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STRONG CONVERGENCE THEOREMS OF BLOCK ITERATIVE METHODS FOR A FINITE FAMILY OF RELATIVELY NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we establish strong convergence theorems of block-iterative methods for a finite family of relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Our results extend and improve the recent ones announced by Matsushita and Takahashi [S. Matsushita, W. Takahashi, A strong convergence theorem for relatively nonexpansive mappings in a Banach space, J. Approx. Theory 134 (2005) 257-266.], Matinez-Yanes and Xu [C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed point iteration processes, Nonlinear Anal. 64 (2006) 2400-2411.], and many others.

1. Introduction

Let H be a Hilbert space and let $\{\Omega_i\}_{i=1}^m$ be a family of closed convex subsets of H with $F := \bigcap_{i=1}^m \Omega_i \neq \varnothing$. Then the problem of *image recovery* is to find an element of F by using the metric projection P_i from H onto Ω_i for each $i = 1, 2, \ldots, m$, where

$$P_i(x) = \arg\min_{y\in\Omega_i} \|y - x\|$$

for all $x \in H$. This problem is connected with the *convex feasibility problem.* In fact, if $\{f_i\}_{i=1}^m$ is a family of continuous convex functions from H into $\mathbb{R}$, then the convex feasibility problem is to find an element of the feasible set

$$\bigcap_{i=1}^m \{x \in H : f_i(x) \leqslant 0\}.$$

We know that each P_i is a nonexpansive retraction from H onto C_i, that is

$$\|P_i x - P_i y\| \leqslant \|x - y\|$$

for all $x, y \in H$ and $P_i^2 = P_i$. Further, it holds that $F = \bigcap_{i=1}^m F(P_i)$, where $F(P_i)$ denotes the set of all fixed points of P_i, $i = 1, 2, \ldots, m$. Thus the problem of image recovery in the setting of Hilbert spaces is a common fixed point problem for a family of nonexpansive mappings.

Two classical iteration processes are often used to approximate a fixed point of a nonexpansive mapping. The first one is introduced in 1953 by Mann [17] which is well-known as Mann's iteration process and is defined as follows:

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$$(1.1)\qquad \begin{cases} x_0 & \in & C \text{ chosen arbitrarily,} \\ x_{n+1} & = & \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geqslant 0, \end{cases}$$

where the sequence $\{\alpha_n\}$ is chosen in $[0,1]$. Fourteen years later, Halpern [13] proposed the new innovation of iteration process which resemble in Mann's iteration (1.1). It is defined by

$$(1.2)\qquad \begin{cases} u & \in & C \text{ chosen arbitrarily,} \\ x_{n+1} & = & \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geqslant 0. \end{cases}$$

For finding a solution of the image recovery problem, *block-iterative projection algorithm* is the one well-known method which was proposed by Aharoni and Censor [1] in finite-dimensional spaces; see also [5, 6, 9, 11] and the references therein. This is an iterative procedure, which generates a sequence $\{x_n\}$ by the rule $x_1 = x \in H$ and

$$(1.3)\qquad x_{n+1} = \sum_{i=1}^{m} \xi_n^{(i)}(\alpha_i x_n + (1-\alpha_i)P_i x_n) \quad (n = 1, 2, \ldots),$$

where $\{\xi_n^{(i)}\}_{i=1}^m \subset [0,1]$ $(n \in \mathbb{N})$ with $\sum_{i=1}^m \xi_n^{(i)} = 1$ $(n \in \mathbb{N})$ and $\{\alpha_i\}_{i=1}^m \subset (-1,1)$. In particular, Butnariu and Censor [6] studied the strong convergence of the process (1.3) to an element of F.

In general not much has been known regarding the convergence of the iteration processes (1.1) and (1.2) unless the underlying space E has elegant properties which we briefly mention here.

Reich [22] proved that if E is a uniformly convex Banach space with a Fréchet differentiable norm and if $\{\alpha_n\}$ is chosen such that $\sum_{n=0}^\infty \alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.1) converges weakly to a fixed point of T. However we note that Mann's iteration process (1.1) has only weak convergence even in a Hilbert space [12].

In both Hilbert spaces [13, 16, 29] and uniformly smooth Banach spaces [23, 26, 31] the iteration process (1.2) has been proved to be strongly convergent if the sequence $\{\alpha_n\}$ satisfies the following conditions:

(i) $\alpha_n \to 0$;

(ii) $\displaystyle\sum_{n=0}^{\infty} \alpha_n = \infty$ and;

(iii) either $\displaystyle\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ or $\displaystyle\lim_{n\to\infty} \frac{\alpha_n}{\alpha_{n+1}} = 1$.

By the restriction of condition (ii), it is widely believed that the Halpern's iteration process (1.2) to have slow convergence though the rate of convergence has not be determined. Halpern [13] proved that conditions (i) and (ii) are necessary in the strong convergence of (1.2) for a nonexpansive mapping T on a closed convex subset C of a Hilbert space H. Moreover, Wittmann [29] showed that (1.2) converges strongly to $P_{F(T)}u$ when $\{\alpha_n\}$ satisfies (i), (ii) and $\sum_{n=0}^\infty |\alpha_n - \alpha_{n+1}| < \infty$ where $P_{F(T)}(\cdot)$ is the metric projection onto $F(T)$.

Some attempts to modify the Mann iteration method so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [21] proposed the following modification of the Mann iteration method for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 &= x \in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n &= \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n\rangle \geqslant 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, ..., \end{cases} \tag{1.4}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded away from one, then $\{x_n\}$ defined by (1.4) converges strongly to $P_{F(T)}x$.

Recently, Martinez-Yanes and Xu [18] has adapted Nakajo and Takahashi's [21] idea to modify the process (1.2) for a single nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 &= x \in C, \\ y_n &= \alpha_n x_0 + (1-\alpha_n)Tx_n, \\ C_n &= \{v \in C : \|y_n - v\|^2 \leqslant \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n &= \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geqslant 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0. \end{cases} \tag{1.5}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. They proved that if $\{\alpha_n\} \subset (0,1)$ and $\lim_{n\to\infty} \alpha_n = 0$, then the sequence $\{x_n\}$ generated by (1.5) converges strongly to $P_{F(T)}x$.

As we all know that if C is a nonempty closed convex subset of a Hilbert space H and $x \in H$ is an arbitrary point, there exists a unique $z \in C$ such that

$$\|x - z\| = \min_{y \in C} \|x - y\|.$$

This idea leads to the definition of the metric projection P_C from H onto C. It is well known that P_C is also nonexpansive. This fact actually characterizes Hilbert spaces. It is not available in more general Banach space. Some attempts to generalize the metric projection from Hilbert spaces to Banach spaces appear in 1996, Alber [2] introduced another generalization of the metric projection operator in Hilbert spaces to that in Banach spaces, which is called the *generalized projection*; see also Kamimura and Takahashi [15]. This projection is known to be the Bregman projection with respect to the Bregman function $\|\cdot\|^2$.

The ideas to generalize the process (1.4) from Hilbert spaces to Banach spaces have recently been made. By using available properties on uniformly convex and uniformly smooth Banach space, Matsushita and Takahashi [20] presented their ideas as the following method for a single relatively nonexpansive mapping T in a

Banach space E:

$$(1.6)\qquad \begin{cases} x_0 &= x \in C \\ y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

where J is the duality mapping on E, and $\Pi_{F(T)}(\cdot)$ is the generalized projection from C onto $F(T)$.

On the other hand, Censor and Reich [7] introduced a convex combination which is based on Bregman distance [4] and studied some iterative schemes for finding a common asymptotic fixed point of a family of operators in finite dimensional spaces. Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let J be the duality mapping from E into E^*, and let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into itself such that the set of all common fixed points of $\{T_i\}_{i=1}^m$ is nonempty. Motivated by the convex combination based on Bregman distances [4] due to Censor and Reich [7], we define an operator $G_n : C \to E$ $(n \in \mathbb{N})$ by

$$G_n := J^{-1}\left(\sum_{i=1}^m \xi_n^{(i)}(\beta_n^{(i)} J + (1-\beta_n^{(i)})JT_i)\right)$$

where $\{\xi_n^{(i)}\}, \{\beta_n^{(i)}\} \subset [0,1]$ with $\sum_{i=1}^m \xi_n^{(i)} = 1$ $(n \in \mathbb{N})$. Such a mapping G_n is called a *block mapping* defined by $T_1, T_2, \ldots, T_m$, $\{\xi_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$.

Inspired and motivated by these facts, we purpose for the paper to improve and generalize the processes (1.5) and (1.6) to the new general processes by using the block iterative methods for a finite family of relatively nonexpansive mappings in Banach spaces. Let C be a closed convex subset of a Banach space E and $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings such that $F := \bigcap_{i=1}^m F(T_i) \neq \varnothing$. Define $\{x_n\}$ in the two following ways:

$$(1.7)\qquad \begin{cases} x_0 &= x \in C \\ y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JG_nx_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

and

(1.8)

$$\begin{cases} x_0 &= x \in C \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JG_nx_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n^{(i)}\}_{i=1}^m$, and $\{\xi_n^{(i)}\}_{i=1}^m$ are sequences in $[0,1]$ with $\sum_{i=1}^m \xi_n^{(i)} = 1$ for all $n \in \mathbb{N} \cup \{0\}$.

We shall prove that both iterations (1.7) and (1.8) converge strongly to a common fixed point of a finite family of relatively nonexpansive mappings T_i, $i = 1, 2, \ldots, m$ provided that $\{\alpha_n\}$, $\{\beta_n^{(i)}\}$, and $\{\xi_n^{(i)}\}$ satisfy some appropriate conditions. Our results extend and improve the corresponding ones announced by Nakajo and Takahashi [21], Martinez-Yanes and Xu [18] and Matsushita and Takahashi [20].

Throughout the paper, we will use the notation:

(1) $\to$ for strong convergence and $\rightharpoonup$ for weak convergence.

(2) $\omega_w(x_n) = \{x : \exists x_{n_r} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual of E. Denote by $\langle \cdot, \cdot \rangle$ the duality product. The normalized duality mapping J from E to E^* is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for $x \in E$.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}, \{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. It is also said to be uniformly smooth if the limit is attained uniformly for $x, y \in U$. It is well known that ℓ^p and L^p $(1 < p < \infty)$ are uniformly convex and uniformly smooth; see Cioranescu [8] or Diestel [10]. We know that if E is smooth, then the duality mapping J is single valued. It is also known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Some properties of the duality mapping have been given in [8, 24, 27, 28]. A Banach space E is said to have the Kadec-Klee property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the Kadec-Klee property; see [8, 27, 28] for more details. Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in E$. It is obvious from the definition of the function ϕ that

(1) $(\|y\| - \|x\|)^2 \leqslant \phi(y, x) \leqslant (\|y\| + \|x\|)^2$,

(2) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$,

(3) $\phi(x, y) = \langle x, Jx - Jy \rangle + \langle y - x, Jy \rangle \leqslant \|x\|\|Jx - Jy\| + \|y - x\|\|y\|$,

for all $x, y, z \in E$. Following Alber [2], we define the *generalized projection* from E onto C by

$$\Pi_C(x) = \arg\min_{y \in C} \phi(y, x)$$

for all $x \in E$; see also Kamimura and Takahashi [15]. If E is a Hilbert space, then $\phi(y, x) = \|y - x\|^2$ for all $x, y \in E$, and hence Π_C is reduced to the metric projection P_C. It should be noted that the mapping ϕ is known to be the *Bregman distance* [4] corresponding to the Bregman function $\|\cdot\|^2$, and hence the projection Π_C is the *Bregman projection* corresponding to ϕ.

This section collects some definitions and lemmas which will be used in the proofs for the main results in the next section. Some of them are known; others are not hard to derive.

Remark 2.1. If E is a strictly convex and smooth Banach space, then for $x, y \in E$, $\phi(y, x) = 0$ if and only if $x = y$. It is sufficient to show that if $\phi(y, x) = 0$ then $x = y$. From (1), we have $\|x\| = \|y\|$. This implies $\langle y, Jx \rangle = \|y\|^2 = \|Jx\|^2$. From the definition of J, we have $Jx = Jy$. Since J is one-to-one, we have $x = y$; see [8, 27, 28] for more details.

Lemma 2.2 (Kamimura and Takahashi [15])**.** *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty closed convex subset of a smooth, strictly convex, and reflexive Banach space E, let T be a mapping from C into itself, and let $F(T)$ be the set of all fixed points of T. Then a point $p \in C$ is said to be an *asymptotic fixed point* of T (see Reich [25]) if there exists a sequence $\{x_n\}$ in C converging weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote the set of all asymptotic fixed points of T by $\hat{F}(T)$ and we say that T is a *relatively nonexpansive mapping* if the following conditions are satisfied:

(R1) $F(T)$ is nonempty;

(R2) $\phi(u, Tx) \leqslant \phi(u, x)$ for all $u \in F(T)$ and $x \in C$;

(R3) $\hat{F}(T) = F(T)$.

Some examples of relatively nonexpansive mappings are listed below; see Reich [25] and Matsushita and Takahashi [19] for more details.

(1) If Ω is a nonempty closed convex subset of a Hilbert space H and T is nonexpansive mapping from Ω into itself such that $F(T)$ is nonempty, then T is a relatively nonexpansive mapping from Ω into itself.
(2) If E is a uniformly smooth and strictly convex Banach space and $A \subset E \times E^*$ is a maximal monotone operator such that $A^{-1}(0)$ is nonempty, then the resolvent $J_r = (J + rA)^{-1}J$ $(r > 0)$ is a relatively nonexpansive mapping from E onto $D(A)$ and $F(J_r) = A^{-1}(0)$.
(3) If Π_Ω is the generalized projection from a smooth, strictly convex, and reflexive Banach space E onto a nonempty closed convex subset C of E, then Π_Ω is a relatively nonexpansive mapping from E onto Ω and $F(\Pi_\Omega) = \Omega$.

(4) If $\{\Omega_i\}_{i=1}^m$ is a finite family of closed convex subset of a uniformly convex and uniformly smooth Banach space E such that $\bigcap_{i=1}^m \Omega_i$ is nonempty and $T = \Pi_{\Omega_1}\Pi_{\Omega_2}\cdots\Pi_{\Omega_m}$ is the composition of the generalized projections Π_{Ω_i} from E onto Ω_i, $i = 1, 2, \dots, m$, then T is a relatively nonexpansive mapping from E into itself and $F(T) = \bigcap_{i=1}^m \Omega_i$.

Lemma 2.3 (Alber [2], Alber and Reich [3], Kamimura and Takahashi [15])**.** *Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$, and let $x_0 \in C$. Then, $x_0 = \Pi_C x$ if and only if $\langle x_0 - y, Jx - Jx_0\rangle \geqslant 0$ for all $y \in C$.*

Lemma 2.4 (Alber [2], Alber and Reich [3], Kamimura and Takahashi [15])**.** *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then $\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leqslant \phi(y, x)$ for all $y \in C$.*

Lemma 2.5. *Let X be a uniformly convex Banach space and $B_r(0) = \{x \in E : \|x\| \leqslant r\}$ be a closed ball of X. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\left\|\sum_{i=1}^m \xi^{(i)} x_i\right\|^2 \leqslant \sum_{i=1}^m \xi^{(i)}\|x_i\|^2 - \xi^{(j)}\xi^{(k)} g(\|x_j - x_k\|), \text{ for any } j, k \in \{1, 2, \dots, m\},$$

where $\{x_i\}_{i=1}^m \subset B_r(0)$ and $\{\xi^{(i)}\}_{i=1}^m \subset [0, 1]$ with $\sum_{i=1}^m \xi^{(i)} = 1$.

Proof. It sufficient to show that

$$\left\|\sum_{i=1}^m \xi^{(i)} x_i\right\|^2 \leqslant \sum_{i=1}^m \xi^{(i)}\|x_i\|^2 - \xi^{(1)}\xi^{(2)} g(\|x_1 - x_2\|). \tag{2.1}$$

It is obvious that (2.1) holds for $m = 1, 2$ (see [30] for more details.). Next, assume that (2.1) is true for $m - 1$. It remains to show that (2.1) holds for m. We observe that

$$\begin{aligned}
\left\|\sum_{i=1}^m \xi^{(i)} x_i\right\|^2 &= \left\|\xi^{(m)} x_m + (1 - \xi^{(m)})\left(\sum_{i=1}^{m-1} \frac{\xi^{(i)}}{1 - \xi^{(m)}} x_i\right)\right\|^2 \\
&\leqslant \xi^{(m)}\|x_m\|^2 + (1 - \xi^{(m)})\left\|\sum_{i=1}^{m-1} \frac{\xi^{(i)}}{1 - \xi^{(m)}} x_i\right\|^2 \\
&\leqslant \xi^{(m)}\|x_m\|^2 + (1 - \xi^{(m)})\left(\sum_{i=1}^{m-1} \frac{\xi^{(i)}}{1 - \xi^{(m)}}\|x_i\|^2 - \frac{\xi^{(1)}\xi^{(2)}}{(1 - \xi^{(m)})^2} g(\|x_1 - x_2\|)\right) \\
&= \sum_{i=1}^m \xi^{(i)}\|x_i\|^2 - \frac{\xi^{(1)}\xi^{(2)}}{(1 - \xi^{(m)})} g(\|x_1 - x_2\|) \\
&\leqslant \sum_{i=1}^m \xi^{(i)}\|x_i\|^2 - \xi^{(1)}\xi^{(2)} g(\|x_1 - x_2\|).
\end{aligned}$$

This completes the proof. □

Lemma 2.6. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a closed convex subset of E and let $w, x, y, z \in E$. Let $a \in \mathbb{R}$. Then the set $K := \{v \in C : \phi(v, y) \leqslant \phi(v, x) + \langle v, Jz - Jw\rangle + a\}$ is closed and convex.*

Proof. As a matter of fact, the defining inequality in K is equivalent to the inequality

$$\langle v, 2(Jx - Jy) - (Jz - Jw)\rangle \leqslant \|x\|^2 - \|y\|^2 + a.$$

This inequality is affine in v and hence the set K is closed and convex. □

3. Main result

In this section, we prove strong convergence theorems for finding a common fixed point of a finite family of relatively nonexpansive mappings in Banach spaces by using the hybrid method in mathematical programming.

Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into itself such that $\bigcap_{i=1}^m F(T_i)$ is nonempty and define $G : C \to E$ by

$$G := J^{-1}\left(\sum_{i=1}^m \xi^{(i)}(\beta^{(i)}J + (1 - \beta^{(i)})JT_i)\right) \tag{3.1}$$

where $\{\xi^{(i)}\}_{i=1}^m, \{\beta^{(i)}\}_{i=1}^m \subset [0, 1]$ with $\sum_{i=1}^m \xi^{(i)} = 1$. The mapping G is called a *block mapping* defined by $\{T_i\}_{i=1}^m$, $\{\xi^{(i)}\}_{i=1}^m$ and $\{\beta^{(i)}\}_{i=1}^m$.

Lemma 3.1. *Let E be a smooth, strictly convex, and reflexive Banach space and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^m F(T_i)$ is nonempty and let G be the block mapping defined by* (3.1)*, where $\{\xi^{(i)}\}_{i=1}^m, \{\beta^{(i)}\}_{i=1}^m \subset [0, 1]$ with $\sum_{i=1}^m \xi^{(i)} = 1$. Then*

$$\phi(u, Gx) \leqslant \phi(u, x)$$

for all $u \in F$ and $x \in C$.

Proof. Let $u \in F$. By the convexity of $\|\cdot\|^2$, we observe that

$$\begin{aligned}
&\phi(u, Gx) = \|u\|^2 - 2\langle u, JGx\rangle + \|Gx\|^2 \\
= \;& \|u\|^2 - 2\left\langle u, \sum_{i=1}^m \xi^{(i)}(\beta^{(i)}Jx + (1 - \beta^{(i)})JT_ix)\right\rangle \\
&+ \left\|\sum_{i=1}^m \xi^{(i)}(\beta^{(i)}Jx + (1 - \beta^{(i)})JT_ix)\right\|^2 \\
\leqslant \;& \sum_{i=1}^m \xi^{(i)}\left(\|u\|^2 - 2\left\langle u, \beta^{(i)}Jx + (1 - \beta^{(i)})JT_ix\right\rangle + \left\|\beta^{(i)}Jx + (1 - \beta^{(i)})JT_ix\right\|^2\right) \\
\leqslant \;& \sum_{i=1}^m \xi^{(i)}\left(\beta^{(i)}\phi(u, x) + (1 - \beta^{(i)})\phi(u, T_ix)\right) \leqslant \phi(u, x)
\end{aligned}$$

for all $x \in C$. □

Applying (3.1), we can define a sequence of mappings $G_n : C \to E$ by

$$G_n := J^{-1}\left(\sum_{i=1}^{m} \xi_n^{(i)}(\beta_n^{(i)} J + (1-\beta_n^{(i)})JT_i)\right), \tag{3.2}$$

for any $n \in \mathbb{N}\cup\{0\}$, where $\{\xi_n^{(i)}\}_{i=1}^m, \{\beta_n^{(i)}\}_{i=1}^m \subset [0,1]$ with $\sum_{i=1}^m \xi_n^{(i)} = 1$. For any $n \in \mathbb{N}\cup\{0\}$ the mapping G_n is called a *block mapping* defined by $\{T_i\}_{i=1}^m$, $\{\xi_n^{(i)}\}_{i=1}^m$ and $\{\beta_n^{(i)}\}_{i=1}^m$.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^m F(T_i)$ is nonempty. Let $\{x_n\}$ be a sequence defined by*

$$\begin{cases} x_0 &= x \in C, \\ y_n &= J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JG_n x_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases} \tag{3.3}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n^{(i)}\} \subset [0,1]$ and $\{\xi_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N}\cup\{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\liminf_{n\to\infty} \beta_n^{(i)}(1-\beta_n^{(i)}) > 0$ *for all* $i = 1, 2, \ldots, m$,

(iii) $\liminf_{n\to\infty} \xi_n^{(i)} > 0$ *for all* $i = 1, 2, 3, \ldots, m$ *and* $\sum_{i=1}^m \xi_n^{(i)} = 1$ *for all* $n \in \mathbb{N}\cup\{0\}$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x$, where Π_F is the generalized projection from C onto F.

Proof. From the definitions of H_n and W_n, it is obvious H_n and W_n are closed and convex for each $n \in \mathbb{N}\cup\{0\}$.

Next, we show that $F \subset H_n \cap W_n$ for each $n \in \mathbb{N}\cup\{0\}$. Let $u \in F$ and let $n \in \mathbb{N}\cup\{0\}$. Then, by Lemma 3.1, we have

$$\phi(u, G_n x_n) \quad \leqslant \quad \phi(u, x_n) \tag{3.4}$$

for all $n \in \mathbb{N}\cup\{0\}$, and then

$$\begin{aligned} &\phi(u, y_n) = \phi(u, J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JG_n x_n)) \\ = \;& \|u\|^2 - 2\langle u, \alpha_n Jx_n + (1-\alpha_n)JG_n x_n\rangle + \|\alpha_n Jx_n + (1-\alpha_n)JG_n x_n\|^2 \\ \leqslant \;& \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2(1-\alpha_n)\langle u, JG_n x_n\rangle + \alpha_n\|x_n\|^2 + (1-\alpha_n)\|G_n x_n\|^2 \\ = \;& \alpha_n(\|u\|^2 - 2\langle u, Jx_n\rangle + \|x_n\|^2) + (1-\alpha_n)(\|u\|^2 - 2\langle u, JG_n x_n\rangle + \|G_n x_n\|^2) \\ = \;& \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, G_n x_n) \leqslant \alpha_n\phi(u, x_n) + (1-\alpha_n)\phi(u, x_n) \\ = \;& \phi(u, x_n). \end{aligned}$$

Thus, we have $u \in H_n$. Therefore we obtain $F \subset H_n$ for each $n \in \mathbb{N}\cup\{0\}$. We note by [20, Proposion 2.4] that each $F(T_i)$ is closed and convex and so is F. Using the same argument presented in the proof of [20, Theorem 3.1; pp. 261-262] we have that $F \subset H_n \cap W_n$ for each $n \in \mathbb{N} \cup \{0\}$, $\{x_n\}$ is well defined and bounded, and

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.5}$$

Since $\|Jx_{n+1} - Jy_n\| = \|Jx_{n+1} - \alpha_n Jx_n - (1-\alpha_n)JG_nx_n\| \geqslant (1-\alpha_n)\|Jx_{n+1} - JG_nx_n\| - \alpha_n\|Jx_n - Jx_{n+1}\|$ for each $n \in \mathbb{N} \cup \{0\}$, we get that

$$\begin{aligned}\|Jx_{n+1} - JG_nx_n\| &\leqslant \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n\|Jx_n - Jx_{n+1}\|)\\ &\leqslant \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \|Jx_n - Jx_{n+1}\|).\end{aligned}$$

From (3.5) and $\limsup_{n\to\infty} \alpha_n < 1$, we have $\lim_{n\to\infty} \|Jx_{n+1} - JG_nx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - G_nx_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_{n+1}) - J^{-1}(JG_nx_n)\| = 0.$$

From $\|x_n - G_nx_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - G_nx_n\|$ we have $\lim_{n\to\infty} \|x_n - G_nx_n\| = 0$.

Next, we show that $\|x_n - T_ix_n\| \to 0$ for all $i = 1, 2, \ldots, m$. Since $\{x_n\}$ is bounded and $\phi(p, T_ix_n) \leqslant \phi(p, x_n)$ for all $i = 1, 2, \ldots, m$, where $p \in F$. We also obtain that $\{Jx_n\}$ and $\{JT_ix_n\}$ are bounded for all $i = 1, 2, \ldots, m$. Then there exists $r > 0$ such that $\{Jx_n\}, \{JT_ix_n\} \subset B_r(0)$ for all $i = 1, 2, \ldots, m$. Therefore Lemma 2.5 is applicable and we observe that

$$\begin{aligned}
\phi(p, G_nx_n) &= \|p\|^2 - 2\left\langle p, \sum_{i=1}^m \xi_n^{(i)}(\beta_n^{(i)}Jx_n + (1-\beta_n^{(i)})JT_ix_n)\right\rangle\\
&\quad + \left\|\sum_{i=1}^m \xi_n^{(i)}(\beta_n^{(i)}Jx_n + (1-\beta_n^{(i)})JT_ix_n)\right\|^2\\
&\leqslant \|p\|^2 - 2\sum_{i=1}^m \xi_n^{(i)}\left\langle p, \beta_n^{(i)}Jx_n + (1-\beta_n^{(i)})JT_ix_n\right\rangle\\
&\quad + \sum_{i=1}^m \xi_n^{(i)}\|\beta_n^{(i)}Jx_n + (1-\beta_n^{(i)})JT_ix_n\|^2\\
&= \sum_{i=1}^m \xi_n^{(i)}\Big(\|p\|^2 - 2\left\langle p, \beta_n^{(i)}Jx_n + (1-\beta_n^{(i)})JT_ix_n\right\rangle\\
&\quad + \|\beta_n^{(i)}Jx_n + (1-\beta_n^{(i)})JT_ix_n\|^2\Big)\\
&\leqslant \sum_{i=1}^m \xi_n^{(i)}(\|p\|^2 - 2\beta_n^{(i)}\langle p, Jx_n\rangle - 2(1-\beta_n^{(i)})\langle p, JT_ix_n\rangle\\
&\quad + \beta_n^{(i)}\|x_n\|^2 + (1-\beta_n^{(i)})\|T_ix_n\|^2 - \beta_n^{(i)}(1-\beta_n^{(i)})g(\|Jx_n - JT_ix_n\|))
\end{aligned}$$

$$\begin{aligned} &\leqslant \sum_{i=1}^{m} \xi_n^{(i)} \Big(\beta_n^{(i)} \phi(p, x_n) + (1-\beta_n^{(i)}) \phi(p, T_i x_n) \\ &\qquad -\beta_n^{(i)}(1-\beta_n^{(i)}) g(\|Jx_n - JT_i x_n\|) \Big) \\ &\leqslant \phi(p, x_n) - \sum_{i=1}^{m} \xi_n^{(i)} \beta_n^{(i)} (1-\beta_n^{(i)}) g(\|Jx_n - JT_i x_n\|), \end{aligned}$$

that is

$$\sum_{i=1}^{m} \xi_n^{(i)} \beta_n^{(i)} (1-\beta_n^{(i)}) g(\|Jx_n - JT_i x_n\|) \leqslant \phi(p, x_n) - \phi(p, G_n x_n), \tag{3.6}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous strictly increasing convex function with $g(0) = 0$ in Lemma 2.5.

Let $\{\|x_{n_l} - T_i x_{n_l}\|\}$ be any subsequence of $\{\|x_n - T_i x_n\|\}$. Since $\{x_{n_l}\}$ is bounded, there exists a subsequence $\{x_{n_r}\}$ of $\{x_{n_l}\}$ such that

$$\lim_{r\to\infty} \phi(p, x_{n_r}) = \limsup_{l\to\infty} \phi(p, x_{n_l}) := a,$$

where $p \in F$. By (2), we have

$$\begin{aligned} \phi(p, x_{n_r}) &= \phi(p, G_{n_r} x_{n_r}) + \phi(G_{n_r} x_{n_r}, x_{n_r}) + 2 \langle p - G_{n_r} x_{n_r}, JG_{n_r} x_{n_r} - Jx_{n_r} \rangle \\ &\leqslant \phi(p, G_{n_r} x_{n_r}) + \phi(G_{n_r} x_{n_r}, x_{n_r}) + M \|JG_{n_r} x_{n_r} - Jx_{n_r}\|, \end{aligned}$$

where $M = \sup_n 2\|p - G_n x_n\|$. Since

$$\lim_{r\to\infty} \phi(G_{n_r} x_{n_r}, x_{n_r}) = 0 = \lim_{r\to\infty} \|JG_{n_r} x_{n_r} - Jx_{n_r}\|,$$

it follows that

$$a = \liminf_{r\to\infty} \phi(p, x_{n_r}) \leqslant \liminf_{r\to\infty} \phi(p, G_{n_r} x_{n_r}).$$

By (3.4), we have

$$\limsup_{r\to\infty} \phi(p, G_{n_r} x_{n_r}) \leqslant \limsup_{r\to\infty} \phi(p, x_{n_r}) = a$$

and hence $\lim_{r\to\infty} \phi(p, x_{n_r}) = a = \lim_{r\to\infty} \phi(p, G_{n_r} x_{n_r})$. By (3.6), we observe that

$$\sum_{i=1}^{m} \xi_{n_r}^{(i)} \beta_{n_r}^{(i)} (1-\beta_{n_r}^{(i)}) g(\|Jx_{n_r} - JT_i x_{n_r}\|) \leqslant \phi(p, x_{n_r}) - \phi(p, G_{n_r} x_{n_r}) \to 0$$

as $r \to \infty$. Since $\liminf_{n\to\infty} \xi_n^{(i)} > 0$ and $\liminf_{n\to\infty} \beta_n^{(i)}(1-\beta_n^{(i)}) > 0$ for all $i \in \{1, 2, \ldots, m\}$, it follows that $\lim_{r\to\infty} g(\|Jx_{n_r} - JT_i x_{n_r}\|) = 0$ for all $i \in \{1, 2, \ldots, m\}$. By the properties of the mapping g, we have $\lim_{r\to\infty} \|Jx_{n_r} - JT_i x_{n_r}\| = 0$ for all $i \in \{1, 2, \ldots, m\}$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{r\to\infty} \|x_{n_r} - T_i x_{n_r}\| = \lim_{r\to\infty} \|J^{-1}(Jx_{n_r}) - J^{-1}(JT_i x_{n_r})\| = 0$$

and then $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0$ for all $i \in \{1, 2, \ldots, m\}$. Then $\omega_w(x_n) \subset \bigcap_{i=1}^{m} \hat{F}(T_i) = \bigcap_{i=1}^{m} F(T_i) = F$.

Finally, we show that $x_n \to \Pi_F x$. Using the same argument as in the proof of [20, Theorem 3.1; pp. 262-263], we have $\{x_n\}$ converges strongly to $\Pi_F x$. □

In the following theorem we deal with the strong convergence of the sequence $\{x_n\}$ by changing the conditions of $\{\xi_n^{(i)}\}_{i=1}^m$ and $\{\beta_n^{(i)}\}_{i=1}^m$.

Theorem 3.3. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^m F(T_i)$ is nonempty. Let a sequence $\{x_n\}$ defined by*

$$\begin{cases} x_0 & = & x \in C, \\ y_n & = & J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JG_n x_n), \\ H_n & = & \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n & = & \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} & = & \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases} \tag{3.7}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n^{(i)}\} \subset [0,1]$ and $\{\xi_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\beta_n^{(i)} =: \beta_n$ *for all* $i = 1, 2, \ldots, m$ *and* $\lim_{n\to\infty} \beta_n = 0$,

(iii) $\liminf_{n\to\infty} \xi_n^{(i)} \xi_n^{(j)} > 0$ *for all* $i \neq j$, $i, j = 1, 2, 3, \ldots, m$ *and* $\sum_{i=1}^m \xi_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then the sequence $\{x_n\}$ converges strongly to $\Pi_F x$, where Π_F is the generalized projection from C onto F.

Proof. From the definition of H_n and W_n, it is obvious H_n and W_n are closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

Next, we show that $F \subset H_n \cap W_n$ for each $n \in \mathbb{N} \cup \{0\}$. Let $u \in F$ and let $n \in \mathbb{N} \cup \{0\}$. Then, as in the proof of Theorem 3.2, we have

$$\phi(u, G_n x_n) \leqslant \phi(u, x_n) \tag{3.8}$$

for all $n \in \mathbb{N} \cup \{0\}$, and then $\phi(u, y_n) \leqslant \phi(u, x_n)$. Thus, we have $u \in H_n$. Therefore we obtain $F \subset H_n$ for each $n \in \mathbb{N} \cup \{0\}$. We note by [20, Proposion 2.4] that each $F(T_i)$ is closed and convex and so is F. Using the same argument presented in the proof of [20, Theorem 3.1; pp. 261-262] we have $F \subset H_n \cap W_n$ for each $n \in \mathbb{N} \cup \{0\}$, $\{x_n\}$ is well defined and bounded, and

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0.$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = 0. \tag{3.9}$$

As in the proof of Theorem 3.2, we also have that

$$\begin{aligned} \|Jx_{n+1} - JG_n x_n\| &\leqslant \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \alpha_n \|Jx_n - Jx_{n+1}\|) \\ &\leqslant \frac{1}{1-\alpha_n}(\|Jx_{n+1} - Jy_n\| + \|Jx_n - Jx_{n+1}\|). \end{aligned}$$

From (3.9) and $\limsup_{n\to\infty} \alpha_n < 1$, we have $\lim_{n\to\infty} \|Jx_{n+1} - JG_nx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - G_nx_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_{n+1}) - J^{-1}(JG_nx_n)\| = 0.$$

From $\|x_n - G_nx_n\| \leqslant \|x_n - x_{n+1}\| + \|x_{n+1} - G_nx_n\|$ we have $\lim_{n\to\infty} \|x_n - G_nx_n\| = 0$.

Next, we show that $\|x_n - T_ix_n\| \to 0$ for all $i = 1, 2, \ldots, m$. Since $\{x_n\}$ is bounded and $\phi(p, T_ix_n) \leqslant \phi(p, x_n)$ for all $i = 1, 2, \ldots, m$, where $p \in F$. We also obtain that $\{Jx_n\}$ and $\{JT_ix_n\}$ are bounded for all $i = 1, 2, \ldots, m$. So, there exists $r > 0$ such that $\{Jx_n\}, \{JT_ix_n\} \subset B_r(0)$ for all $i = 1, 2, \ldots, m$. Therefore Lemma 2.5 is applicable and we observe that

$$\begin{aligned}
\phi(p, G_nx_n) &= \|p\|^2 - 2\left\langle p, \sum_{i=1}^m \xi_n^{(i)}(\beta_nJx_n + (1-\beta_n)JT_ix_n)\right\rangle \\
&\quad + \left\|\sum_{i=1}^m \xi_n^{(i)}(\beta_nJx_n + (1-\beta_n)JT_ix_n)\right\|^2 \\
&\leqslant \|p\|^2 - 2\sum_{i=1}^m \xi_n^{(i)} \langle p, \beta_nJx_n + (1-\beta_n)JT_ix_n\rangle \\
&\quad + \sum_{i=1}^m \xi_n^{(i)}\|\beta_nJx_n + (1-\beta_n)JT_ix_n\|^2 \\
&\quad - \xi_n^{(j)}\xi_n^{(k)}g((1-\beta_n)\|JT_jx_n - JT_kx_n\|) \\
&= \sum_{i=1}^m \xi_n^{(i)}\Big(\|p\|^2 - 2\langle p, \beta_nJx_n + (1-\beta_n)JT_ix_n\rangle \\
&\quad + \|\beta_nJx_n + (1-\beta_n)JT_ix_n\|^2\Big) \\
&\quad - \xi_n^{(j)}\xi_n^{(k)}g((1-\beta_n)\|JT_jx_n - JT_kx_n\|) \\
&\leqslant \sum_{i=1}^m \xi_n^{(i)}\Big(\beta_n\phi(p, x_n) + (1-\beta_n)\phi(p, T_ix_n) \\
&\quad - \xi_n^{(j)}\xi_n^{(k)}g((1-\beta_n)\|JT_jx_n - JT_kx_n\|)\Big) \\
&\leqslant \phi(p, x_n) - \xi_n^{(j)}\xi_n^{(k)}g((1-\beta_n)\|JT_jx_n - JT_kx_n\|),
\end{aligned}$$

that is

$$\xi_n^{(j)}\xi_n^{(k)}g((1-\beta_n)\|JT_jx_n - JT_kx_n\|) \leqslant \phi(p, x_n) - \phi(p, G_nx_n), \tag{3.10}$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing convex function with $g(0) = 0$ in Lemma 2.5.

Let $\{\|T_jx_{n_l} - T_kx_{n_l}\|\}$ be any subsequence of $\{\|T_jx_n - T_kx_n\|\}$. Since $\{x_{n_l}\}$ is bounded, there exists $\{x_{n_r}\}$ a subsequence of $\{x_{n_l}\}$ such that

$$\lim_{r\to\infty} \phi(p, x_{n_r}) = \limsup_{l\to\infty} \phi(p, x_{n_l}) := a$$

where $p \in F$. As in the proof of Theorem 3.2, $\lim_{r\to\infty} \phi(p, x_{n_r}) = a = \lim_{r\to\infty} \phi(p, G_{n_r} x_{n_r})$. By (3.10), we observe that

$$\xi_{n_r}^{(j)} \xi_{n_r}^{(k)} g((1-\beta_{n_r})\|JT_j x_{n_r} - JT_k x_{n_r}\|) \leqslant \phi(p, x_{n_r}) - \phi(p, G_{n_r} x_{n_r}) \to 0$$

as $r \to \infty$. Since $\liminf_{n\to\infty} \xi_n^{(j)} \xi_n^{(k)} > 0$, it follows that $\lim_{r\to\infty} g((1-\beta_{n_r})\|JT_j x_{n_r} - JT_k x_{n_r}\|) = 0$. By the properties of the mapping g, we have $\lim_{r\to\infty}(1-\beta_{n_r})\|JT_j x_{n_r} - JT_k x_{n_r}\| = 0$ and then $\lim_{r\to\infty} \|JT_j x_{n_r} - JT_k x_{n_r}\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{r\to\infty} \|T_j x_{n_r} - T_k x_{n_r}\| = \lim_{r\to\infty} \|J^{-1}(JT_j x_{n_r}) - J^{-1}(JT_k x_{n_r})\| = 0$$

and then $\lim_{n\to\infty} \|T_j x_n - T_k x_n\| = 0$ for all $j \neq k$. Next, we observe from $\beta_n \to 0$ and (3) that

$$\begin{aligned}
\phi(T_j x_n, G_n x_n) &= \|T_j x_n\|^2 - 2\left\langle T_j x_n, \sum_{i=1}^m \xi_n^{(i)}(\beta_n J x_n + (1-\beta_n) JT_i x_n)\right\rangle \\
&\quad + \left\|\sum_{i=1}^m \xi_n^{(i)}(\beta_n J x_n + (1-\beta_n) JT_i x_n)\right\|^2 \\
&\leqslant \sum_{i=1}^m \xi_n^{(i)} \Big(\|T_j x_n\|^2 - 2\langle T_j x_n, \beta_n J x_n + (1-\beta_n) JT_i x_n\rangle \\
&\quad + \|\beta_n J x_n + (1-\beta_n) JT_i x_n\|^2 \Big) \\
&\leqslant \sum_{i=1}^m \xi_n^{(i)} \left(\beta_n \phi(T_j x_n, x_n) + (1-\beta_n)\phi(T_j x_n, T_i x_n)\right) \to 0 \\
&\quad \text{as} \quad n \to \infty.
\end{aligned}$$

By Lemma 2.2, we have $\lim_{n\to\infty} \|T_j x_n - G_n x_n\| = 0$ for all $j = 1, 2, \ldots, m$ and hence

$$\|T_j x_n - x_n\| \leqslant \|T_j x_n - G_n x_n\| + \|G_n x_n - x_n\| \to 0 \quad \text{as} \quad n \to \infty,$$

for all $j = 1, 2, \ldots, m$. Then $\omega_w(x_n) \subset \bigcap_{i=1}^m \hat{F}(T_i) = \bigcap_{i=1}^m F(T_i) = F$.

Finally, we show that $x_n \to \Pi_F x$. Using the same argument as in the proof of [20, Theorem 3.1; pp. 262-263], we have $\{x_n\}$ converges strongly to $\Pi_F x$. □

If $\beta_n = 0$ and $T_1 = T_2 = \ldots = T_m =: T$ for all $n \in \mathbb{N} \cup \{0\}$, then Theorem 3.3 reduces to the following corollary.

Corollary 3.4 (Matsushita and Takahashi [20, Theorem 4.1]). *Let E be a uniformly convex and uniformly smooth Banach space, let C be a nonempty closed convex subset of E, let T be a relatively nonexpansive mapping from C into itself, and let $\{\alpha_n\}$ be sequence of real numbers such that $0 \leqslant \alpha_n < 1$ and $\limsup_{n\to\infty} \alpha_n < 1$. If*

$F(T)$ is nonempty, then the sequence $\{x_n\}$ generated by

$$\begin{cases} x_0 & = & x \in C \\ y_n & = & J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ H_n & = & \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n)\}, \\ W_n & = & \{z \in C : \langle x_n - z, Jx - Jx_n \rangle \geqslant 0\}, \\ x_{n+1} & = & \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

converges strongly to $\Pi_{F(T)}x$, where $\Pi_{F(T)}$ is the generalized projection from C onto $F(T)$.

If E in Theorem 3.3 is a Hilbert space, then we have the following corollary.

Corollary 3.5. *Let C be a nonempty closed convex subset of a Hilbert space H, and let $\{T_i\}_{i=1}^m$ be a finite family of nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^m F(T_i)$ is nonempty. Suppose that $\{x_n\}$ is given by*

$$\begin{cases} x_0 & = & x \in C, \\ y_n & = & \alpha_n x_n + (1-\alpha_n) z_n, \\ z_n & = & \sum_{i=1}^m \xi_n^{(i)} (\beta_n x_n + (1-\beta_n) T_i x_n), \\ C_n & = & \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\}, \\ Q_n & = & \{z \in C : \langle x_n - z, x - x_n \rangle \geqslant 0\}, \\ x_{n+1} & = & P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n\} \subset [0,1]$ and $\{\xi_n^{(i)}\} \subset [0,1]$ satisfy the following conditions:

(i) $0 \leqslant \alpha_n < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $\limsup_{n\to\infty} \alpha_n < 1$,

(ii) $\lim_{n\to\infty} \beta_n = 0$,

(iii) $\liminf_{n\to\infty} \xi_n^{(i)} \xi_n^{(j)} > 0$ *for all* $i \neq j$, $i, j = 1, 2, 3, \ldots, m$ *and* $\sum_{i=1}^m \xi_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

where $P_{C_n \cap Q_n}$ is the metric projection from C onto $C_n \cap Q_n$. Then $\{x_n\}$ converges strongly to $P_F x$, where P_F is the metric projection from C onto F.

Proof. By the proof of [20, Theorem 4.1], we have each T_i is relatively nonexpansive for all $i = 1, 2, \ldots, m$. Using Theorem 3.3, we obtain the desired result. □

In the case that $\beta_n = 0$ and $T_1 = T_2 = \ldots = T_m =: T$ for all $n \in \mathbb{N} \cup \{0\}$, Corollary 3.5 reduces to the following corollary.

Corollary 3.6 (Nakajo and Takahashi [21]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a nonexpansive mapping such that $F(T)$ is not empty. Assume that $\{\alpha_n\} \subset [0, a]$ for some $a \in [0, 1)$. Then the* *sequence* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 &= x \in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n &= \{z \in C : \|z - y_n\| \leqslant \|z - x_n\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n\rangle \geqslant 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

converges in norm to the fixed point $P_{F(T)}(x_0)$*, where* $P_{F(T)}$ *is the metric projection from* C *onto* $F(T)$*.*

Finally, we prove two strong convergence theorems of Halpern's type for a finite family of relatively nonexpansive mappings by using the hybrid method in mathematical programming.

Theorem 3.7. *Let* E *be a uniformly convex and uniformly smooth Banach space, and let* C *be a nonempty closed convex subset of* E*. Let* $\{T_i\}_{i=1}^m$ *be a finite family of relatively nonexpansive mappings from* C *into itself such that* $F := \bigcap_{i=1}^m F(T_i)$ *is nonempty. Let a sequence* $\{x_n\}$ *defined by*

(3.11)

$$\begin{cases} x_0 &= x \in C, \\ y_n &= J^{-1}(\alpha_n Jx_0 + (1-\alpha_n)JG_n x_n), \\ H_n &= \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\}, \\ W_n &= \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

where $\{\alpha_n\} \subset [0,1]$*,* $\{\beta_n^{(i)}\} \subset [0,1]$ *and* $\{\xi_n^{(i)}\} \subset [0,1]$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\liminf_{n\to\infty} \beta_n^{(i)}(1-\beta_n^{(i)}) > 0$ *for all* $i = 1, 2, \ldots, m$*,*

(iii) $\liminf_{n\to\infty} \xi_n^{(i)} > 0$ *for all* $i = 1, 2, 3, \ldots, m$ *and* $\sum_{i=1}^m \xi_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$*.*

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x$*, where* Π_F *is the generalized projection from* C *onto* F*.*

Proof. We first show that H_n and W_n are closed and convex for each $n \in \mathbb{N} \cup \{0\}$. From the definition of W_n, it is obvious that W_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. By Lemma 2.6, H_n is also closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

We claim that $F \subset H_n$ for all $n \in \mathbb{N} \cup \{0\}$. Let $p \in F$. By the same argument as in the proof of Theorem 3.3, we have $\phi(p, G_n x_n) \leqslant \phi(p, x_n)$. Then, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \phi(p, y_n) &= \|p\|^2 - 2\langle p, \alpha_n Jx_0 + (1-\alpha_n)JG_n x_n\rangle + \|\alpha_n Jx_0 + (1-\alpha_n)JG_n x_n\|^2 \\ &\leqslant \|p\|^2 - 2\alpha_n\langle p, Jx_0\rangle - 2(1-\alpha_n)\langle p, JG_n x_n\rangle \\ &\quad + \alpha_n\|x_0\|^2 + (1-\alpha_n)\|G_n x_n\|^2 \end{aligned}$$

$$\begin{aligned}
&= \alpha_n\phi(p,x_0)+(1-\alpha_n)\phi(p,G_nx_n) \leqslant \alpha_n\phi(p,x_0)+(1-\alpha_n)\phi(p,x_n)\\
&= \phi(p,x_n)+\alpha_n(\phi(p,x_0)-\phi(p,x_n))\\
&= \phi(p,x_n)+\alpha_n(\|x_0\|^2-\|x_n\|^2+2\langle p, Jx_n-Jx_0\rangle)\\
&\leqslant \phi(p,x_n)+\alpha_n(\|x_0\|^2+2\langle p, Jx_n-Jx_0\rangle).
\end{aligned}$$

This implies that $p \in H_n$ and hence $F \subset H_n$ for all $n \in \mathbb{N}\cup\{0\}$. By the same argument as in the proof of [20, Theorem 3.1, pp. 261-262], we obtain $F \subset H_n \cap W_n$ for all $n \in \mathbb{N}\cup\{0\}$, $\{x_n\}$ is well defined and bounded, and $\|x_{n+1}-x_n\| \to 0$. Since $x_{n+1} = \Pi_{H_n\cap W_n}x \in H_n$, we have

$$\phi(x_{n+1},y_n) \leqslant \phi(x_{n+1},x_n)+\alpha_n(\|x_0\|^2+2\langle x_{n+1}, Jx_n-Jx\rangle) \to 0 \quad \text{as } n\to\infty.$$

By Lemma 2.2, we have $\|x_{n+1}-y_n\| \to 0$ and then

$$\|y_n-x_n\| \leqslant \|y_n-x_{n+1}\|+\|x_{n+1}-x_n\| \to 0.$$

We observe that

$$\begin{aligned}
\phi(G_nx_n,x_n) &= \phi(G_nx_n,y_n)+\phi(y_n,x_n)+2\langle G_nx_n-y_n, Jy_n-Jx_n\rangle\\
&\leqslant \phi(G_nx_n,y_n)+\phi(y_n,x_n)+2\|G_nx_n-y_n\|\|Jy_n-Jx_n\|.
\end{aligned}$$

Further, from $\alpha_n \to 0$, we have that

$$\begin{aligned}
\phi(G_nx_n,y_n) &= \|G_nx_n\|^2-2\langle G_nx_n, \alpha_nJx_0+(1-\alpha_n)JG_nx_n\rangle\\
&\quad +\|\alpha_nJx_0+(1-\alpha_n)JG_nx_n\|^2\\
&\leqslant \|G_nx_n\|^2-2\alpha_n\langle G_nx_n, Jx_0\rangle-2(1-\alpha_n)\langle G_nx_n, JG_nx_n\rangle\\
&\quad +\alpha_n\|x_0\|^2+(1-\alpha_n)\|G_nx_n\|^2\\
&= \alpha_n\phi(G_nx_n,x_0)+(1-\alpha_n)\phi(G_nx_n,G_nx_n)\\
&= \alpha_n\phi(G_nx_n,x_0) \to 0.
\end{aligned}$$

Since $\lim_{n\to\infty}\phi(y_n,x_n) = 0 = \lim_{n\to\infty}\|G_nx_n-y_n\|\|Jy_n-Jx_n\|$, it follows that $\lim_{n\to\infty}\phi(G_nx_n,x_n) = 0$. Using Lemma 2.2 we have that $\|G_nx_n-x_n\| \to 0$. By the same argument as in the proof of Theorem 3.2, we have $\lim_{n\to\infty}\|x_n-T_ix_n\| = 0$ for all $i = \{1,2,\ldots,m\}$. Using the same argument as in the last part of proof of Theorem 3.2, we have $\{x_n\}$ converges strongly to Π_Fx. □

In the following theorem we deal with the strong convergence of the sequence $\{x_n\}$ by changing the conditions of $\{\xi_n^{(i)}\}_{i=1}^m$ and $\{\beta_n^{(i)}\}_{i=1}^m$.

Theorem 3.8. *Let E be a uniformly convex and uniformly smooth Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^m$ be a finite family of relatively nonexpansive mappings from C into itself such that $F := \bigcap_{i=1}^m F(T_i)$ is nonempty. Let a sequence $\{x_n\}$ defined by*

(3.12)

$$\begin{cases}
x_0 &= x \in C,\\
y_n &= J^{-1}(\alpha_nJx_0+(1-\alpha_n)JG_nx_n),\\
H_n &= \{z \in C : \phi(z,y_n) \leqslant \phi(z,x_n)+\alpha_n(\|x_0\|^2+2\langle z, Jx_n-Jx\rangle)\},\\
W_n &= \{z \in C : \langle x_n-z, Jx-Jx_n\rangle \geqslant 0\},\\
x_{n+1} &= \Pi_{H_n\cap W_n}x, \quad n = 0,1,2,...,
\end{cases}$$

where $\{\alpha_n\} \subset [0,1]$, $\{\beta_n^{(i)}\} \subset [0,1]$ *and* $\{\xi_n^{(i)}\} \subset [0,1]$ *satisfy the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$,

(ii) $\beta_n^{(i)} =: \beta_n$ *for all* $i = 1, 2, \dots, m$ *and* $\lim_{n\to\infty} \beta_n = 0$,

(iii) $\liminf_{n\to\infty} \xi_n^{(i)} \xi_n^{(j)} > 0$ *for all* $i \neq j$, $i, j = 1, 2, 3, \dots, m$ *and* $\sum_{i=1}^m \xi_n^{(i)} = 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then the sequence $\{x_n\}$ *converges strongly to* $\Pi_F x$, *where* Π_F *is the generalized projection from* C *onto* F.

Proof. As in the proofs of Theorem 3.3 and Theorem 3.7, we have the desired result. □

If $\beta_n = 0$ and $T_1 = T_2 = \dots = T_m =: T$, then Theorem 3.8 reduces to the following result.

Corollary 3.9. *Let* E *be a uniformly convex and uniformly smooth Banach space, let* C *be a nonempty closed convex subset of* E, *let* T *be a relatively nonexpansive mapping from* C *into itself, and* $\{\alpha_n\} \subset [0,1]$ *is such that* $\lim_{n\to\infty} \alpha_n = 0$. *Suppose that* $\{x_n\}$ *is given by*

$$\begin{cases} x_0 & = & x \in C \\ y_n & = & J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J T x_n), \\ H_n & = & \{z \in C : \phi(z, y_n) \leqslant \phi(z, x_n) + \alpha_n(\|x_0\|^2 + 2\langle z, Jx_n - Jx\rangle)\}, \\ W_n & = & \{z \in C : \langle x_n - z, Jx - Jx_n\rangle \geqslant 0\}, \\ x_{n+1} & = & \Pi_{H_n \cap W_n} x, \quad n = 0, 1, 2, ..., \end{cases}$$

where J *is the duality mapping on* E. *If* $F(T)$ *is nonempty, then* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)} x$, *where* $\Pi_{F(T)}$ *is the generalized projection from* C *onto* $F(T)$.

If E in Corollary 3.9 is a Hilbert space, we have the following result.

Corollary 3.10 (Martinez-Yanes and Xu [18, Theorem 3.1])**.** *Let* H *be a real Hilbert space,* C *a closed convex subset of* H *and* $T : C \to C$ *a nonexpansive mapping. Assume that* $\{\alpha_n\} \subset (0,1)$ *is such that* $\lim_{n\to\infty} \alpha_n = 0$. *If* $F(T) \neq \varnothing$, *then the sequence* $\{x_n\}$ *generated by*

$$\begin{cases} x_0 & = & x \in C, \\ y_n & = & \alpha_n x_0 + (1-\alpha_n) T x_n, \\ C_n & = & \{v \in C : \|y_n - v\|^2 \leqslant \|x_n - v\|^2 + \alpha_n(\|x_0\|^2 + 2\langle x_n - x_0, v\rangle)\}, \\ Q_n & = & \{v \in C : \langle x_n - v, x_0 - x_n\rangle \geqslant 0\}, \\ x_{n+1} & = & P_{C_n \cap Q_n} x_0 \end{cases}$$

converges strongly to $P_{F(T)} x$.

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