \in \mathbb{N}} r_n < 2\alpha$. Let $\{x_n\}$ be a sequence in C defined by $x_1 = x \in C$ and $x_{n+1} = P_C(x_n - r_n A x_n)$ for $n \in \mathbb{N}$, where P_C is the metric projection H onto C. Then $\{x_n\}$ converges weakly to some point $v \in \mathrm{VI}(C, A)$, and moreover, $v = \lim_{n \to \infty} Px_n$, where P is the metric projection of H onto $\mathrm{VI}(C, A)$.*

Proof. Let $N_C(x)$ be the normal cone of C at $x \in C$, that is, $N_C(x) = \{z \in H : \langle y - x, z \rangle \leq 0 \text{ for all } y \in C\}$. Define a multi-valued mapping $B \subset H \times H$ by $Bx = N_C(x)$ for $x \in C$ and $Bx = \emptyset$ for $x \notin C$. It is known that B is a maximal monotone operator and the resolvent of B is the metric projection P_C. Also it is not hard to check that $(A + B)^{-1}0 = \mathrm{VI}(C, A)$. Hence we deduce the conclusion from Theorem 5.9. □

References

[1] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space*, Nonlinear Anal. **67** (2007), 2350-2360.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, *Finding common fixed points of a countable family of nonexpansive mappings in a Banach space*, Sci. Math. Jpn. **66** (2007), 89-99.

[3] F. E. Browder, *Convergence theorems for sequences of nonlinear operators in Banach spaces*, Math. Z. **100** (1967), 201-225.

[4] F. E. Browder, *Nonlinear maximal monotone operators in Banach space*, Math. Ann. **175** (1968), 89-113.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197-228.
[6] R. E. Bruck, *Nonexpansive projections on subsets of Banach spaces*, Pacific J. Math. **47** (1973), 341-355.
[7] R. E. Bruck and S. Reich, *Nonexpansive projections and resolvents of accretive operators in Banach spaces*, Houston J. Math. **3** (1977), 459-470.
[8] K. Eshita and W. Takahashi, *Approximating zero points of accretive operators in general Banach spaces*, JP J. Fixed Point Theory Appl. **2** (2007), 105-116.
[9] K. Goebel and W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, Vol. 28, Cambridge University Press, Cambridge, 1990.
[10] G. J. Minty, *Monotone (nonlinear) operators in Hilbert space*, Duke Math. J. **29** (1962), 341-346.
[11] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 591-597.
[12] G. B. Passty, *Ergodic convergence to a zero of the sum of monotone operators in Hilbert space*, J. Math. Anal. Appl. **72** (1979), 383-390.
[13] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM J. Control Optim. **14** (1976), 877-898.
[14] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75-88.
[15] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.
[16] W. Takahashi, Introduction to Nonlinear and Convex Analysis, Yokohama Publishers, Yokohama, 2005 (in Japanese).
[17] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003) 417-428.
[18] K.-K. Tan and H.-K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. **178** (1993), 301-308.
[19] H.-K. Xu and R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim. **22** (2001), 767-773.
[20] Q. Yang and J. Zhao, *Generalized KM theorems and their applications*, Inverse Problems **22** (2006), 833-844.

Manuscript received December 20, 2006
revised September 30, 2007

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