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ON STRICTLY QUASI-MONOTONE OPERATORS AND VARIATIONAL INEQUALITIES

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Abstract. Let E be a real reflexive Banach space. A mapping $T : D \subseteq E \to 2^{E^*}$ is said to be *strictly quasi-monotone* if $(g, x-y) > 0$ for some $g \in Ty$ implies that $(f, x-y) > 0$ for $f \in Tx$, where $x, y \in D$. In this paper, we first study variational inequality problems for strictly quasi-monotone operators, then we obtain a surjective result for strictly quasi-monotone operator equation.

1. Introduction

The class of monotone operators is an important class of nonlinear operators that has many applications in nonlinear partial differential equations, nonlinear semi-group theory, variational inequality and so on (see [1], [4], [9], [10], [16]). The theory for monotone operators has been well developed and has many other generalizations (see [2], [4], [16]). In [14], an open problem concerning the existence of zero points of nonlinear operators related to the generalization of monotone operator was suggested by Bott, and we recall this problem as follows:

Problem B. *Let B be the unit ball in a real Hilbert space and $A : B \to H$ be finite dimensional continuous. Suppose that*

$$(Ax, x) \geq -\theta \|Ax\| \|x\|$$

for $x \in \partial B$ and

$$(Ax - Ay, x - y) \geq -\theta \|Ax - Ay\| \|x - y\|$$

for all $x, y \in \overline{B}$, where $\theta \in (0, 1)$. Does $Ax = 0$ has a solution in $\overline{B}$?

Now, we introduce the following definition:

Definition 1.1. Let E be a real Banach space and $T : D \subseteq E \to 2^{E^*}$ be a mapping.

(1) T is said to be *quasi-monotone* if, for any $u, v \in D$, $(g, u-v) > 0$ for some $g \in Tv$ implies that $(f, u-v) \geq 0$ for all $f \in Tu$ (see [11]).
(2) T is said to be *strictly quasi-monotone* if, for any $u, v \in D$, $(g, u-v) > 0$ for some $g \in Tv$ implies that $(f, u-v) > 0$ for all $f \in Tu$.

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One can easily see that the class of strictly quasi-monotone operators is a subclass of quasi-monotone operators, and we will know that the class of strictly quasi-monotone operators has better properties than that of quasi-monotone operators in studying variational inequality as showing in Lemma 2.5. For other works related to quasi-monotone operators, we refer the reader to [2] and [13]. One may also see that the operator $A : R \to R$ given by $Ax = x^2+1$ for all $x \in R$ is strictly quasi-monotone but not quasi-monotone, thus the concept of strictly quasi-monotone operator is a generalization of quasi-monotone operator.

In this paper, we obtain some existence results for variational inequalities of strictly quasi-monotone operators and thereby we obtain some existence results for strictly quasi-monotone operator equations. Our results may be viewed as partial solutions related to the Problem B.

2. Main results

In this section, we study the variational inequality problems and the existence problems of operator equations for strictly quasi-monotone operators. First, we give the following result on the of sum of a strictly quasi-monotone operator and the sub-differential of an indicator function of a closed convex set.

Proposition 2.1. *Let E be a real reflexive Banach space and K a closed convex subset of E. If $A : K \to 2^{E^*}$ is strictly quasi-monotone, then $A + \partial I_K$ is strictly quasi-monotone, where $I_K(x) = 0$ if $x \in K$ and $I_K(x) = +\infty$, otherwise.*

Proof. If $(f + g, y - x) > 0$ for some $f \in Ay$, $g \in \partial I_K(y)$ and $x, y \in D(\partial I_K)$, then

$$(f, y - x) > -(g, y - x).$$

However $(g, y - x) \leq 0$ for $g \in \partial I_k(y)$ and $x, y \in K$, so we have $(f, y - x) > 0$. Hence $(f', x - y) > 0$ for all $f' \in Ax$ and $x, y \in K$. Consequently, it follows that

$$(f' + g', x - y) > 0$$

for all $f' \in Ax$, $g' \in \partial I_K(x)$ and $x, y \in D(\partial I_K)$, i.e., $A + \partial I_K$ is strictly quasi-monotone. □

Remark 2.2. The conclusion for Proposition 2.1 is generally not true if ∂I_K is replaced by a monotone operator, and this can be seen by the following example.

Example 2.3. Let $A, B : R \to R$ be given by $Ax = 2x$ for all $x \in R$ and $Bx = x^2+1$ for all $x \in R$. Then A is monotone and B is strictly quasi-monotone. Further, $A + B$ is not strictly quasi-monotone. In fact, if we take $x = -1$ and $y = -2$, then $(Ay + By)(-1 + 2) > 0$, but $(Ax + Bx)(-1 + 2) = 0$.

If K is closed convex, then the normal cone $N_K(x) = \{f \in E^* : f(x - y) \geq 0 \text{ for all } y \in K\}$ equals to $\partial I_K(x)$ (see [7]). Therefore, Proposition 2.1 can be stated as the following equivalent form:

Proposition 2.4. *Let E be a real reflexive Banach space and K a closed subset of E. If $A : K \to 2^{E^*}$ is strictly quasi-monotone, then $A + N_K$ is strictly quasi-monotone.*

Lemma 2.5. *Let E be a real reflexive Banach space and C a non-empty bounded closed convex subset of E. If $A : C \to 2^{E^*}$ is an finite dimensional weakly upper semi-continuous (i.e., for each finite dimensional subspace F of E with $F \cap C \neq \emptyset$, $A : C \cap F \to 2^{E^*}$ is upper semi-continuous in the weak topology) and strictly quasi-monotone mapping with bounded closed convex values, then $(f_v, u_0 - v) \leq 0$ for all $v \in C$ and some $f_v \in Tu_0$ if and only if $(g, u_0 - v) \leq 0$ for all $v \in C$ and $g \in Tv$.*

Proof. For the only if part, if $(g, u_0 - v) > 0$ for some $v \in C$ and $g \in Tv$, then, by Definition 1.1, we have $(f, u_0 - v) > 0$ for all $f \in Tu_0$, which is a contradiction.

Now, we prove the if part. For $v \in C$, we put $v_t = tu_0 + (1-t)v$ for any $t \in (0, 1)$. Then we have

$$(g_t, u_0 - v_t) \leq 0$$

for all $g_t \in Tv_t$, i.e., $(g_t, u_0 - v) \leq 0$ for all $g_t \in Tv_t$. By letting $t \to 1_-$, then the finite dimensional weakly upper semi-continuity of T and bounded closed convexity of Tu_0 imply that there exists $f_v \in Tu_0$ such that

$$(f_v, u_0 - v) \leq 0.$$

□

Remark 2.6. For the results of Lemma 2.5 in monotone case, we refer the reader to [6] and [10], and we do not know whether Lemma 2.5 is still true for quasi-monotone operators.

Theorem 2.7. *Let E be a real reflexive Banach space and C a non-empty closed convex bounded subset of E. If $A : C \to 2^{E^*}$ is an finite dimensional weakly upper semi-continuous and strictly quasi-monotone mapping with bounded closed convex values, then there exists $u_0 \in C$ such that*

$$(f_v, u_0 - v) \leq 0$$

for all $v \in C$ and some $f_v \in Tu_0$.

Proof. For any finite dimensional subspace F of E with $F \cap C \neq \emptyset$, let $j_F : F \to E$ be the natural inclusion and j_F^* be the conjugate mapping of j_F. Consider the following variational inequality problem:

Find $u \in F \cap C$ such that

$$(j_F^* f_v, u - v) \leq 0$$

for all $v \in C \cap F$ and some $f_v \in Tu$. Since T is finite dimensional weakly upper semi-continuous and $j_F^* T$ is upper semi-continuous on $F \cap C$, there exists $u_F \in F \cap C$ such that

$$(j_F^* f_v, u_F - v) \leq 0$$

for all $v \in C \cap F$ and some $f_v \in Tu_F$, i.e., $(f_v, u_F - v) \leq 0$ for all $v \in C \cap F$ and some $f_v \in Tu_F$. By Lemma 2.5, we get

$$(g, u_F - v) \leq 0$$

for all $v \in C \cap F$ and $g \in Tv$. Now, we put

$$W_F = \{u \in C : (g, u - v) \leq 0 \text{ for all } v \in F \cap C \text{ and } g \in Tv\}.$$

It is obvious that W_F is closed convex. One can easily check that

$$W_{\cup_{i=1}^n F_i} \subseteq W_{F_i},\ dim(F_i) < +\infty,\ F_i \cap C \neq \emptyset$$

for $i = 1, 2, \dots, n$. Hence $\cap_{F\in\mathcal{F}} W_F \neq \emptyset$, where

$$\mathcal{F} = \{F \subset E : F \cap C \neq \emptyset \text{ and } dim(F) < +\infty\}.$$

Take $u_0 \in \cap_{F\in\mathcal{F}} W_F$. We claim that u_0 satisfies the conclusion of Theorem 2.7. In fact, $(g, u_0 - v) \leq 0$ for all $v \in C$ and $g \in Tv$. By Lemma 2.5, we know that

$$(f_v, u_0 - v) \leq 0$$

for all $v \in C$ and some $f_v \in Tu_0$. □

From Theorem 2.7, we have the following:

Corollary 2.8. *Let E be a real reflexive Banach space and C a non-empty closed convex unbounded subset of E. If $A : C \to 2^{E^*}$ is an finite dimensional weakly upper semi-continuous and strictly quasi-monotone mapping with bounded closed convex values and there exist $v_0 \in C$, $r > 0$ such that*

$$(f, u - v_0) > 0$$

for all $f \in Tu$ and $u \in C$ with $\|u\| > r$, then there exists $u_0 \in C$ such that

$$(f_v, u_0 - v) \leq 0$$

for all $v \in C$ and some $f_v \in Tu_0$.

Proof. For $C_n = C \cap B(0, n)$, by Theorem 2.7, there exists $u_n \in C_n$ such that

$$(f, u_n - v) \leq 0$$

for all $v \in C_n$ and some $f_v \in Tu_n$.By Lemma 2.5, we know that

$$(g, u_n - v) \leq 0$$

for all $v \in C_n$ and some $g \in Tv$. By assumption, we know that $\|u_n\| \leq r$ for $n = 1, 2, \dots$ and thus we may assume that $u_n \rightharpoonup u_0$ as $n \to \infty$. Otherwise, we take subsequence. Consequently, it follows that

$$(g, u_0 - v) \leq 0$$

for all $v \in C$ and $g \in Tv$. Again, we use Lemma 2.5 to conclude the proof. □

Corollary 2.9. *Let E be a real reflexive Banach space, C a non-empty closed convex unbounded subset of E. If $A : B(0, R) \to E^*$ is an finite dimensional weakly continuous and strictly quasi-monotone mapping and*

$$(Au, u) > -\|Au\|\|u\|$$

for all $u \in \partial B(0, r)$, then there exists $u_0 \in B(0, r)$ such that $Au_0 = 0$.

Proof. By Theorem 2.7, there exists $u_0 \in B(0, R)$ such that

$$(Au_0, u_0 - v) \leq 0$$

for all $v \in B(0, R)$. Now, we claim that $Au_0 = 0$. We first prove that $\|u_0\| < R$. In fact, if $\|u_0\| = R$, then, by assumption, $\|Au_0\| \neq 0$ and thus there exists $v_0 \in \partial B(0, r)$ such that $(Au_0, v_0) = -\|Au_0\|\|v_0\|$. But we have

$$-\|Au_0\|\|u_0\| < (Au_0, u_0) \leq (Au_0, v_0) = -\|Au_0\|\|v_0\|,$$

which is a contradiction. Therefore, we have $\|u_0\| < R$. Since there exists $r > 0$ such that $u_0 + v \in B(0, R)$ for all $v \in E$ with $\|v\| \leq r$, we have

$$(Au_0, v) \geq 0$$

for all $v \in B(0, r)$ and so $Au_0 = 0$. □

We do not know whether the conclusion of Corollary 2.8 is true if A is a multi-valued strictly quasi-monotone mapping and so we give an open question as follows:

Question. *Let E be a real reflexive Banach space and C a non-empty closed convex unbounded subset of E. If $A : B(0, R) \to 2^{E^*}$ is an finite dimensional weakly upper semi-continuous and strictly quasi-monotone mapping with bounded closed convex values and*

$$(f, u) > -\|f\|\|u\|$$

for all $u \in \partial B(0, r)$ and $f \in Au$. Does $0 \in Au_0$ has a solution in $B(0, R)$?

Theorem 2.10. *Let E be a real reflexive Banach space and let $A : E \to 2^{E^*}$ be an finite dimensional weakly upper semi-continuous mapping with bounded closed convex values. Suppose the following conditions are satisfied:*

(i) *$A - p^*$ is strictly quasi-monotone for each p^* in E^*,*

(ii) $\liminf_{\|x\| \to \infty, f \in Ax} \frac{(f,x)}{\|x\|^2} = +\infty$.

Then $A(E) = E^*$.

Proof. For each $p^* \in E^*$, by assumption (ii), there exists $R > 0$ such that

$$(f - p^*, u) > 0$$

for all $u \in E$ with $\|u\| \geq R$. By the assumption (i), $A - p^*$ is strictly quasi-monotone and so Corollary 2.8 implies that there exists $u_0 \in E$ such that

$$(f_v - p^*, u_0 - v) \leq 0$$

for all $v \in E$ and some $f_v \in Au_0$. Put $v = u_0 - w$ for all $w \in E$, we get

$$(f_w - p^*, w) \leq 0$$

for all $w \in E$ and some $f_w \in Au_0$. But Au_0 is bounded closed convex and thus the separation theorem of Mazur for convex subsets implies that $0 \in Au_0 - p^*$ (see [17]). □

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