Journal of Nonlinear and Convex Analysis
Volume 8, Number 3, 2007, 451–462
Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

# STRONG CONVERGENCE OF AVERAGED APPROXIMANTS FOR ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS IN BANACH SPACES

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Abstract. Let $D$ be a nonempty closed convex subset of a real Banach space $E$ which is both uniformly convex and $q$-uniformly smooth. Let $T : D \to D$ be a uniformly $L$-Lipschitzian, asymptotically nonexpansive-type and asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$. Assume that the set $F(T)$ of fixed points of $T$ is nonempty. Then $F(T)$ is a sunny nonexpansive retract of $D$. If $U$ is the sunny nonexpansive retraction of $D$ onto $F(T)$, $w$ is any given point of $D$ and $\{a_n\}$ is a real sequence in $(0, 1]$ satisfying some restrictions, then the sequence $\{x_n\}$ in $D$ defined by

$$x_n = a_n w + (1 - a_n)\frac{1}{n+1}\sum_{j=0}^{n}[(1 - a_j)I + a_j T^j]x_n, \quad \forall n \geq 0$$

converges strongly to $Uw$. No boundedness assumption is made on the set $D$.

## 1. Introduction

Let $E$ be a real Banach space with dual $E^*$. Given a gauge function $\Phi : [0, \infty) \to [0, \infty)$, the mapping $J_\Phi : E \to 2^{E^*}$ defined by

$$J_\Phi(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \Phi(\|x\|)\}$$

is said to be the generalized duality mapping with gauge fuction $\Phi$. In particular, if $\Phi(t) = t$, $\forall t \geq 0$, the duality mapping $J = J_\Phi$ is called the normalized duality mapping. For $q > 1$, let $\Phi(t) = t^{q-1}$ be a gauge function. We define the generalized duality mapping $J_q : E \to 2^{E^*}$ by

$$J_q(x) = \{x^* \in E^* : \langle x, x^*\rangle = \|x\|\|x^*\|, \|x^*\| = \|x\|^{q-1}\}$$

and observe that for $q = 2, J_2 = J$ (the normalized duality mapping). It is well known that if $E$ is smooth, then $J_\Phi$ is single-valued. In the sequel, we shall denote the single-valued generalized duality mapping and single-valued normalized duality mapping by $J_q$ and $J$, respectively.

Let $D$ be a nonempty subset of $E$. A mapping $T : D \to D$ is said to be nonexpansive if for all $x, y \in D$ we have $\|Tx - Ty\| \leq \|x - y\|$. It is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ and $\lim k_n = 1$ such that

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2000 *Mathematics Subject Classification.* Primary 47H05, 47H10, 47H17.
*Key words and phrases.* Asymptotically pseudocontractive mapping, uniformly Lipschitzian mapping, $q$-uniformly smooth space, uniformly convex space, sunny nonexpansive retraction.
*This research was partially supported by grant from the National Science Council of Taiwan.
** This research was partially supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China and the Dawn Program Foundation in Shanghai.

$\|T^n x - T^n y\| \le k_n \|x - y\|$ for all integers $n \ge 0$ and all $x, y \in K$. Clearly every nonexpansive mapping is asymptotically nonexpansive with a sequence $k_n = 1\ \forall n \ge 0$. Conversely, Gobel and Kirk [5] gave an example that asymptotically nonexpansive mappings are not nonexpansive and proved that if $D$ is a nonempty bounded closed convex subset of a uniformly convex Banach space, then every asymptotically nonexpansive selfmapping of $D$ has a fixed point.

An important class of nonlinear mappings generalizing the class of asymptotically nonexpansive mappings has been introduced by Schu [13] in 1991. Let $D$ be a nonempty subset of a real Banach space $E$. A mapping $T : D \to D$is said to be asymptotically pseudocontracitve if there is a sequence $\{k_n\} \subset (0, \infty)$ with $\lim k_n = 1$ and for any $x, y \in D$ there exists $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \le k_n \|x - y\|^2$$

for all integers $n \ge 0$. $T$ is called uniformly $L$-Lipschitzian if there is $L > 0$ such that

$$\|T^n x - T^n y\| \le L\|x - y\|$$

for all $x, y \in D$ and for each integer $n \ge 1$. Also recall ([6]; see also [19]) that a mapping $T : D \to D$ is said to be asymptotically nonexpansive-type if

$$\limsup_{n\to\infty}\{\sup_{y\in D}(\|T^n x - T^n y\| - \|x - y\|)\} \le 0$$

for each $x \in D$. It is clear that every asymptotically nonexpansive selfmapping of $D$ is both asymptotically pseudocontractive and asymptotically nonexpansive-type.

The iterative approximation problems for nonexpansive mappings, asymptotically nonexpansive (-type) mappings and asymptotically pseudocontracitive mappings were studied extensively by many authors; for example, Gobel and Kirk [5], Kirk [6], Rhoads [12], Liu [7], Schu [13], Xu [18], [19], Xu and Roach [20], Chang [4], Shimizu and Takahashi [14], Shioji and Takahasi [15], Moore and Nnoli [8] and Zeng [22], [23] in the setting of Hilbert spaces or uniformly convex Banach spaces.

In 2000, Chang [4] proved the following strong convergence theorem of modified Ishikawa iterative sequences with errors for asymptotically pseudocontractive mappings.

**Theorem 1.1** ([4]). *Let $E$ be a real uniformly smooth Banach space, $D$ be a nonempty bounded closed convex subset of $E$, $T : D \to D$ be an asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset (0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and let $F(T) \neq \emptyset$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$and $\{\delta_n\}$ be four sequences in $[0, 1]$ satisfying the following conditions:*

(i) $\alpha_n + \gamma_n \le 1, \beta_n + \delta_n \le 1;$

(ii) $\alpha_n \to 0, \beta_n \to 0, \delta_n \to 0 (n \to \infty);$

(iii) $\sum_{n=0}^{\infty} \alpha_n = \infty, \sum_{n=0}^{\infty} \gamma_n < 0.$

*Let $x_0 \in D$ be any given point and let $\{x_n\}, \{y_n\}$ be the modified Ishikawa iterative sequence with errors defined by*

$$\begin{cases} x_{n+1} = (1 - \alpha_n - \gamma_n)x_n + \alpha_n T^n y_n + \gamma_n u_n \\ y_n = (1 - \beta_n - \delta_n)x_n + \beta_n T^n x_n + \delta_n v_n \end{cases} \quad \text{for all} \quad n \ge 0.$$

(1) *If* $\{x_n\}$ *converges strongly to a fixed point* $q$ *of* $T$ *in* $D$*, then there exists a nondecreasing function* $\phi : [0,\infty) \to [0,\infty)$ *,* $\phi(0) = 0$ *such that for all* $n \geq 0$*,*

$$\langle T^n y_n - q, J(y_n - q)\rangle \leq k_n \|y_n - q\|^2 - \phi(\|y_n - q\|). \tag{1.1}$$

(2) *Conversely, if there is a strictly increasing function* $\phi : [0,\infty) \to [0,\infty)$*,* $\phi(0) = 0$ *satisfying condition (1.1), then* $x_n \to q \in F(T)$*.*

On the other hand, a mapping $T : D \to D$ is called pseudocontractive if for each $x, y \in D$ there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$, where $I$ denotes the identity operator.

Recently using an idea of Browder [2], Shimizu and Takahashi [14] proved that in Hilbert spaces,the approximation sequence

$$x_n = a_n w + (1-a_n)\frac{1}{n}\sum_{j=1}^{n} T^j x_n \quad for \quad n = 1, 2, 3, ...$$

for an asymptotically nonexpansive mapping $T$ with a sequence $\{k_n\}$ converges strongly to the element of$F(T)$ nearest to $w$ (where $a_n = \frac{b_n - 1}{b_n - 1 + a}, 0 < a < 1, b_n = \frac{1}{n}\sum_{j=1}^{n}(1 + |1-k_j| + e^{-j})$). Shioji and Takahashi [15] extended this result to uniformly convex Banach spaces with uniformly Gateaux differentiable norms.

In 2001 by the techniques of [14] and [15], Moore and Nnoli [8] established the following strong convergence theorem of averaged approximants for Lipschitzian pseudocontractive self-mappings.

**Theorem 1.2** ([8])**.** *Let* $D$ *be a nonempty bounded closed convex subset of a real Banach space which is both uniformly convex and q-uniformly smooth. Let* $T$ *be pseudocontractive and Lipschitzian and let* $U$ *be the sunny nonexpansive retraction from* $D$ *onto* $F(T)$*.Let* $\{a_n\}$ *be a real sequence satisfying the conditions:*

(i) $0 < a_n \leq 1$;

(ii) $\lim_{n\to\infty} a_n = 0$;

(iii) $\lim_{n\to\infty} \frac{b_n - 1}{a_n} = 0$ *where* $b_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j$ *and* $k_n = [1 + a_n^q(1+L)^q]^{\frac{1}{q}}, n \geq 0$.

*Let* $w$ *be an element of* $D$ *and let* $x_n$ *be the unique point of* $D$ *which conforms to*

$$x_n = a_n w + (1-a_n)\frac{1}{n+1}\sum_{j=0}^{n}[(1-a_j)I + a_j T]x_n$$

*for* $n \geq N_0$ *(where* $N_0$ *is a sufficiently large natural number). Then* $\{x_n\}$ *converges strongly to* $Uw$*.*

Let $D$ be a nonempty closed convex subset of a real Banach space $E$ which is both uniformly convex and $q$-uniformly smooth. In this paper the techniques of Shimizu and Takahashi [14], Shioji and Takahashi [15] and Moore and Nnoli [8] are extended to develop the iteration proecss for finding fixed points of uniformly $L$-Lipschitzian, asymptotically nonexpansive-type and asymptotically pseudocontractive selfmappings of $D$. Our theorems improve several important known results in [4, 5, 6, 7, 8, 12, 13, 14, 15, 18, 19, 20, 22, 23].

## 2. Preliminaries

Let $E$ be a real Banach space. The modulus of smoothness of $E$ is the function$\rho_E$ : $[0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le \tau\right\}.$$

$E$ is uniformly smooth if and only if $\lim_{\tau\to 0}(\rho_E(\tau)/\tau) = 0$. Let $q > 1$. $E$ is called $q$-uniformly smooth (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \le c\tau^q$. Hilbert spaces, $L_p$ (or $l_p$) spaces, $1 < p < \infty$ and the Sobolev spaces $W_m^p, 1 < p < \infty$ are $q$-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p \text{ (or } l_p) \quad \text{or} \quad W_m^p \quad \text{is} \begin{cases} p - \text{uniformly smooth if} & 1 < p \le 2, \\ 2 - \text{uniformly smooth if} & p \ge 2. \end{cases}$$

**Theorem 2.1** ([18], p. 1130)**.** *Let $q > 1$ and let $E$ be a real smooth Banach space. Then the following are equivalent:*

(1) *$E$ is $q$-uniformly smooth;*

(2) *There exists a constant $c_q > 0$ such that for all $x, y \in E$*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q; \tag{2.1}$$

(3) *There exists a constant $d_q$ such that for all $x, y \in E$ and $t \in [0,1]$*

$$\|(1-t)x + ty\|^q \ge (1-t)\|x\|^q + t\|y\|^q - \omega_q(t)d_q\|x-y\|^q$$

*where $\omega_q(t) = t^q(1-t) + t(1-t)^q$.*

Recall that $E$ is said to be uniformly convex if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|(x+y)/2\| \le 1-\delta$ for each $x, y \in B_1$ with $\|x-y\| \ge \varepsilon$ where $B_1 = \{x \in E : \|x\| \le 1\}$.

In the sequel we shall also need the following definitions and results. Let $\mu$ be a continuous linear functional on $l_\infty$ and let $(a_0, a_1, ...) \in l_\infty$. We write $\mu_n(a_n)$ instead of $\mu((a_0, a_1, ...))$. We call $\mu$ a Banach limit (see e.g., [1]) when $\mu$ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for each $n \ge 0$. For a Banach limit ,we know (see e.g., [15]) that

$$\text{liminf}_{n\to\infty} a_n \le \mu_n(a_n) \le \text{limsup}_{n\to\infty} a_n \quad \forall (a_0, a_1, ...) \in l_\infty. \tag{2.2}$$

**Lemma 2.2** (Reich [10], [11]; see also Takahashi and Jeong [16])**.** *Let $D$ be a nonempty closed convex subset of a uniformly convex Banach space $E$. Let $\{x_n\}$ be a bounded sequence in $E$. Let $\mu$ be a Banach limit and let $g$ be a real-valued function on $D$ defined by*

$$g(y) = \mu_n(\|x_n - y\|^2) \quad \text{for} \quad \text{each} \quad y \in D.$$

*Then $g$ is continuous, convex and $\lim_{\|y\|\to\infty} g(y) = \infty$. Moreover, for each $R > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$g\left(\frac{y+z}{2}\right) \le \frac{1}{2}(g(y) + g(z)) - \delta$$

*for all $y, z \in K \cap B_R$ with $\|y - z\| \ge \varepsilon$.*

**Lemma 2.3** (Takahashi and Ueda [17]). *Let $D$ be a nonempty convex subset of a Banach spaceE whose norm is uniformly Gateaux differentiable. Let $\{x_n\}$ be a bounded sequence of $D$. Let $z$ be a point of $D$ and let $\mu$ be a Banach limit. Then*

$$\mu_n(\|x_n - z\|^2) = \min_{y\in D} \mu_n(\|x_n - y\|^2) \Longleftrightarrow \mu_n(\langle y - z, J_q(x_n - z)\rangle) \le 0, \quad \forall y \in D.$$

**Lemma 2.4** (Bruck [3]; see also Reich [9]). *Let $D$ be a nonempty convex subset of a smooth Banach space. Let $K$ be a nonempty subset of $D$ and let $U$ be a retraction from $D$ into $K$. Then $U$ is sunny nonexpansive if and only if $\forall x \in D$ and $y \in K$,*

$$\langle x - Ux, J(y - Ux)\rangle \le 0.$$

**Lemma 2.5.** *Let $D$ be a nonempty subset of a q-uniformly smooth Banach space $E$. Let $T : D \to D$ be uniformly L-Lipschitzian and asymptotically pseudocontractive with a sequence $\{k_n\} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$,and let $\{a_n\}$ be a real sequence in $(0, 1]$ with $\lim_{n\to\infty} a_n = 0$.For each $n \ge 1$, define $S_n = (1 - a_n)I + a_nT^n$. Then for each $n \ge 1$,*

$$\|S_nx - S_ny\| \le \kappa_n\|x - y\| \quad \forall x, y \in D,$$

*where $\kappa_n = [1 + a_n^q c_q(L + \sup_{n\ge1} k_n)^q]^{\frac{1}{q}} + a_n(k_n - 1)\forall n \ge 1$ and $c_q$ is the constant appearing in (2.1).*

*Proof.* Define

$$A_n = I - a_n(k_nI - T^n) \quad \forall n \ge 1.$$

Since $T : D \to D$ is asymptotically pseudocontractive, for each $x, y \in D$,

$$\langle T^nx - T^ny, J_q(x - y)\rangle \le k_n\|x - y\|^q$$

that is,

$$\langle (k_nI - T^n)x - (k_nI - T^n)y, J_q(x - y)\rangle \ge 0.$$

Utilizing Theorem 2.1(2), we deduce that for each $x, y \in D$

$$\begin{aligned}
\|A_nx - A_ny\|^q &= \|x - y - a_n[(k_nI - T^n)x - (k_nI - T^n)y]\|^q \\
&\le \|x - y\|^q - qa_n\langle (k_nI - T^n)x - (k_nI - T^n)y, J_q(x - y)\rangle \\
&\quad + c_qa_n^q\|(k_nI - T^n)x - (k_nI - T^n)y\|^q \\
&\le \|x - y\|^q + c_qa_n^q(k_n\|x - y\| + \|T^nx - T^ny\|)^q \\
&\le \|x - y\|^q + a_n^q c_q(k_n + L)^q\|x - y\|^q \\
&\le [1 + a_n^q c_q(L + \sup\nolimits_{n\ge1} k_n)^q]\|x - y\|^q.
\end{aligned}$$

This implies that

$$\|A_nx - A_ny\| \le [1 + a_n^q c_q(L + \sup_{n\ge1} k_n)^q]^{\frac{1}{q}}\|x - y\|. \tag{2.3}$$

Observe that

$$\begin{aligned}
S_n &= (1 - a_n)I + a_nT^n = I - a_n(I - T^n) \\
&= I - a_n(k_nI - T^n) + a_n(k_n - 1)I \\
&= A_n + a_n(k_n - 1)I.
\end{aligned}$$

Hence, it follows from (2.3) that for each $n \ge 1$

$$\begin{aligned}
\|S_nx - S_ny\| &= \|A_nx - A_ny + a_n(k_n - 1)(x - y)\| \\
&\le \|A_nx - A_ny\| + a_n(k_n - 1)\|x - y\| \\
&\le [1 + a_n^q c_q(L + \sup\nolimits_{n\ge1} k_n)^q]^{\frac{1}{n}}\|x - y\| + a_n(k_n - 1)\|x - y\| \\
&= \kappa_n\|x - y\|
\end{aligned}$$

where $\kappa_n = [1 + a_n^q c_q(L + \sup_{n\geq 1} k_n)^q]^{\frac{1}{q}} + a_n(k_n - 1)$. □

## 3. Main Results

In order to prove the main results in this paper, we also need the following lemmas.

**Lemma 3.1.** *Let $D$ be a nonempty closed convex subset of a real Banach space $E$ which is both uniformly convex and $q$-uniformly smooth. Let $T : D \to D$ be a uniformly $L$-Lipschitzian, asymptotically nonexpansive-type and asymptotically pseudocontractive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$.*

*Assume that $F(T) \neq \emptyset$. Let $\{a_n\}$ be a real sequence in $(0, 1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} a_n = 0$;

(ii) $\lim_{n\to\infty} \frac{b_n - 1}{a_n} = 0$ *where* $b_n = \frac{1}{n+1}\sum_{j=0}^{n} \kappa_j$ *for each* $n \geq 0$ *and* $\{\kappa_n\}$ *is the sequence appearing in Lemma 2.5.*

*Let $w$ be any given point in $D$. Then the following hold;*

(1) *For each $n \geq N_0$ (where $N_0$ is a sufficiently large nonegative integer) there exists exactly one $x_n \in D$ such that*

$$x_n = a_n w + (1 - a_n)\frac{1}{n+1}\sum_{j=0}^{n} S_j x_n, \tag{3.1}$$

*where $S_n = (1 - a_n)I + a_n T^n$ for each $n \geq 0$.*

(2) *If $\mu$ is a Banach limit and $\{x_n\}$ is a bounded sequence such that $\|Tx_n - x_n\| \to 0$ as $n \to \infty$, then for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a unique element $x^* \in D$ satisfying*

$$\mu_i(\|x_{n_i} - x^*\|^2) = \min_{y\in D} \mu_i(\|x_{n_i} - y\|^2) \tag{3.2}$$

*and the point $x^*$ is a fixed point of $T$.*

*Proof.* (1) For each $n \geq 0$, we define $T_n : D \to D$ by

$$T_n x = a_n w + (1 - a_n)\frac{1}{n+1}\sum_{j=0}^{n} S_j x \quad \forall x \in D.$$

Observe that for each $u, v \in D$

$$\begin{aligned}\|T_nu - T_nv\| &= \left\|a_nw + (1-a_n)\tfrac{1}{n+1}\sum_{j=0}^{n} S_ju - a_nw - (1-a_n)\frac{1}{n+1}\sum_{j=0}^{n} S_jv\right\| \\ &= (1-a_n)\tfrac{1}{n+1}\left\|\sum_{j=0}^{n} S_ju - \sum_{j=0}^{n} S_jv\right\| \\ &\le (1-a_n)\tfrac{1}{n+1}\|\sum_{j=0}^{n}\kappa_j\|u-v\| \\ &= (1-a_n)b_n\|u-v\|.\end{aligned}$$

Condition (ii) implies that

$$\lim_{n\to\infty}\frac{b_n-1}{a_n} = 0 < 1 = \lim_{n\to\infty} b_n,$$

and hence there exist nonnegative integers $N_1, N_2 \ge 0$ such that

$$\frac{b_n-1}{a_n} < \frac{1}{2} \quad \forall n \ge N_1 \quad \text{and } \frac{1}{2} < b_n \ \forall n \ge n_2.$$

Put $N_0 = \max\{N_1, N_2\}$. Then we get $\frac{b_n-1}{a_n} < b_n \ \forall n \ge N_0$; i.e., $(1-a_n)b_n < 1 \ \forall n \ge N_0$. Thus for each $n \ge N_0$, $T_n : D \to D$ is contractive. By the Banach Contraction Principle, for each $n \ge N_0$ there exists a unique $x_n \in D$ such that $T_nx_n = x_n$. This establishes (3.1).

(2) Next we set $x_n = w$ for $n = 0, 1, ...., N_0 - 1$. From Lemma 2.2, it is easy to see that there exists a unique element $x^* \in D$ satisfying (3.2). Now we claim that $\lim_{n\to\infty} T^nx^* = x^*$. Indeed suppose that $\lim_{n\to\infty} T^nx^* \ne x^*$. Then there is $\varepsilon > 0$ such that for each $m \ge 1$, there exists $l \ge m$ such that $\|T^lx^* - x^*\| \ge \varepsilon$. Since $F(T) \ne \emptyset$ and $\{x_n\}$ is bounded, it follows from the uniformly $L$-Lipschitzian continuity of $T$ that both $\{T^mx^*\}$ and $\{T^mx_n : m, n \ge 0\}$ are bounded. Putting

$$\sup\{\|T^mx^*\| : m \ge 0\} = R > 0,$$

then $T^lx^*, x^* \in D \cap B_R$. By Lemma 2.2, we conclude that there exists $\delta > 0$ such that

$$\begin{aligned}&\mu_i(\|x_{n_i} - \frac{T^lx^* + x^*}{2}\|^2) \\ &\le \frac{1}{2}[\mu_i(\|x_{n_i} - T^lx^*\|^2) + \mu_i(\|x_{n_i} - x^*\|^2)] - \delta \qquad (3.3) \\ &\le \frac{1}{2}[\mu_i(\|x_{n_i} - T^lx_{n_i}\| + \|T^lx_{n_i} - T^lx^*\|)^2 + \ mu_i(\|x_{n_i} - x^*\|)] - \delta.\end{aligned}$$

Again set $M = \sup\{\|T^mx_n - T^mx^*\| + \|x_n - x^*\| : m, n \ge 0\}$. Note that $\|Tx_n - x_n\| \to 0 (n \to \infty)$ implies that

$$\begin{aligned}\|x_{n_i} - T^lx_{n_i}\| &\le \|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - T^2x_{n_i}\| + \cdots + \|T^{l-1}x_{n_i} - T^lx_{n_i}\| \\ &\le \|x_{n_i} - Tx_{n_i}\| + L\|x_{n_i} - Tx_{n_i}\| + \cdots + L\|x_{n_i} - Tx_{n_i}\| \\ &\le l \cdot \max\{1, L\} \cdot \|x_{n_i} - Tx_{n_i}\| \to 0 \quad (i \to \infty).\end{aligned}$$

Hence, it follows from (2.2) that

$$\mu_i(\|x_{n_i} - T^l x_{n_i}\|) = \mu_i(\|x_{n_i} - Tx_{n_i}\|^2) = 0. \tag{3.4}$$

Since $T : D \to D$ is asymptotically nonexpansive-type, we have

$$\begin{aligned}
&\operatorname{limsup}_{l\to\infty}\{\sup_{i\geq 0}(\|T^l x_{n_i} - T^l x^*\| - \|x_{n_i} - x^*\|)\}\\
&\leq \operatorname{limsup}_{l\to\infty}\{\sup_{y\in D}(\|T^l y - T^l x^*\| - \|y - x^*\|)\}\\
&\leq 0 < \frac{2\delta}{M+1}.
\end{aligned}$$

Thus there exists $l_0 \geq 1$ such that for all $l \geq l_0$

$$\|T^l x_{n_i} - T^l x^*\| - \|x_{n_i} - x^*\| < \frac{2\delta}{M+1} \quad \text{for} \quad i = 0, 1, 2, ...$$

This implies that for all $l \geq l_0$

$$\|T^l x_{n_i} - T^l x^*\|^2 - \|x_{n_i} - x^*\|^2 < 2\delta$$

and hence

$$\begin{aligned}
\|T^l x_{n_i} - T^l x^*\|^2 &= \|T^l x_{n_i} - T^l x^*\|^2 - \|x_{n_i} - x^*\|^2 + \|x_{n_i} - x^*\|^2\\
&< 2\delta + \|x_{n_i} - x^*\|^2
\end{aligned}$$

for $i = 0, 1, 2, ....$ So for the above $\delta > 0$, we deduce that for sufficiently large $l$

$$\mu_i(\|T^l x_{n_i} - T^l x^*\|^2) < \mu_i(\|x_{n_i} - x^*\|^2) + 2\delta. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\begin{aligned}
&\mu_i(\|x_{n_i} - T^l x_{n_i}\| + \|T^l x_{n_i} - T^l x^*\|)^2\\
&= \mu_i(\|x_{n_i} - T^l x_{n_i}\|^2) + 2\mu_i(\|x_{n_i} - T^l x_{n_i}\| \cdot \|T^l x_{n_i} - T^l x^*\|)\\
&\quad + \mu_i(\|T^l x_{n_i} - T^l x^*\|^2)\\
&\leq \mu_i(\|x_{n_i} - T^l x_{n_i}\|^2) + 2M \cdot \mu_i(\|x_{n_i} - T^l x_{n_i}\|) + \mu_i(\|T^l x_{n_i} - T^l x^*\|^2)\\
&< \mu_i(\|x_{n_i} - x^*\|^2) + 2\delta.
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.3), we get

$$\begin{aligned}
&\mu_i(\|x_{n_i} - \frac{T^l x^* + x^*}{2}\|^2)\\
&\leq \frac{1}{2}[\mu_i(\|x_{n_i} - T^l x_{n_i}\| + \|T^l x_{n_i} - T^l x^*\|)^2 + \mu_i(\|x_{n_i} - x^*\|^2)] - \delta\\
&< \frac{1}{2}[\mu_i(\|x_{n_i} - x^*\|^2) + 2\delta + \mu_i(\|x_{n_i} - x^*\|^2)] - \delta\\
&= \mu_i(\|x_{n_i} - x^*\|^2)
\end{aligned}$$

for sufficiently large $l$, which is a contradiction to the uniqueness of $x^*$. □

**Lemma 3.2.** *Let $D, T, \{\kappa_n\}, \{a_n\}, \{b_n\}, \omega$ and $\{x_n\}$ be as in Lemma 3.1. Let $x_n = \omega$ for $n = 0, 1, ....N_0 - 1$. Then for all $n \geq N_0$ and $x^* \in F(T)$*

$$\langle x_n - \omega, J_q(x_n - x^*)\rangle \leq \left(\frac{b_n - 1}{a_n}\right)\|x_n - x^*\|^q.$$

*Proof.* Let $n \geq N_0$ and let $x^* \in F(T)$. Then we derive for all $n \geq 0$

$$S_n x^* = (1 - a_n)x^* + a_n T^n x^* = x^*.$$

From (3.1) we get

$$a_n(x_n - \omega) = (1 - a_n)\frac{1}{n+1}\sum_{j=0}^{n} S_j x_n - (1 - a_n)x_n.$$

Observe that

$$\begin{aligned}
\langle x_n - \omega, J_q(x_n - x^*)\rangle &= \left(\frac{1 - a_n}{a_n}\right)\left\langle \frac{1}{n+1}\sum_{j=0}^{n} S_j x_n - x_n, J_q(x_n - x^*)\right\rangle \\
&= \left(\frac{1 - a_n}{a_n}\right)\left\langle \frac{1}{n+1}\sum_{j=0}^{n} S_j x_n - \frac{1}{n+1}\sum_{j=0}^{n} S_j x^*, J_q(x_n - x^*)\right\rangle \\
&+ \left(\frac{1 - a_n}{a_n}\right)\langle x^* - x_n, J_q(x_n - x^*)\rangle \\
&\leq \left(\frac{1 - a_n}{a_n}\right)\left(\frac{1}{n+1}\sum_{j=0}^{n} \kappa_j \|x_n - x^*\|^q - \|x_n - x^*\|^q\right) \\
&\leq \frac{b_n - 1}{a_n}\|x_n - x^*\|^q.
\end{aligned}$$

□

**Lemma 3.3.** *Let $D, T\{\kappa_n\}, \{a_n\}, \{b_n\}, \omega$ and $\{x_n\}$ be as in Lemma 3.1. Let $x_n = \omega$ for $n = 0, 1, 2, ..., N_0 - 1$. Then each subsequence $\{x_{n_j}\}$ of $\{x_n\}$ contains a subsequence converging strongly to an element of $F(T)$.*

*Proof.* Let $\{x_{n_j}\}$ be a subsequence of $\{x_n\}$ and let $\mu$ be a Banach limit. By Lemma 3.1, there exists $x^* \in F(T)$ satisfying (3.2). By Lemma 3.2, we obtain

$$\mu_i(\langle x_{n_i} - \omega, J_q(x_{n_i} - x^*)\rangle) \leq \left(\frac{b_{n_i} - 1}{a_{n_i}}\right)\mu_i(\|x_{n_i} - x^*\|^q). \tag{3.7}$$

Subtracting $\mu_i(\|x_{n_i} - x^*\|^q)$ from both sides of (3.7), we get

$$\mu_i(\|x_{n_i} - x^*\|^q) \leq \left(\frac{1}{1 - (\frac{b_{n_i} - 1}{a_{n_i}})} \cdot \mu_i(\langle \omega - x^*, J_q(x_{n_i} - x^*)\rangle)\right) \leq 0$$

by Lemma 2.3. From (1.2) it follows that $\liminf_{i\to\infty}\|x_{n_i} - x^*\|^q \leq \mu_i(\|x_{n_i} - x^*\|^q) \leq 0$. This implies that there exists a subsequence of $\{x_{n_i}\}$ converging strongly to $x^*$. □

**Theorem 3.4.** *Let $D$ be a nonempty closed convex subset of a real Banach space $E$ which is both uniformly convex and q-uniformly smooth. Let $T : D \to D$ be a uniformly L-Lipschitzian, asymptotically nonexpansive-type and asymptotically pseudocontarctive mapping with a sequence $\{k_n\} \subset [1, \infty)$, $\lim_{n\to\infty} k_n = 1$. Assume that $F(T) \neq \emptyset$. Let $\{a_n\}$ be a real sequence in $(0, 1]$ satisfying the following conditions:*

(i) $\lim_{n\to\infty} a_n = 0$;

(ii) $\lim_{n\to\infty} \frac{b_n - 1}{a_n} = 0$ *where* $b_n = \frac{1}{n+1}\sum_{j=0}^{n} \kappa_j$ *for each* $n \geq 0$ *and* $\{\kappa_n\}$ *is the sequence appearing in Lemma 2.5.*

Let $w$ be any given point in $D$. For each $n \geq N_0$ (where $N_0$ is a sufficiently large nonnegative integer), $x_n$ is the unique point in $D$ satisfying (3.1). For $n = 0, 1, ..., N_0 - 1$, let $x_n = w$. Then $\{x_n\}$ converges strongly to an element of $F(T)$ if and only if $\{x_n\}$ is a bounded sequence such that $\|Tx_n - x_n\| \to 0$ as $n \to \infty$.

*Proof.* "Necessity". Let $x_n \to x^* \in F(T)$. Then $\{x_n\}$ is bounded. Moreover it is easy to see that $x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = Tx^*$.

"Sufficiency". Suppose that $\{x_n\}$ is a bounded sequence such that $\|Tx_n - x_n\| \to 0$ as $n \to \infty$. Then by Lemma 3.3, each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ contains a subsequence which converges strongly to an element of $F(T)$. Let $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be two subsequences of $\{x_n\}$ converging strongly to the elements $y$ and $z$ of $F(T)$, respectively. Now we claim that $y = z$. Indeed from Lemma 3.2, we have

$$\langle x_{n_i} - w, J_q(x_{n_i} - z)\rangle \leq \left(\frac{b_{n_i} - 1}{a_{n_i}}\right) \|x_{n_i} - z\|^q.$$

Passing to the limit and using condition (iii), we conclude that $\langle y-w, J_q(y-z)\rangle \leq 0$. Similarly we also get $\langle z - w, J_q(z - y)\rangle \leq 0$. Thus, it follows immediately that $\|y - z\|^q \leq 0$, i.e., $y = z$. This shows that $\{x_n\}$ converges strongly to an element of $F(T)$. □

**Theorem 3.5.** *Assume that all conditions in Theorem 3.4 are satisfied. If for each* $w \in D, \{x_n\}$ *is the bounded sequence defined by (3.2) and satisfying* $\|Tx_n - x_n\| \to 0$ *as* $n \to \infty$*, then* $F(T)$ *is a sunny nonexpansive retract of* $D$*.*

*Proof.* In view of Theorem 3.4, for each $w \in D$ the sequence $\{x_n\}$ converges strongly to an element of $F(T)$. Now we define a mapping $U$ from $D$ into $F(T)$ by

$$Uw = \lim_{n\to\infty} x_n, \ \forall w \in D.$$

By Lemma 3.2, we have

$$\langle x_n - w, J_q(x_n - z)\rangle \leq \left(\frac{b_n - 1}{a_n}\right) \|x_n - z\|^q, \quad \forall\ n \geq N_0 \quad \text{and} \quad z \in F(T).$$

Letting $n \to \infty$, we obtain $\langle w - Uw, J_q(z - Uw)\rangle \leq 0$, $\forall w \in D$ and $z \in F(T)$. Therefore, by Lemma 2.4, $U$ is a sunny nonexpansive retraction of $D$. □

**Theorem 3.6.** *Assume that all conditions in Theorem 3.4 are satisfied. Let* $U$ *be the sunny nonexpansive retraction from* $D$ *onto* $F(T)$*. If for any given* $w \in D, \{x_n\}$ *is the bounded sequence defined by (3.2) and satisfying* $\|Tx_n - x_n\| \to 0$ *as* $n \to \infty$*, then* $\{x_n\}$ *converges strongly to* $Uw$*.*

*Proof.* By Theorem 3.4, for any given $w \in D$, the sequence $\{x_n\}$ converges strongly to an element $y$ of $F(T)$. Now,we claim that $y = Uw$. Indeed, by Lemma 3.2 we have

$$\langle x_n - w, J_q(x_n - Uw)\rangle \leq \left(\frac{b_n - 1}{a_n}\right) \|x_n - Uw\|^q, \quad \forall \quad n \geq N_0.$$

Letting $n \to \infty$, we obtain $\langle y - w, J_q(y - Uw) \rangle \le 0$. Therefore, we have

$$\langle y - Uw, J_q(y - Uw) \rangle + \langle Uw - w, J_q(y - Uw) \rangle = \langle y - w, J_q(y - Uw) \rangle \le 0$$

and thus by Lemma 2.4,

$$\|y - Uw\|^q \le \langle w - Uw, J_q(y - Uw) \rangle \le 0.$$

This implies that $y = Uw$. □

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*Manuscript received October 30, 2006*
*revised March 8, 2007*

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