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OPTIMALITY CONDITIONS AND DUALITY IN NONSMOOTH MULTIOBJECTIVE OPTIMIZATION

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Abstract. In this paper, a new class of generalized $(F, \rho, \sigma)-$type I functions are introduced for a nonsmooth multiobjective optimization problem. Based upon these generalized functions, Karush-Kuhn-Tucker type sufficient optimality conditions are derived for a feasible point to be an efficient or properly efficient solution. Appropriate duality theorems are also proved for a general Mond-Weir type dual.

1. Introduction

Multiobjective optimization is a useful mathematical model in order to investigate real-world problems with conflicting objectives, arising from economics, engineering and human decision making. Various optimality conditions and approaches to duality for the multiobjective optimization problems may be found in the literature. The case involving nonlinear functions has been of much interest in the recent past and many contributions have been made to this development.

It is well known that convexity plays a vital role in many aspects of mathematical programming including sufficient optimality conditions and duality theorems, but does no longer suffice. To relax convexity assumptions imposed on sufficient optimality conditions and duality theorems, various generalized convexity notions have been proposed. One of the useful generalizations is (F, ρ)-convexity was introduced by Preda [13] as an extension of F-convexity [7], and ρ - convexity [15], and he used this concept to obtain duality results for efficient solutions. Recently, Aghezzaf [1] and Ahmad [3] obtained sufficiency and duality theorems for efficient and properly efficient solutions under generalized $(F, \rho)-$convexity.

Hanson and Mond [8] introduced two new classes of functions, called type I and type II functions for scalar optimization problem, which were further generalized to pseudo-type I and quasi-type I by Rueda and Hanson [14]. Other classes of generalized type I functions have been introduced in [2, 6, 9]. Zhao [16] gave optimality conditions and duality results in nondifferentiable scalar optimization assuming Clarke [4] generalized subgradients under type I functions. Recently, Kuk and Tanino [10] established sufficient optimality conditions and duality theorems under generalized type I functions in terms of Clarke subgradients.

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In the present paper, we consider a nonsmooth multiobjective optimization problem and define a new class of generalized (F, ρ, σ) - type I functions in order to establish the Karush-Kuhn-Tucker type sufficient optimality conditions for a feasible point to be efficient or properly efficient. Moreover, weak, strong and strict converse duality theorems are obtained for a general Mond-Weir type dual.

2. Definitions and preliminaries

Following conventions of vectors in R^n will be followed throughout the paper: $x \geqq y \Longleftrightarrow x_i \geqq y_i,\ i = 1, 2, \dots, n;\ x \geq y \Longleftrightarrow x_i \geqq y_i,\ i = 1, 2, \dots, n,$ but $x \neq y;\ x > y \Longleftrightarrow x_i > y_i,\ i = 1, 2, \dots, n$. Let $K = \{1, 2, \dots, k\}$, and $M = \{1, 2, \dots, m\}$ be index sets.

A function $f : R^n \longrightarrow R$ is said to be locally Lipschitz at $\bar{x} \in R^n$, if there exist scalars $\delta > 0$ and $\epsilon > 0$ such that

$$| f(x^1) - f(x^2) | \leqq \delta \parallel x^1 - x^2 \parallel, \text{ for all } x^1, x^2 \in \bar{x} + \epsilon B,$$

where $\bar{x} + \epsilon B$ is the open ball of radius ϵ about $\bar{x}$.

The generalized directional derivative [4] of a locally Lipschitz function f at x in the direction v, denoted by $f^0(x; v)$, is as follows :

$$f^0(x; v) = \lim \sup_{\substack{y \to x \\ t \downarrow 0}} \left[\frac{f(y + tv) - f(y)}{t} \right].$$

The Clarke generalized gradient [4] of f at x is denoted by

$$\partial f(x) = \left\{ \xi : f^0(x; v) \geqq \xi^T v, \text{ for all } v \in R^n \right\}.$$

The function f at x is regular in the sense of Clarke [4], if $f^0(x; v) = f'(x; v)$, where $f'(x; v)$ is the directional derivative

$$f'(x; v) = \lim_{t \downarrow 0} \left[\frac{f(x + tv) - f(x)}{t} \right].$$

We now consider the following multiobjective optimization problem :

$$\text{(MP)} \qquad \text{Minimize } f(x) = [f_1(x), f_2(x), \dots, f_k(x)]$$

$$\text{subject to } x \in X = \{x \in S : g_j(x) \leqq 0,\ j \in M\},$$

where S is a non-empty open convex subset of R^n, and $f_i : S \to R$, $i \in K$ and $g_j : S \to R$, $j \in M$, are locally Lipschitz functions.

The following two definitions are from Geoffrion [5].

Definition 2.1. A point $\bar{x} \in X$ is said to be an efficient solution of (MP) if there exists no $x \in X$ such that $f(x) \leq f(\bar{x})$.

Definition 2.2. An efficient solution $\bar{x}$ is said to be a properly efficient solution of (MP) if there exists a scalar $N > 0$ such that for each $i, f_i(x) < f_i(\bar{x})$ and $x \in X$ imply that

$$\frac{f_i(\bar{x}) - f_i(x)}{f_j(x) - f_j(\bar{x})} \leqq N$$

for at least one j satisfying $f_j(\bar{x}) < f_j(x)$.

Definition 2.3. A functional $F : S \times S \times R^n \longrightarrow R$ is said to be sublinear in its third component, if for all $x, \bar{x} \in S$,

(i) $F(x, \bar{x}; a + b) \leqq F(x, \bar{x}; a) + F(x, \bar{x}; b),$ for all $a, b \in R^n$,

(ii) $F(x, \bar{x}; \alpha a) = \alpha F(x, \bar{x}; a),\ \forall\ \alpha \in R, \alpha \geqq 0,$ and for all $a \in R^n$.

From (ii), it is clear that $F(x, \bar{x}; 0) = 0$.

We next introduce generalized $(F, \rho, \sigma)-$type I functions.

Let F be a sublinear functional, and let the functions f and g be locally Lipschitz at a given point $\bar{x} \in S$. Let $\rho_i \in R$, $i \in K$, $\sigma_j \in R$, $j \in M$, and $d(\cdot, \cdot) : S \times S \to R$.

Definition 2.4. (f, g) is said to be (F, ρ, σ)-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$(2.1)\quad \begin{aligned} f_i(x) - f_i(\bar{x}) &\geqq F(x, \bar{x}; \xi_i) + \rho_i d^2(x, \bar{x}),\ \text{for all } \xi_i \in \partial f_i(\bar{x}),\ i \in K, \\ -g_j(\bar{x}) &\geqq F(x, \bar{x}; \zeta_j) + \sigma_j d^2(x, \bar{x}),\ \text{for all } \zeta_j \in \partial g_j(\bar{x}),\ j \in M. \end{aligned}$$

If in the above definition, (2.1) is a strict inequality, then we say that (f, g) is (F, ρ, σ)-semistrictly-type I at $\bar{x}$.

Remark 2.5. If $\rho_i = 0$, $F(x, \bar{x}; \xi_i) = \xi_i^T \eta(x, \bar{x})$, $i \in K$, and $\sigma_j = 0$, $F(x, \bar{x}; \zeta_j) = \zeta_j^T \eta(x, \bar{x})$, $j \in M$, for a certain mapping $\eta : X \times S \to R^n$, then above definition reduces to one of type I functions defined in [10].

Example 2.6. Consider the following multiobjective optimization problem:

$$\text{Minimize } f(x) = [f_1(x), f_2(x)]$$

$$\text{subject to } g(x) \leqq 0,\ x \in S,$$

where $f = (f_1, f_2) : S \to R^2$, and $g : S \to R$ are given by

$$f_1(x) = \begin{cases} x^3 + x\ ;\ -1 \leqq x < 0, \\ 3x\ ;\ 0 \leqq x \leqq 1, \end{cases}$$

$$f_2(x) = \begin{cases} x^2\ ;\ -1 \leqq x < 0, \\ x\ ;\ 0 \leqq x \leqq 1, \end{cases}$$

and

$$g(x) = |x| - 1 \leqq 0.$$

The feasible region is $X = \{x :\ -1 \leqq x \leqq 1\}$.

The Clarke generalized gradients of f_1, f_2, and g at $\bar{x} = 0$ are

$$\partial f_1(0) = \{\xi_1 \ : 1 \leqq \xi_1 \leqq 3\}, \partial f_2(0) = \{\xi_2 \ : 0 \leqq \xi_2 \leqq 1\},$$

and

$$\partial g(0) = \{\zeta \ : -1 \leqq \zeta \leqq 1\}.$$

It can be easily seen that (f, g) is (F, ρ, σ)-type I at $\bar{x} = 0 \in S$, for the sublinear functional $F(x, \bar{x}; a) = a^T(x^3 + \bar{x})$, $d(x, \bar{x}) = \sqrt{x + \bar{x} - 2}$, $\rho = 2$, and $\sigma = 3$. But (f, g) is not type I [10] as can be verified by taking $\rho = 0$, and $\sigma = 0$.

Definition 2.7. (f, g) is said to be (F, ρ, σ)-quasi-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$f_i(x) \leqq f_i(\bar{x}) \Longrightarrow F(x, \bar{x}; \xi_i) \leqq -\rho_i d^2(x, \bar{x}), \text{ for all } \ \xi_i \in \partial f_i(\bar{x}), i \in K,$$

$$-g_j(\bar{x}) \leqq 0 \Longrightarrow F(x, \bar{x}; \zeta_j) \leqq -\sigma_j d^2(x, \bar{x}), \text{ for all } \zeta_j \in \partial g_j(\bar{x}), j \in M.$$

Definition 2.8. (f, g) is said to be (F, ρ, σ)-pseudo-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$F(x, \bar{x}; \xi_i) \geqq -\rho_i d^2(x, \bar{x}) \Longrightarrow f_i(x) \geqq f_i(\bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), i \in K,$$

$$F(x, \bar{x}; \zeta_j) \geqq -\sigma_j d^2(x, \bar{x}) \Longrightarrow -g_j(\bar{x}) \geqq 0, \text{ for all } \zeta_j \in \partial g_j(\bar{x}), j \in M.$$

Definition 2.9. (f, g) is said to be (F, ρ, σ)-quasipseudo-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$f_i(x) \leqq f_i(\bar{x}) \Longrightarrow F(x, \bar{x}; \xi_i) \leqq -\rho_i d^2(x, \bar{x}), \text{ for all } \ \xi_i \in \partial f_i(\bar{x}), i \in K,$$

$$F(x, \bar{x}, \zeta_j) \geqq -\sigma_j d^2(x, \bar{x}) \Longrightarrow -g_j(\bar{x}) \geqq 0, \text{ for all } \zeta_j \in \partial g_j(\bar{x}), j \in M. \tag{2.2}$$

If in the above definition, inequality (2.2) is satisfied as

$$F(x, \bar{x}; \zeta_j) \geqq -\sigma_j d^2(x, \bar{x}) \Longrightarrow -g_j(\bar{x}) > 0, \text{ for all } \zeta_j \in \partial g_j(\bar{x}), j \in M,$$

then we say that (f, g) is (F, ρ, σ)-quasistrictly-pseudo-type I at $\bar{x}$.

Definition 2.10. (f, g) is said to be (F, ρ, σ)-pseudoquasi-type I at $\bar{x} \in S$, if for all $x \in X$, we have

$$F(x, \bar{x}; \xi_i) \geqq -\rho_i d^2(x, \bar{x}) \Longrightarrow f_i(x) \geqq f_i(\bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), i \in K, \tag{2.3}$$

$$-g_j(\bar{x}) \leqq 0 \Longrightarrow F(x, \bar{x}; \zeta_j) \leqq -\sigma_j d^2(x, \bar{x}), \text{ for all } \zeta_j \in \partial g_j(\bar{x}), j \in M.$$

If in the above definition, inequality (2.3) is satisfied as

$$F(x, \bar{x}; \xi_i) \geqq -\rho_i d^2(x, \bar{x}) \Longrightarrow f_i(x) > f_i(\bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), i \in K,$$

or equivalently,

$$f_i(x) \leqq f_i(\bar{x}) \Longrightarrow F(x, \bar{x}; \xi_i) < -\rho_i d^2(x, \bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), i \in K,$$

then we say that (f, g) is (F, ρ, σ)-strictly-pseudoquasi-type I at $\bar{x}$.

3. Sufficiency

In this section, we obtain Karush-Kuhn-Tucker type sufficient optimality conditions for a feasible point of (MP) to be efficient or properly efficient. Let $J(\bar{x}) = \{j \in M : g_j(\bar{x}) = 0\}$, and $g_J(\bar{x})$ denotes the vector of active constraints.

Theorem 3.1. *Suppose that there exists a feasible solution $\bar{x}$ of (MP) and scalars $\lambda_i \geqq 0$, $i \in K$, $\sum_{i=1}^{k} \lambda_i = 1$ and $\mu_j \geqq 0, j \in J(\bar{x})$ such that*

$$0 \in \sum_{i=1}^{k} \lambda_i \partial f_i(\bar{x}) + \sum_{j \in J(\bar{x})} \mu_j \partial g_j(\bar{x}). \tag{3.1}$$

If (f, g_J) is $(F, \rho, \sigma)-$semistrictly-type I at $\bar{x}$ and $\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j \in J(\bar{x})} \mu_j \sigma_j \geqq 0$, then $\bar{x}$ is an efficient solution of (MP).

Proof. Condition (3.1) implies that there exist $\xi_i \in \partial f_i(\bar{x})$, $i \in K$, and $\zeta_j \in \partial g_j(\bar{x})$, $j \in J(\bar{x})$ satisfying

$$\sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j \in J(\bar{x})} \mu_j \zeta_j = 0. \tag{3.2}$$

Now suppose that $\bar{x}$ is not an efficient solution of (MP), then there exists a feasible solution x of (MP), and an index r such that

$$f_r(x) < f_r(\bar{x}),$$

and

$$f_i(x) \leqq f_i(\bar{x}), \text{ for all } i \neq r.$$

These two inequalities lead to

$$\sum_{i=1}^{k} \lambda_i f_i(x) \leqq \sum_{i=1}^{k} \lambda_i f_i(\bar{x}).$$

Also, we have $-\sum_{j \in J(\bar{x})} \mu_j g_j(\bar{x}) = 0$.

Since (f, g_J) is $(F, \rho, \sigma)-$semistrictly-type I at $\bar{x}$, we have

$$f_i(x) - f_i(\bar{x}) > F(x, \bar{x}; \xi_i) + \rho_i d^2(x, \bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), i \in K, \tag{3.3}$$

and

$$-g_j(\bar{x}) \geqq F(x, \bar{x}; \zeta_j) + \sigma_j d^2(x, \bar{x}), \text{ for all } \zeta_j \in \partial g_j(\bar{x}), j \in J(\bar{x}). \tag{3.4}$$

On summing the inequalities obtained on multiplying (3.3) by $\lambda_i \geqq 0$, $i \in K$, and (3.4) by $\mu_j \geqq 0$, $j \in M$, respectively, we get

$$0 \geqq \sum_{i=1}^{k} \lambda_i f_i(x) - \sum_{i=1}^{k} \lambda_i f_i(\bar{x})$$

$$> F(x,\bar{x};\sum_{i=1}^{k}\lambda_i\xi_i) + \sum_{i=1}^{k}\lambda_i\rho_i d^2(x,\bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}), \tag{3.5}$$

$$0 = -\sum_{j\in J(\bar{x})}\mu_j g_j(\bar{x})$$

$$\geqq F(x,\bar{x};\sum_{j\in J(\bar{x})}\mu_j\zeta_j) + \sum_{j\in J(\bar{x})}\mu_j\sigma_j d^2(x,\bar{x}), \text{ for all } \zeta_j \in \partial g_j(\bar{x}). \tag{3.6}$$

Now relations (3.5), (3.6), and the sublinearity of F imply

$$F(x,\bar{x};\sum_{i=1}^{k}\lambda_i\xi_i + \sum_{j\in J(\bar{x})}\mu_j\zeta_j) \leqq F(x,\bar{x};\sum_{i=1}^{k}\lambda_i\xi_i) + F(x,\bar{x};\sum_{j\in J(\bar{x})}\mu_j\zeta_j)$$

$$< -(\sum_{i=1}^{k}\lambda_i\rho_i + \sum_{j\in J(\bar{x})}\mu_j\sigma_j)d^2(x,\bar{x}) \leqq 0.$$

Therefore

$$\sum_{i=1}^{k}\lambda_i\xi_i + \sum_{j\in J(\bar{x})}\mu_j\zeta_j \neq 0,$$

which is a contradiction to (3.2). Hence $\bar{x}$ is an efficient solution of (MP). □

Theorem 3.2. *Suppose that there exists a feasible solution $\bar{x}$ of (MP) and scalars $\lambda_i \geqq 0$, $i \in K$, $\sum_{i=1}^{k}\lambda_i = 1$ and $\mu_j \geqq 0, j \in J(\bar{x})$ such that (3.1) in Theorem 3.1 holds. If $(\sum_{i=1}^{k}\lambda_i f_i, \sum_{j\in J(\bar{x})}\mu_j g_j)$ is (F,ρ_1,σ_1)-strictly-pseudoquasi-type I at $\bar{x}$ and $\rho_1+\sigma_1 \geqq 0$, then $\bar{x}$ is an efficient solution of (MP).*

Proof. Following the proof of Theorem 3.1, we obtain

$$\sum_{i=1}^{k}\lambda_i f_i(x) \leqq \sum_{i=1}^{k}\lambda_i f_i(\bar{x}),$$

and

$$-\sum_{j\in J(\bar{x})}\mu_j g_j(\bar{x}) = 0.$$

As $(\sum_{i=1}^{k}\lambda_i f_i, \sum_{j\in J(\bar{x})}\mu_j g_j)$ is (F,ρ_1,σ_1)-strictly-pseudoquasi-type I at $\bar{x}$, it follows that

$$F(x,\bar{x};\sum_{i=1}^{k}\lambda_i\xi_i) < -\rho_1 d^2(x,\bar{x}), \text{ for all } \xi_i \in \partial f_i(\bar{x}),\ i \in K,$$

$$F(x,\bar{x};\sum_{j\in J(\bar{x})}\mu_j\zeta_j) \leqq -\sigma_1 d^2(x,\bar{x}), \text{ for all } \zeta_j \in \partial g_j(\bar{x}),\ j \in J(\bar{x}).$$

Using the sublinearity of F and $\rho_1 + \sigma_1 \geqq 0$, we again reach a contradiction like Theorem 3.1. □

The following theorem can be proved along the similar lines of the proof of Theorem 3.2.

Theorem 3.3. *Suppose that there exists a feasible solution $\bar{x}$ of (MP) and scalars $\lambda_i \geqq 0, i \in K, \sum_{i=1}^{k} \lambda_i = 1$ and $\mu_j \geqq 0, j \in J(\bar{x})$ such that (3.1) in Theorem 3.1 holds. If $(\sum_{i=1}^{k} \lambda_i f_i, \sum_{j\in J(\bar{x})} \mu_j g_j)$ is (F, ρ_2, σ_2)-quasistrictly-pseudo-type I at $\bar{x}$ and $\rho_2+\sigma_2 \geqq 0$, then $\bar{x}$ is an efficient solution of (MP).*

Theorem 3.4. *Suppose that there exists a feasible solution $\bar{x}$ of (MP) and scalars $\lambda_i > 0, i \in K$, and $\mu_j \geqq 0, j \in J(\bar{x})$ such that (3.1) in Theorem 3.1 holds. If any one of the following two sets of hypotheses is satisfied:*

(a) (i) *(f, g_J) is (F, ρ, σ)-type I at $\bar{x}$,*

(ii) $\sum_{i=1}^{k} \lambda_i \rho_i + \sum_{j\in J(\bar{x})} \mu_j \sigma_j \geqq 0,$

(b) (i) *$(\sum_{i=1}^{k} \lambda_i f_i, \sum_{j\in J(\bar{x})} \mu_j g_j)$ is (F, ρ_3, σ_3)-pseudoquasi-type I at $\bar{x}$,*

(ii) $\rho_3 + \sigma_3 \geqq 0,$

then $\bar{x}$ is a properly efficient solution of (MP).

Proof. Condition (3.1) implies that there exist $\xi_i \in \partial f_i(\bar{x})$, and $\zeta_j \in \partial g_j(\bar{x})$ satisfying

$$0 = \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j\in J(\bar{x})} \mu_j \zeta_j. \tag{3.7}$$

(a) Since (f, g_J) is (F, ρ, σ)-type I at $\bar{x}$, we have for all $x \in X$,

$$\begin{aligned} \sum_{i=1}^{k} \lambda_i f_i(x) - \sum_{i=1}^{k} \lambda_i f_i(\bar{x}) \quad &\geqq \quad F(x, \bar{x}; \sum_{i=1}^{k} \lambda_i \xi_i) \\ &\quad + \sum_{i=1}^{k} \lambda_i \rho_i d^2(x, \bar{x}), \ \forall \ \xi_i \in \partial f_i(\bar{x}), \end{aligned} \tag{3.8}$$

$$\begin{aligned} 0 = -\sum_{j\in J(\bar{x})} \mu_j g_j(\bar{x}) \quad &\geqq \quad F(x, \bar{x}; \sum_{j\in J(\bar{x})} \mu_j \zeta_j) \\ &\quad + \sum_{j\in J(\bar{x})} \mu_j \sigma_j d^2(x, \bar{x}), \ \forall \ \zeta_j \in \partial g_j(\bar{x}). \end{aligned} \tag{3.9}$$

Now relations (3.8), (3.9), and the sublinearity of F imply that

$$\sum_{i=1}^{k} \lambda_i f_i(x) - \sum_{i=1}^{k} \lambda_i f_i(\bar{x}) \quad \geqq \quad F(x, \bar{x}; \sum_{i=1}^{k} \lambda_i \xi_i) + \sum_{i=1}^{k} \lambda_i \rho_i d^2(x, \bar{x})$$

$$\begin{aligned}
&+F(x,\bar{x};\sum_{j\in J(\bar{x})}\mu_j\zeta_j)+\sum_{j\in J(\bar{x})}\mu_j\sigma_j d^2(x,\bar{x})\\
\geqq\ & F(x,\bar{x};\sum_{i=1}^{k}\lambda_i\xi_i+\sum_{j\in J(\bar{x})}\mu_j\zeta_j)\\
&+(\sum_{i=1}^{k}\lambda_i\rho_i+\sum_{j\in J(\bar{x})}\mu_j\sigma_j)d^2(x,\bar{x})\\
\geqq\ & (\sum_{i=1}^{k}\lambda_i\rho_i+\sum_{j\in J(\bar{x})}\mu_j\sigma_j)d^2(x,\bar{x}),\ \text{(by (3.7))}\\
\geqq\ & 0,\text{(by (ii))}\,.
\end{aligned}$$

Hence by Theorem 1 of Geoffrion [5], $\bar{x}$ is a properly efficient solution of (MP).
(b) Since $g_J(\bar{x})=0, \mu_J \geqq 0$, the second part of assumption (i) gives

$$F(x,\bar{x};\sum_{j\in J(\bar{x})}\mu_j\zeta_j)\leqq -\sigma_3 d^2(x,\bar{x}),\ \text{for all } \zeta_j\in\partial g_j(\bar{x}),\ j\in J(\bar{x}).$$

The above inequality together with the sublinearity of F, (3.7), and assumption (ii) imply

$$F(x,\bar{x};\sum_{i=1}^{k}\lambda_i\xi_i)\geqq -\rho_3 d^2(x,\bar{x}),\ \text{for all } \xi_i\in\partial f_i(\bar{x}),\ i\in K,$$

which on applying the first part of assumption (i) yields

$$\sum_{i=1}^{k}\lambda_i f_i(x)\geqq\sum_{i=1}^{k}\lambda_i f_i(\bar{x}).$$

Hence by Theorem 1 of Geoffrion [5], $\bar{x}$ is a properly efficient solution of (MP). □

The so-called Cottle constraint qualification is used in the following theorem:

Proposition 3.5. *Let $f_i,\ i\in K$ and $g_j,\ j\in M$, be locally Lipschitz functions at a point $\bar{x}\in X$. Problem (MP) satisfies the Cottle constraint qualification at $\bar{x}$ if either $g_j(\bar{x})<0$, for all $j\in M$, or $0\notin$ conv $\{\partial g_j(\bar{x}) : g_j(\bar{x})=0\}$.*

Assuming the Cottle constraint qualification, we obtain the Karush-Kuhn-Tucker type necessary conditions for efficiency (see, for example, [11], Theorem 3.2.9).

Theorem 3.6. *Assume that $\bar{x}$ is an efficient solution of (MP) at which the Cottle constraint qualification is satisfied. Then there exist scalars $\lambda_i\geqq 0,\ i\in K,\ \sum_{i=1}^{k}\lambda_i=1,\ \mu_j\geqq 0,\ j\in M$, such that*

$$0\in\sum_{i=1}^{k}\lambda_i\partial f_i(\bar{x})+\sum_{j=1}^{m}\mu_j\partial g_j(\bar{x}), \tag{3.10}$$

$$\mu_j g_j(\bar{x}) = 0,\ j \in M. \tag{3.11}$$

The above conditions are also necessary for weak efficiency of $\bar{x}$ for (MP) (see [11], Corollary 3.2.10).

4. Generalized Mond-Weir type duality

In this section, we discuss weak, strong, and strict converse duality theorems between the problem (MP), and its corresponding general Mond-Weir [12] type dual:

(MD) $$\text{Maximize } f(y) + \sum_{j \in J_0} \mu_j g_j(y) e$$

subject to

$$0 \in \sum_{i=1}^{k} \lambda_i \partial f_i(y) + \sum_{j=1}^{m} \mu_j \partial g_j(y), \tag{4.1}$$

$$\sum_{j \in J_\alpha} \mu_j g_j(y) \geqq 0,\ \alpha = 1, 2, \ldots, p, \tag{4.2}$$

$$\lambda_i \geqq 0,\ i \in K, \tag{4.3}$$

$$\mu_j \geqq 0,\ j \in M, \tag{4.4}$$

$$\sum_{i=1}^{k} \lambda_i = 1, \tag{4.5}$$

where, $e = (1, 1, \ldots, 1) \in R^k$, and $J_\alpha \subseteq M,\ \alpha = 0, 1, 2, \ldots, p$, with $J_\alpha \cap J_\beta = \phi$, $\alpha \neq \beta$, and $\bigcup_{\alpha=0}^{p} J_\alpha = M$.

Theorem 4.1 (Weak Duality)**.** *Assume that for all feasible x for (MP) and all feasible (y, λ, μ) for (MD), any one of the following two sets of hypotheses is satisfied:*

(a) (i) $(\sum_{i=1}^{k} \lambda_i f_i + \sum_{j \in J_0} \mu_j g_j,\ \sum_{j \in J_\alpha} \mu_j g_j)$ *is $(F, \rho_1, \sigma_{1\alpha})$-strictly-pseudoquasi-type I at y for any $\alpha = 1, 2, \ldots, p$,*

(ii) $\rho_1 + \sum_{\alpha=1}^{p} \sigma_{1\alpha} \geqq 0,$

(b) (i) $(\sum_{i=1}^{k} \lambda_i f_i + \sum_{j \in J_0} \mu_j g_j,\ \sum_{j \in J_\alpha} \mu_j g_j)$ *is $(F, \rho_2, \sigma_{2\alpha})$-quasi-strictlypseudo-type I at y for any $\alpha = 1, 2, \ldots, p$,*

(ii) $\rho_2 + \sum_{\alpha=1}^{p} \sigma_{2\alpha} \geqq 0;$

then the following cannot hold

$$f(x) \leq f(y) + \sum_{j \in J_0} \mu_j g_j(y) e.$$

Proof. (a) Suppose to the contrary that

$$f(x) \leq f(y) + \sum_{j \in J_0} \mu_j g_j(y) e, \tag{4.6}$$

holds. Since x is feasible for (MP), $\lambda_i \geqq 0, i \in K$, $\sum_{i=1}^{k} \lambda_i = 1$ and $\mu_j \geqq 0, j \in M$, then (4.6) implies

$$\sum_{i=1}^{k} \lambda_i f_i(x) + \sum_{j \in J_0} \mu_j g_j(x) \leqq \sum_{i=1}^{k} \lambda_i f_i(y) + \sum_{j \in J_0} \mu_j g_j(y). \tag{4.7}$$

Also, from (4.2), we have

$$- \sum_{j \in J_\alpha} \mu_j g_j(y) \leqq 0, \ \alpha = 1, 2, \ldots, p. \tag{4.8}$$

Using hypothesis (i), we see that (4.7) and (4.8) together give

$$F(x, y; \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j \in J_0} \mu_j \zeta_j) < -\rho_1 d^2(x, y), \ \text{for all } \xi_i \in \partial f_i(y), \ i \in K, \tag{4.9}$$

(4.10)
$$F(x, y; \sum_{j \in J_\alpha} \mu_j \zeta_j) \leqq -\sigma_{1\alpha} d^2(x, y), \ \alpha = 1, 2, \ldots, p, \ \text{for all } \zeta_j \in \partial g_j(y), \ j \in J_\alpha.$$

By the sublinearity of F, we summarize to get

$$F(x, y; \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j) \leqq F(x, y; \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j \in J_0} \mu_j \zeta_j) + \sum_{\alpha=1}^{p} F(x, y; \sum_{j \in J_\alpha} \mu_j \zeta_j)$$
$$< -(\rho_1 + \sum_{\alpha=1}^{p} \sigma_{1\alpha}) d^2(x, y).$$

Since $(\rho_1 + \sum_{\alpha=1}^{p} \sigma_{1\alpha}) \geqq 0$, we have

$$F(x, y; \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j) < 0. \tag{4.11}$$

From condition (4.1), there exist $\xi_i \in \partial f_i(y)$, $i \in K$, and $\zeta_j \in \partial g_j(y)$, $j \in M$, such that

$$\sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j = 0,$$

which implies

$$F(x, y; \sum_{i=1}^{k} \lambda_i \xi_i + \sum_{j=1}^{m} \mu_j \zeta_j) = 0,$$

contradicting (4.11). Hence $f(x) \leq f(y) + \sum_{j\in J_0} \mu_j g_j(y)e$ cannot hold.

(b). Under this hypothesis, inequality (4.9) holds as $\leqq$ inequality (with $\rho_1 = \rho_2$), and (4.10) holds as strict inequality (with $\sigma_{1\alpha} = \sigma_{2\alpha}$). Therefore (4.11) also holds as strict inequality, again a contradiction. □

Theorem 4.2 (Strong Duality). *Let $\bar{x}$ be an efficient solution of (MP) at which the Cottle constraint qualification is satisfied. Then there exist $\bar{\lambda} \in R^k$, and $\bar{\mu} \in R^m$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (MD) and the objective values of (MP) and (MD) are equal. If the assumptions of weak duality (Theorem 4.1) are satisfied, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of (MD).*

Proof. Using Theorem 3.6, we obtain scalars $\bar{\lambda}_i \geqq 0$, $i \in K$, $\sum_{i=1}^{k} \bar{\lambda}_i = 1$ and $\bar{\mu}_j \geqq 0, j \in M$ such that (3.10) and (3.11) hold. Therefore $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible solution of (MD), and the objective values of (MP) and (MD) are equal. If $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is not an efficient solution of (MD), then there exists a feasible solution (y, λ, μ) of (MD) such that

$$f(\bar{x}) + \sum_{j\in J_\circ} \bar{\mu}_j g_j(\bar{x})e \leq f(y) + \sum_{j\in J_\circ} \mu_j g_j(y)e.$$

From the above inequality and (3.11), we have

$$f(\bar{x}) \leq f(y) + \sum_{j\in J_\circ} \mu_j g_j(y)e,$$

which contradicts weak duality (Theorem 4.1). Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is an efficient solution of (MD). □

Remark 4.3. It may be noted that the above strong duality theorem is also valid for weakly efficient solution as the Karush-Kuhn-Tucker type necessary conditions used in the theorem holds for weak efficiency as well.

Theorem 4.4 (Strict Converse Duality). *Let $\bar{x}$ be a feasible solution of (MP) and let $(\bar{y}, \bar{\lambda}, \bar{\mu})$ be a feasible solution of (MD) such that*

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) \leqq \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j\in J_0} \bar{\mu}_j g_j(\bar{y}). \tag{4.12}$$

If $(\sum_{i=1}^{k} \bar{\lambda}_i f_i + \sum_{j\in J_0} \bar{\mu}_j g_j, \sum_{j\in J_\alpha} \bar{\mu}_j g_j)$ is $(F, \rho_1, \sigma_{1\alpha})$-strictly-pseudoquasi-type I at $\bar{y}$ for any $\alpha = 1, 2, \dots p$, and $\rho_1 + \sum_{\alpha=1}^{p} \sigma_{1\alpha} \geqq 0$, then $\bar{x} = \bar{y}$.

Proof. We assume that $\bar{x} \neq \bar{y}$, and exhibit a contradiction. By imposing the $(F, \rho_1, \sigma_{1\alpha})$-strictly-pseudoquasi-type I assumption on $(\sum_{i=1}^{k} \bar{\lambda}_i f_i + \sum_{j\in J_0} \bar{\mu}_j g_j, \sum_{j\in J_\alpha} \bar{\mu}_j g_j)$ at $\bar{y}$, we have

$$(4.13)\quad F(\bar{x},\bar{y};\sum_{i=1}^{k}\bar{\lambda}_i\xi_i+\sum_{j\in J_0}\bar{\mu}_j\zeta_j)\geqq -\rho_1 d^2(\bar{x},\bar{y})$$
$$\Longrightarrow \sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})+\sum_{j\in J_0}\bar{\mu}_j g_j(\bar{x})>\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j\in J_0}\bar{\mu}_j g_j(\bar{y}),$$

$$(4.14)\quad -\sum_{j\in J_\alpha}\bar{\mu}_j g_j(\bar{y})\leqq 0\Longrightarrow F(\bar{x},\bar{y};\sum_{j\in J_\alpha}\bar{\mu}_j\zeta_j)\leqq -\sigma_{1\alpha}d^2(\bar{x},\bar{y}),\ \alpha=1,2,\dots,p.$$

Since $(\bar{y},\bar{\lambda},\bar{\mu})$ is feasible for (MD), we obtain

$$(4.15)\quad -\sum_{j\in J_\alpha}\bar{\mu}_j g_j(\bar{y})\leqq 0,\ \alpha=1,2,\dots,p.$$

Relation (4.14) along with (4.15) yields

$$(4.16)\quad F(\bar{x},\bar{y};\sum_{j\in J_\alpha}\bar{\mu}_j\zeta_j)\leqq -\sigma_{1\alpha}d^2(\bar{x},\bar{y}),\ \alpha=1,2,\dots,p.$$

From condition (4.1), there exist $\xi_i\in\partial f_i(\bar{y})$ and $\zeta_j\in\partial g_j(\bar{y})$, such that

$$\sum_{i=1}^{k}\bar{\lambda}_i\xi_i+\sum_{j=1}^{m}\bar{\mu}_j\zeta_j=0,$$

which along with the sublinearity of F gives

$$(4.17)\quad 0 = F(\bar{x},\bar{y};\sum_{i=1}^{k}\bar{\lambda}_i\xi_i+\sum_{j=1}^{m}\bar{\mu}_j\zeta_j)$$
$$\leqq F(\bar{x},\bar{y};\sum_{i=1}^{k}\bar{\lambda}_i\xi_i+\sum_{j\in J_\circ}\bar{\mu}_j\zeta_j)+\sum_{\alpha=1}^{p}F(\bar{x},\bar{y};\sum_{j\in J_\alpha}\bar{\mu}_j\zeta_j).$$

The inequality (4.17) together with (4.16) reveals

$$(4.18)\quad F(\bar{x},\bar{y};\sum_{i=1}^{k}\bar{\lambda}_i\xi_i+\sum_{j\in J_\circ}\bar{\mu}_j\zeta_j) \geqq -\sum_{\alpha=1}^{p}F(\bar{x},\bar{y};\sum_{j\in J_\alpha}\bar{\mu}_j\zeta_j)$$
$$\geqq \sum_{\alpha=1}^{p}\sigma_{1\alpha}d^2(\bar{x},\bar{y})$$
$$\geqq -\rho_1 d^2(\bar{x},\bar{y}),\ (\text{by } \rho_1+\sum_{\alpha=1}^{p}\sigma_{1\alpha}\geqq 0).$$

The inequalities (4.13) and (4.18) imply

$$\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{x})+\sum_{j\in J_\circ}\bar{\mu}_j g_j(\bar{x})>\sum_{i=1}^{k}\bar{\lambda}_i f_i(\bar{y})+\sum_{j\in J_0}\bar{\mu}_j g_j(\bar{y}).$$

Since $\bar{\mu}_j \geqq 0$ and $g_j(\bar{x}) \leqq 0$, $j \in M$, we obtain

$$\sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{x}) > \sum_{i=1}^{k} \bar{\lambda}_i f_i(\bar{y}) + \sum_{j \in J_0} \bar{\mu}_j g_j(\bar{y}),$$

contradicting (4.12). Hence $\bar{x} = \bar{y}$. □

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