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NONAUTONOMOUS SECOND ORDER PERIODIC SYSTEMS: EXISTENCE AND MULTIPLICITY OF SOLUTIONS

GIUSEPPINA BARLETTA AND NIKOLAOS S. PAPAGEORGIOU

Abstract. In this paper we study the existence and multiplicity of solutions for a second order nonautonumous periodic system with a nonsmooth potential. We prove two existence theorems and a multiplicity result. In the first existence theorem the Euler functional is coercive and the solution is a minimizer of it. In the second existence theorem the Euler functional is unbounded and the solution is obtained using the saddle point theorem. Finally for the multiplicity result we employ a nonsmooth version of the local linking theorem.

1. Introduction

In this paper we study the following second order periodic system with a nonsmooth potential:

$$\begin{cases} -x''(t) - A(t)x(t) \in \partial j(t, x(t)) \quad \text{a.e. on } T = [0, b], \\ x(0) = x(b),\ x'(0) = x'(b)\,. \end{cases} \tag{1.1}$$

Here $A : T \to \mathbb{R}^{N\times N}$ is a continuous map and for every $t \in T$, $A(t)$ is a symmetric $N \times N$-matrix. Also $j : T \times \mathbb{R}^N \to \mathbb{R}$ is a measurable function, which is locally Lipschitz and in general nonsmooth in the $x \in \mathbb{R}^N$ variable. By $\partial j(t, x)$ we denote the generalized subdifferential of the locally Lipschitz function $x \to j(t, x)$ (see Section 2).

Problem (1.1) was firstly studied by Rabinowitz [14] under the assumptions that for every $t \in T$, $A(t)$ is a negative definite matrix, $j(t, \cdot) \in C^1(\mathbb{R}^N)$ and $x \to j(t, x)$ exhibits a strictly subquadratic growth (more precisely, it satisfies the well known Ambrosetti-Rabinowitz condition). He proved the existence of a solution using variational methods based on the saddle point theorem. Later Mawhin [9] (see also Mawhin-Willem [10], p.89), considered problem (1.1) with A being a time-independent symmetric $N \times N$-matrix, but he did not impose any sign condition on it. Moreover, he assumed that the potential function $j(t, x)$ is measurable, continuously differentiable in $x \in \mathbb{R}^N$ and satisfies

$$|j(t, x)| \le h(t),\ \|\nabla j(t, x)\| \le h(t) \text{ for a.a. } t \in T \text{ and all } x \in \mathbb{R}^N\,,$$

where $h \in L^1(T)_+$. He proved an existence result using the saddle point theorem. In the book of Mawhin-Willem [10], p.63, the problem was studied with $A(t) = m^2\omega^2 I$ for all $t \in T$, where $m \ge 1$ is a positive integer, $\omega = \frac{2\pi}{b}$ and I is the identity

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$N \times N$-matrix. The potential function $j(t,x)$ is measurable and continuously differentiable and convex in $x \in \mathbb{R}^N$. In this case, the authors approach the problem using the dual action principle and they prove an existence theorem. Recently Tang-Wu [19] examined problem (1.1) with A being time-dependent, no sign condition is imposed on the matrix $A(t)$, $t \in T$ and the potential function $j(t,x)$ is measurable, continuously differentiable in $x \in \mathbb{R}^N$ and exhibits a strictly subquadratic growth. The authors prove an existence theorem using the saddle point theorem. Motreanu-Motreanu-Papageorgiou [11] considered the complementary situation to that in the work of Tang-Wu [19] and assumed that $x \to j(t,x)$ exhibits a strictly superquadratic growth. Moreover, in their work $x \to j(t,x)$ is only locally Lipschitz and in general nonsmooth. Multiplicity results were proved by Barletta-Livrea [1], Bonanno-Livrea [2], Cordaro [4], Faraci [5] (problems with a smooth potential) and by Motreanu-Motreanu-Papageorgiou [11] (problems with a nonsmooth potential). In the works of Barletta-Livrea [1], Bonanno-Livrea [2], Cordaro [4] and Faraci [5], for every $t \in T$ $A(t)$ is negative definite and the method of the proof is based on an abstract multiplicity result of Ricceri [15] or variants of it. In Motreanu-Motreanu-Papageorgiou [11], the nonsmooth potential is quadratic or symmetric (even).

In this paper we prove two existence theorems and a multiplicity result using nonsmooth critical point theory (see the books of Gasinski-Papageorgiou [6] and Motreanu-Radulescu [12]).

2. Mathematical background

We make the following assumption on the matrix-valued map $t \to A(t)$.
$\underline{H(A)}$: $A : T = [0,b] \to \mathbb{R}^{N\times N}$ is continuous and for every $t \in T$, $A(t)$ is symmetric.
In our analysis of problem (1.1), we use the following space:

$$W^{1,2}_{per}((0,b),\mathbb{R}^N) = \{x \in W^{1,2}((0,b),\mathbb{R}^N) \,:\, x(0) = x(b)\}\,.$$

Since $W^{1,2}((0,b),\mathbb{R}^N)$ is compactly embedded into $C(T,\mathbb{R}^N)$, in the above definition the evaluations at $t = 0$ and $t = b$ make sense. By $\|\cdot\|$ we denote the norm of $W^{1,2}_{per}((0,b),\mathbb{R}^N)$ and of $\mathbb{R}^N$. It will always be clear from the context which one is in use. Let $\widehat{A} \in \mathcal{L}(C(T,\mathbb{R}^N), C(T,\mathbb{R}^N))$ be defined by

$$\widehat{A}(x)(t) = A(t)x(t) \text{ for all } t \in T,\ x \in C(T,\mathbb{R}^N)$$

(the Nemytskii operator corresponding to $A(\cdot)$). As in Mawhin-Willem [10], p.89 and Showalter [16], p.78, using the spectral theorem for compact self-adjoint operators on a Hilbert space, for the differential operator $x \to -x'' - \widehat{A}x$, we obtain a sequence of eigenfunctions which form an orthonormal basis for $\mathcal{L}^2(T,\mathbb{R}^N)$ and an orthogonal basis for $W^{1,2}_{per}((0,b),\mathbb{R}^N)$. Then we can consider the following orthogonal direct sum decomposition

$$W^{1,2}_{per}((0,b),\mathbb{R}^N) = H_- \oplus H_0 \oplus H_+ \tag{2.1}$$

where

$$H_- = span\{x \in W^{1,2}_{per}((0,b),\mathbb{R}^N) : -x'' - \widehat{A}(x) = \lambda x \text{ for some } \lambda < 0\}$$

$$H_0 = ker(-x'' - \widehat{A}x)$$

and

$$H_+ = \overline{span}\{x \in W^{1,2}_{per}((0,b),\mathbb{R}^N) : -x'' - \widehat{A}(x) = \lambda x \text{ for some } \lambda > 0\}.$$

Note that both H_- and H_0 are finite dimensional subspaces of $W^{1,2}_{per}((0,b),\mathbb{R}^N)$. Nonsmooth critical point theory relies on the subdifferential theory for locally Lipschitz functions due to Clarke [3]. So, let X be a Banach space, X^* its topological dual and denote by $\langle\cdot,\cdot\rangle$ the duality brackets for the pair (X^*,X). Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$ the generalized directional derivative $\varphi^0(x;h)$ of φ at $x \in X$ in the direction $h \in X$, is defined by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $\varphi^0(x;\cdot)$ is sublinear continuous. Therefore it is the support function of a nonempty, w^*-compact and convex set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi^0(x;h) \ \forall\, h \in X\}.$$

The multifunction $x \to \partial\varphi(x)$ is called the "generalized subdifferential" of φ. If $\varphi : X \to \mathbb{R}$ is continuous convex, then φ is locally Lipschitz and the generalized subdifferential of φ coincides with the subdifferential $\partial_C\varphi(\cdot)$ of φ in the sense of convex analysis, defined by

$$\partial_C\varphi(x) = \{x^* \in X^* : \langle x^*, h\rangle \le \varphi(x+h) - \varphi(x) \ \forall\, h \in X\}.$$

Also, if $\varphi \in C^1(X)$, then φ is locally Lipschitz and

$$\partial\varphi(x) = \{\varphi'(x)\}.$$

We say that $x \in X$ is a critical point of the locally Lipschitz function $\varphi : X \to \mathbb{R}$, if $0 \in \partial\varphi(x)$. In the present nonsmooth setting, the Palais-Smale condition takes the following form:
"A locally Lipschitz function $\varphi : X \to \mathbb{R}$ satisfies the "nonsmooth Palais-Smale condition at level $c \in \mathbb{R}$" (nonsmooth PS_c-condition for short), if every sequence $\{x_n\}_{n\ge1} \subseteq X$ such that

$$\varphi(x_n) \to c \ \text{ and } \ m(x_n) = \inf[\|x^*\| : x^* \in \partial\varphi(x_n)] \to 0 \ \text{ as } n \to \infty,$$

has a strongly convergent subsequence. If this property holds at every level $c \in \mathbb{R}$, then we simply say that φ satisfies the nonsmooth PS-condition".
The following is a nonsmooth version of the saddle point theorem.

Theorem 2.1. *If $X = Y \oplus V$ with $dimY < +\infty$, there exists $r > 0$ such that*

$$\max_{\substack{x \in Y \\ \|x\| = r}} \varphi(x) \le \inf_{x \in V} \varphi(x)$$

and φ satisfies the PS_c-condition where

$$c = \inf_{\gamma\in\Gamma} \sup_{x \in \overline{B}_r} \varphi(\gamma(x))$$

with $\Gamma = \{\gamma \in C(\overline{B}_r, X) : \gamma|_{\partial \overline{B}_r} = id|_{\partial \overline{B}_r}\}$, $\overline{B}_r = \{x \in X : \|x\| \le r\}$ *and* $\partial \overline{B}_r = \{x \in X : \|x\| = r\}$, *then* $c \ge \inf_V \varphi$ *and* c *is a critical value of* φ. *Moreover, if* $c = inf_V \varphi$, *then* $V \cap K_c \neq \emptyset$, *where* $K_c = \{x \in X : \varphi(x) = c, 0 \in \partial\varphi(x)\}$ *(the critical set of* φ *at level* $c \in \mathbb{R}$*).*

For the multiplicity theorem, we will use the following nonsmooth version of the local linking theorem due to Kandilakis-Kourogenis-Papageorgiou [8].

Theorem 2.2. *If* $X = Y \oplus V$ *with* $dim Y < +\infty$, $\varphi : X \to \mathbb{R}$ *is Lipschitz continuous on bounded sets, it satisfies the nonsmooth* PS*-condition,* $\varphi(0) = 0$, φ *is bounded from below,* $\inf_X \varphi < 0$ *and there exists* $r > 0$ *such that*

$$\varphi(y) \le 0 \text{ if } y \in Y,\ \|y\| \le r, \text{ and } \varphi(v) \ge 0 \text{ if } v \in V,\ \|v\| \le r\,,$$

then φ *has at least two nontrivial critical points.*

3. Existence theorems

For the first existence theorem, our hypotheses on the nonsmooth potential function $j(z,x)$, are the following:
$\underline{H(j)_1}$: $j : T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that

- (i) for all $x \in \mathbb{R}^N$, $t \to j(t,x)$ is measurable;
- (ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;
- (iii) there exist functions $a \in L^1(T)_+$ and $c \in C(\mathbb{R}_+)_+$ such that for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have
$$|u| \le a(t)c(\|x\|)\,;$$
- (iv) there exist a measurable set $C \subseteq T$ with $|C| > 0$ ($|\cdot|$ being the Lebesgue measure on $\mathbb{R}$) and $\xi \in L^1(T)_+$ such that
$$j(t,x) \to -\infty \text{ for a.a. } t \in C \text{ as } \|x\| \to \infty$$
and
$$j(t,x) \le \xi(t) \text{ for a.a. } t \in T \text{ and all } x \in \mathbb{R}^N\,;$$
- (v) there exists $\delta_0 > 0$, such that for almost $t \in T$ and all $0 < \|x\| \le \delta_0$
$$j(t,x) > 0\,.$$

Examples : The following potential functions satisfy hypothesis $\underline{H(j)_1}$:

$$j_1(x) = \min\left\{\frac{1}{2}\|x\|^2,\ \frac{1}{r}\|x\|^r\right\} - \frac{1}{s}\|x\|^s \text{ with } 2, r < s,$$

$$j_2(x) = \frac{\|x\|}{\|x\|+1} - \frac{1}{r}\chi_C(t)\|x\|^r \text{ with } 1 < r,$$

$$\text{and } j_3(x) = \frac{1}{2}\|x\|^2 - \frac{1}{3}\|x\|^3\,.$$

Evidently $j_3 \in C^1(\mathbb{R}^N)$.

We start with two auxiliary results which will be used in the proof of the first existence theorem.

By virtue of decomposition (2.1), given any $x \in W^{1,2}_{per}((0,b),\mathbb{R}^N)$ we can write in a unique way

$$x = \overline{x} + x^0 + \widehat{x} \text{ with } \overline{x} \in H_-,\ x^0 \in H_0 \text{ and } \widehat{x} \in H_+ .$$

Lemma 3.1. *Given $\varepsilon > 0$, we can find $\eta_\varepsilon > 0$*

$$|\{t \in T : \|x^0(t)\| < \eta_\varepsilon \|x^0\|\}| < \varepsilon \text{ for all } x^0 \in H_0 .$$

Proof. We argue by contradiction. So suppose that the lemma is not true. Then we can find $\varepsilon > 0$ and a sequence $\{x^0_n\}_{n\geq 1} \subseteq H_0$ such that

$$\left|\left\{t \in T : \|x^0_n(t)\| < \frac{1}{n}\|x^0_n\|\right\}\right| \geq \varepsilon \quad \text{for all } n \geq 1 .$$

Let $y^0_n = \frac{x^0_n}{\|x^0_n\|}$, $n \geq 1$. Since H_0 is finite dimensional, by passing to a suitable subsequence if necessary, we may assume that

$$y^0_n \to y^0 \text{ in } W^{1,2}_{per}((0,b),\mathbb{R}^N) \text{ and in } C(T,\mathbb{R}^N) \text{ as } n \to \infty .$$

Hence $y^0 \in H_0$, $\|y^0\| = 1$ and so $y^0 \neq 0$. Also, if

$$D_n = \left\{t \in T : \|y^0_n(t)\| < \frac{1}{n}\right\} \text{ and } \widehat{C}_0 = \{t \in T : y^0(t) = 0\},$$

then clearly $\limsup_{n\to\infty} D_n \subseteq \widehat{C}_0$. So we obtain

$$\varepsilon \leq \limsup_{n\to\infty} |D_n| \leq |\limsup_{n\to\infty} D_n| \leq |\widehat{C}_0| . \tag{3.1}$$

But because $y^0 \in H_0 \setminus \{0\}$, we have $y^0(t) \neq 0$ for almost all $t \in T$. This contradicts (3.1). ☐

The next Lemma gives useful information for the component spaces H_- and H_+. Its proof can be found in Motreanu-Motreanu-Papageorgiou [11].

Lemma 3.2. (*a*) *There exists $\xi_+ > 0$ such that*

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt \geq \xi_+ \|x\|^2 \text{ for all } x \in H_+.$$

(*b*) *There exists $\xi_- > 0$ such that*

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt \leq -\xi_- \|x\|^2 \text{ for all } x \in H_-.$$

Remark 3.3. *Let $\{\lambda_n\}_{n\geq 1}$ denote the eigenvalues of the differential operator $x \to -x'' - \widehat{A}x$ repeated according to multiplicity. Using Lemma 3.2 (a), we see that there is a smallest positive eigenvalue $\lambda_m > 0$ and in fact $\xi_+ = \lambda_m$. Moreover, by virtue of the finite dimensionality of H_- (see also Lemma 3.2 (b)), there is a biggest negative eigenvalue $\lambda_k < 0$. In fact $-\xi_- = \lambda_k$.*

Now we are ready for the first existence theorem concerning problem (1.1).

Theorem 3.4. *If hypotheses $H(A)$ and $H(j)_1$ hold and $dim H_- = 0$, then problem (1.1) admits a nontrivial solution $x_0 \in C^1(T,\mathbb{R}^N) \cap W^{2,1}((0,b),\mathbb{R}^N)$.*

Proof. Consider the integral functional $\widehat{\psi} : C_{per}(T, \mathbb{R}^N) \to \mathbb{R}$ defined by

$$\widehat{\psi}(x) = \int_0^b j(t, x(t))dt\,.$$

We claim that $\widehat{\psi}$ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz. To this end, let $x \in C_{per}(T, \mathbb{R}^N)$ and $\|x\|_\infty \le r$. Then for any $u \in C_{per}(T, \mathbb{R}^N)$ with $\|u\|_\infty \le r$, from the mean value theorem for locally Lipschitz functions (see Clarke [3], p.41), for almost all $t \in T$ we have

$$j(t, x(t)) - j(t, u(t)) = (v_t^*(t), x(t) - u(t))_{\mathbb{R}^N} \tag{3.2}$$

with $v_t^* \in \partial j(t, y_t)$, where $y_t = (1 - \lambda_t)x(t) + \lambda_t u(t)$, $\lambda_t \in (0, 1)$. Assuming without any loss of generality that $c \in C(\mathbb{R}_+)_+$ in hypothesis $H(j)_1(iii)$ is increasing, from (3.2) we have

$$j(t, x(t)) - j(t, u(t)) \le a(t)c(\eta)\|x(t) - u(t)\| \text{ with } \eta = \max\{\|x\|_\infty, \|v\|_\infty\},$$

$$\text{so } \int_0^b |j(t, x(t)) - j(t, u(t))|dt \le \|a\|_1 c(\eta)\|x - u\|_\infty\,,$$

and finally

$$|\widehat{\psi}(x) - \widehat{\psi}(u)| \le \|a\|_1 c(\eta)\|x - u\|_\infty\,.$$

Therefore $\widehat{\psi}$ is Lipschitz continuous on $\overline{B}_r^C = \{y \in C_{per}(T, \mathbb{R}^N) : \|y\|_\infty \le r\}$, hence it is locally Lipschitz. Since $W^{1,2}_{per}((0, b), \mathbb{R}^N)$ is embedded continuously and densely into $C_{per}(T, \mathbb{R}^N)$, from Clarke [3], pp.47 and 83, we have $\psi = \widehat{\psi}_{|W^{1,2}_{per}((0,b),\mathbb{R}^N)}$ is locally Lipschitz and

$$\partial\psi(x) \subseteq \{u^* \in L^1(T, \mathbb{R}^N) : u^*(t) \in \partial j(t, x(t)) \text{ a.e. on } T\}.$$

Since $x \to \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\,dt$ is a C^1-convex function, then the Euler functional $\varphi : W^{1,2}_{per}((0, b), \mathbb{R}^N) \to \mathbb{R}$ for problem (1.1) defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t, x(t))dt\,,$$

is locally Lipschitz. Because of hypothesis $H(j)(iv)$ and Lemmata 1 and 3 of Tang-Wu [18], given $\varepsilon > 0$, we can find $D_\varepsilon \subseteq C$ measurable such that $|C \setminus D_\varepsilon| < \varepsilon$, $j(t, x) \to -\infty$ uniformly in $t \in D_\varepsilon$ as $\|x\| \to \infty$ and

$$j(t, x) \le h(t) - g(x) \text{ for a.a. } t \in D_\varepsilon \text{ and all } x \in \mathbb{R}^N\,, \tag{3.3}$$

where $h \in L^1(T)_+$ and $g \in C(\mathbb{R}^N)$, $g \ge 0$ with the following properties:

$$g \text{ is subadditive;} \tag{3.4}$$

$$g \text{ is coercive;} \tag{3.5}$$

$$g(x) \le 4 + \|x\| \text{ for all } x \in \mathbb{R}^N. \tag{3.6}$$

We consider the integral functional $G : W^{1,2}_{per}((0, b), \mathbb{R}^N) \to \mathbb{R}_+$ defined by

$$G(y) = \int_{D_\varepsilon} g(y(t))dt \quad \text{for all } y \in W^{1,2}_{per}((0, b), \mathbb{R}^N)\,.$$

Evidently G is continuous.
Claim I: $G_{|H_0}$ is coercive.
Let $\{x_n^0\}_{n\geq 1} \subseteq H_0$ be a sequence such that $\|x_n^0\| \to \infty$. By virtue of Lemma 3.1, given $\delta > 0$, we can find a measurable set $E_\delta \subseteq D_\varepsilon$, with $|D_\varepsilon \setminus E_\delta| < \delta$ such that

$$\|x_n^0(t)\| \to +\infty \text{ uniformly in } t \in E_\delta\,.$$

We can always choose $\delta > 0$ small so that $|E_\delta| > 0$. Because g is coercive (see (3.5)), given $\theta > 0$, we can find $R = R(\theta) > 0$ large such that

$$g(x) \geq \theta \text{ for all } \|x\| \geq R\,. \tag{3.7}$$

So we can find $n_0 = n_0(R) \geq 1$ such that

$$\|x_n^0(t)\| \geq R \text{ for all } t \in E_\delta \text{ and all } n \geq n_0\,,$$

so

$$g(x_n^0(t)) \geq \theta \text{ for all } t \in E_\delta \text{ and all } n \geq n_0 \text{ (see (3.7))}\,. \tag{3.8}$$

Bearing in mind that $g \geq 0$, we have

$$\begin{aligned}\int_{D_\varepsilon} g(x_n^0(t))dt &= \int_{E_\delta} g(x_n^0(t))dt + \int_{D\varepsilon\setminus E_\delta} g(x_n^0(t))dt \\ &\geq \int_{E_\delta} g(x_n^0(t))dt \geq \theta |E_\delta| > 0 \;\text{ for all } n \geq n_0\,. \end{aligned} \tag{3.9}$$

Because $\theta > 0$ was arbitrary, from (3.9) we deduce that

$$\lim_{n\to\infty} G(x_n^0) = \lim_{n\to\infty} \int_{D_\varepsilon} g(x_n^0(t))dt = +\infty\,,$$

so $G_{|H_0}$ is coercive.
Claim II: φ is coercive.
From (3.3) and hypothesis $H(j)_1(iv)$, for all $x \in W^{1,2}_{per}((0,b), \mathbb{R}^N)$, we have

$$\int_0^b j(t,x(t))dt = \int_{D_\varepsilon} j(t,x(t))dt + \int_{T\setminus D_\varepsilon} j(t,x(t))dt \leq \|h\|_1 - \int_{D_\varepsilon} g(x(t))dt + \|\xi\|_1\,. \tag{3.10}$$

We write $x = x^0 + \widehat{x}$, with $x^0 \in H_0$ and $\widehat{x} \in H_+$ (recall we assume that $dimH_- = 0$). Exploiting the subadditivity of g (see (3.4)), we have

$$g(x^0(t)) = g(x(t) - \widehat{x}(t)) \leq g(x(t)) + g(-\widehat{x}(t)),\text{ for all } t \in T,\text{ hence}$$

$$-g(x(t)) \leq g(-\widehat{x}(t)) - g(x^0(t)) \text{ for all } t \in T\,. \tag{3.11}$$

So, returning to (3.10) and using (3.11) and (3.6), we obtain

$$\begin{aligned}\int_0^b j(t,x(t))dt &\leq \|h\|_1 + \int_{D_\varepsilon} g(-\widehat{x}(t))dt - \int_{D_\varepsilon} g(x^0(t))dt + \|\xi\|_1 \\ &\leq \|h\|_1 + \int_0^b (4 + \|\widehat{x}(t)\|)dt - \int_{D_\varepsilon} g(x^0(t))dt + \|\xi\|_1 \\ &\leq c_1 + c_2\|\widehat{x}\| - \int_{D_\varepsilon} g(x^0(t))dt \;\text{ for some } c_1, c_2 > 0. \end{aligned} \tag{3.12}$$

Then, exploiting the orthogonality of the component spaces, using Lemma 3.2 and (3.12), for all $x \in W^{1,2}_{per}((0,b),\mathbb{R}^N)$ we have

$$\varphi(x) = \frac{1}{2}\|\widehat{x}'\|_2^2 - \frac{1}{2}\int_0^b (A(t)\widehat{x}(t),\widehat{x}(t))_{\mathbb{R}^N}\,dt - \int_0^b j(t,x(t))dt$$

$$\geq \frac{\xi_+}{2}\|\widehat{x}\|^2 - c_2\|\widehat{x}\| + \int_{D_\varepsilon} g(x^0(t))dt - c_1\,. \tag{3.13}$$

From (3.13) and *Claim I*, it follows that φ is coercive.
Claim III: φ is sequentially weakly lower semicontinuous.
Let $x_n \rightharpoonup x$ in $W^{1,2}_{per}((0,b),\mathbb{R}^N)$. We may also assume that $x_n \to x$ in $C(T,\mathbb{R}^N)$. Then by virtue of hypothesis $H(j)_1(ii)$, we have

$$j(t,x_n(t)) \to j(t,x(t)) \text{ a.e. on } T\,.$$

Then hypothesis $H(j)_1(iv)$ and Fatou's lemma, imply

$$\limsup_{n\to\infty}\int_0^b j(t,x_n(t))dt \leq \int_0^b j(t,x(t))dt\,. \tag{3.14}$$

Also, because $x_n' \rightharpoonup x'$ in $L^2(T,\mathbb{R}^N)$, from the weak lower semicontinuity of the norm functional, we have

$$\|x'\|_2^2 \leq \liminf_{n\to\infty}\|x_n'\|_2^2\,. \tag{3.15}$$

Finally, since $x_n \to x$ in $C(T,\mathbb{R}^N)$, we see that

$$\int_0^b (A(t)x_n(t),x_n(t))_{\mathbb{R}^N}\,dt \to \int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\,dt\,. \tag{3.16}$$

From (3.14), (3.15) and (3.16), it follows that

$$\varphi(x) \leq \liminf_{n\to\infty}\varphi(x_n)\,,$$

hence φ is sequentially weakly lower semicontinuous.
Because of *Claims I, II* and *III*, we can apply the theorem of Weierstrass and find $x_0 \in W^{1,2}_{per}((0,b),\mathbb{R}^N)$ such that

$$\varphi(x_0) = \inf\left[\varphi(x):\ x \in W^{1,2}_{per}((0,b),\mathbb{R}^N)\right]\,. \tag{3.17}$$

Let $\delta_0 > 0$ be as in hypothesis $H(j)_1(v)$ and $x^0 \in H_0$ such that $\|x^0\|_\infty \leq \delta_0$. We know that $x^0(t) \neq 0$ for a.a. $t \in T$. Then

$$\varphi(x^0) = \frac{1}{2}\|(x^0)'\|_2^2 - \frac{1}{2}\int_0^b \left(A(t)x^0(t),x^0(t)\right)_{\mathbb{R}^N}dt - \int_0^b j(t,x^0(t))dt =$$

$$= -\int_0^b j(t,x^0(t))dt < 0,$$

so $\varphi(x_0) \leq \varphi(x^0) < 0 = \varphi(0)$ and $x_0 \neq 0$.
From (3.17), we have $0 \in \partial\varphi(x_0)$, then

$$V(x_0) - \widehat{A}(x_0) = u_0^*\,, \tag{3.18}$$

where $V \in \mathcal{L}\Big(W^{1,2}_{per}((0,b),\mathbb{R}^N), W^{1,2}_{per}((0,b),\mathbb{R}^N)^*\Big)$ is defined by

$$\langle V(x), y\rangle = \int_0^b \big(x'(t), y'(t)\big)_{\mathbb{R}^N}\, dt \quad \text{for all } x,\, y \in W^{1,2}_{per}((0,b),\mathbb{R}^N)\,,$$

and $u_0^* \in L^1(T,\mathbb{R}^N)$ is such that $u_0^*(t) \in \partial j(t, x_0(t))$ a.e. on T (see, Clarke [3], p.83).
From the representation theorem for the elements in $W^{-1,2}((0,b),\mathbb{R}^N) = W^{1,2}_0((0,b),\mathbb{R}^N)^*$ (see for example Gasinski-Papageorgiou [7], p.212), we know that $x'' \in W^{-1,2}((0,b),\mathbb{R}^N)$. By $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $\Big(W^{-1,2}((0,b),\mathbb{R}^N), W^{1,2}_0((0,b),\mathbb{R}^N)\Big)$. Acting on (3.18) with the test function $\psi \in C^1_c((0,b),\mathbb{R}^N)$ and performing an integration by parts, we obtain

$$\langle -x_0'', \psi\rangle_0 - \int_0^b (A(t)x_0(t), \psi(t))_{\mathbb{R}^N}\, dt = \int_0^b (u_0^*(t), \psi(t))_{\mathbb{R}^N}\, dt\,. \tag{3.19}$$

Since $C^1_c((0,b),\mathbb{R}^N)$ is dense in $W^{1,2}_0((0,b),\mathbb{R}^N)$, from (3.19) it follows that

$$-x_0''(t) - A(t)x_0(t) = u_0^*(t) \text{ a.e. on } T,\; x_0(0) = x_0(b)\,, \tag{3.20}$$

so $x_0'' \in L^1(T,\mathbb{R}^N)$, and $x_0 \in C^1(T,\mathbb{R}^N) \cap W^{2,1}((0,b),\mathbb{R}^N)$. Acting on (3.18) with $v \in W^{1,2}_{per}((0,b),\mathbb{R}^N)$, after integrating by parts, we have

$$\big(x_0'(b), v(b)\big)_{\mathbb{R}^N} - \big(x_0'(0), v(0)\big)_{\mathbb{R}^N} - \int_0^b \big(x_0''(t), v(t)\big)_{\mathbb{R}^N}\, dt - \int_0^b (A(t)x_0(t), v(t))_{\mathbb{R}^N}\, dt$$
$$= \int_0^b (u_0^*(t), v(t))_{\mathbb{R}^N}\, dt$$

and using (3.20) we deduce

$$\big(x_0'(b), v(b)\big)_{\mathbb{R}^N} = \big(x_0'(0), v(0)\big)_{\mathbb{R}^N} \text{ for all } v \in W^{1,2}_{per}((0,b),\mathbb{R}^N)\,,$$

so we conclude that $x_0'(0) = x_0'(b)$, hence $x_0 \in C^1(T,\mathbb{R}^N) \cap W^{2,1}((0,b),\mathbb{R}^N)$ is a nontrivial solution of (1.1). □

In the above existence theorem, the Euler functional of the problem was bounded from below (in fact coercive). In the next existence theorem, the Euler functional is indefinite. Moreover, we do not assume that $dim\, H_- = 0$ and so the problem has an indefinite linear part.

The new hypotheses on the nonsmooth potential $j(t,x)$ are the following:
$\underline{H(j)_2}$: $j : T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that hypotheses $H(j)_2(i)$ and (ii) are the same as hypotheses $H(j)_1(i)$ and (ii), and

(iii) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have

$$|u| \le a(t) + c(t)\|x\|^\theta \text{ with } a,\, c \in L^1(T)_+,\; 0 < \theta < 1\,;$$

(iv)

$$\frac{1}{\|x\|^{2\theta}} \int_0^b j(t,x(t))dt \to \pm\infty \text{ as } \|x\| \to \infty,\; x \in H_0.$$

Remark 3.5. *Hypothesis $H(j)_2(iv)$ is weaker than the analogous condition employed in Theorems 2 and 3 by Tang-Wu* [19]. *Note that, if $A \equiv 0$, then hypothesis $H(j)_2(iv)$ becomes*

$$\lim_{\|x\|\to\infty} \frac{1}{\|x\|^{2\theta}} \int_0^b j(t,x)dt = \pm\infty \ .$$

This condition was used by Tang [17] *and by Papageorgiou-Papageorgiou* [13], *in the latter in the context of problems driven by the ordinary vector $p-$Laplacian.*

As before, we consider the Euler functional $\varphi: W^{1,2}_{per}((0,b),\mathbb{R}^N) \to \mathbb{R}$ for problem (1.1), defined by

$$\varphi(x) = \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t),x(t))_{\mathbb{R}^N}\, dt - \int_0^b j(t,x(t))dt\,.$$

Due to hypotheses $H(j)_2(i)$, $(ii)\,(iii)$ we know that φ is Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [3], p.83).
First we check that φ satisfies the nonsmooth $PS-$condition.

Proposition 3.6. *If hypotheses $H(A)$ and $H(j)_2$ hold, then φ satisfies the nonsmooth $PS-$condition.*

Proof. We do the proof for the case $\frac{1}{\|x\|^{2\theta}}\int_0^b j(t,x(t))dt \to +\infty$ as $\|x\| \to \infty$, $x \in H_0$, the proof being similar if the limit is $-\infty$. So let $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)$ be a sequence such that

$$|\varphi(x_n) \leq M_1 \text{ for some } M_1 > 0, \text{ all } n \geq 1 \text{ and } m(x_n) \to 0\,.$$

Since $\partial\varphi(x_n) \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)^*$ is weakly compact and the norm functional in a Banach space is weakly lower semicontinuous, by the Weierstrass theorem, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$, $n \geq 1$. We write

$$x_n = \overline{x}_n + x_n^0 + \widehat{x}_n \text{ with } \overline{x}_n \in H_-,\ x_n^0 \in H_0,\ \widehat{x}_n \in H_+ \text{ for all } n \geq 1\,.$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)$, we have

$$\langle x_n^*, \widehat{x}_n\rangle \leq \varepsilon_n\|\widehat{x}_n\| \text{ with } \varepsilon_n \downarrow 0\,,$$

hence we can find $u_n^* \in L^1(T,\mathbb{R}^N)$, $u_n^*(t) \in \partial j(t,x_n(t))$ a.e. on T such that

$$\langle V(x_n),\widehat{x}_n\rangle - \langle \widehat{A}(x_n),\widehat{x}_n\rangle - \int_0^b (u_n^*,\widehat{x}_n)_{\mathbb{R}^N} dt \leq \varepsilon_n\|\widehat{x}_n\|\,. \tag{3.21}$$

Exploiting the orthogonality of the component spaces, and using Lemma 3.2 (a), we have

$$\langle V(x_n),\widehat{x}_n\rangle - \langle \widehat{A}(x_n),\widehat{x}_n\rangle = \|\widehat{x}_n'\|_2^2 - \int_0^b (A(t)\widehat{x}_n(t),\widehat{x}_n(t))_{\mathbb{R}^N} dt \geq \xi_+\|\widehat{x}_n\|^2\,. \tag{3.22}$$

The use of (3.22) and of hypothesis $H(j)_2(iii)$ in (3.21), yields

$$\begin{aligned}\xi_+\|\widehat{x}_n\|^2 &\leq \varepsilon_n\|\widehat{x}_n\| + \int_0^b \|u_n^*(t)\|\|\widehat{x}_n(t)\|dt \leq \varepsilon_n\|\widehat{x}_n\| + \left(c_3 + c_4\|x_n\|^\theta\right)\|\widehat{x}_n\| \\ &\leq \varepsilon_n\|x_n\| + \left(c_3 + c_4\|\overline{x}_n\|^\theta + c_4\|x_n^0\|^\theta\right)\|\widehat{x}_n\| + c_4\|\widehat{x}_n\|^{\theta+1} \\ &\qquad \text{for some } c_3,\, c_4 > 0.\end{aligned} \tag{3.23}$$

Using Young's inequality with suitably small $\varepsilon > 0$ on each summand of the right hand side in (3.23), we obtain

$$\|\widehat{x}_n\|^2 \leq c_5\|\overline{x}_n\|^{2\theta} + c_6\|x_n^0\|^{2\theta} + c_7 \text{ for some } c_5,\, c_6,\, c_7 > 0,\ \text{all } n \geq 1. \tag{3.24}$$

In a similar fashion, using as a test function $-\overline{x}_n^- \in W^{1,2}_{per}((0,b),\mathbb{R}^N)$, we have

$$\|\overline{x}_n\|^2 \leq c_8\|x_n^0\|^{2\theta} + c_9\|\widehat{x}_n\|^{2\theta} + c_{10} \text{ for some } c_8,\, c_9,\, c_{10} > 0,\ \text{all } n \geq 1. \tag{3.25}$$

From (3.24) and (3.25) and if $y_n = \overline{x}_n + \widehat{x}_n$, then

$$\|y_n\|^2 = \|\overline{x}_n\|^2 + \|\widehat{x}_n\|^2 \leq c_{11}\|y_n\|^{2\theta} + c_{12}\|x_n^0\|^{2\theta} + c_{13} \quad \text{for some } c_{11},\, c_{12},\, c_{13} > 0,\ \text{all } n \geq 1,$$

so, via Young's inequality with $\varepsilon > 0$,

$$\|y_n\|^2 \leq c_{14}\|x_n^0\|^{2\theta} + c_{15} \ \text{ for some } c_{14},\, c_{15} > 0,\ \text{all } n \geq 1,$$

hence

$$\limsup_{n\to\infty} \frac{\|y_n\|}{\|x_n^0\|^\theta} < +\infty\,. \tag{3.26}$$

From the mean value theorem for locally Lipschitz functions (see Clarke [3], p.41), for almost all $t \in T$, we can find $v_n^*(t) \in \partial j(t, v_n(t))$ with $v_n(t) = \lambda_n(t)x_n(t) + (1 - \lambda_n(t))x_n^0(t)$, $\lambda_n(t) \in (0,1)$, such that

$$\begin{aligned} |j(t,x_n(t)) - j(t,x_n^0(t))| &= |(v_n^*(t), \overline{x}_n(t) + \widehat{x}_n(t))_{\mathbb{R}^N}| \leq \|v_n^*(t)\|\|y_n(t)\| \\ &\leq \Big(a(t) + \widehat{c}(t)\left(\|x_n^0(t)\|^\theta + \|y_n(t)\|^\theta\right)\Big)\|y_n(t)\| \ \text{ with } \widehat{c} \in L^1(T)_+\,. \end{aligned} \tag{3.27}$$

From the choice of the sequence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)$, we have

$$\varphi(x_n) + \int_0^b j(t,x_n^0(t))dt \geq -M_1 + \int_0^b j(t,x_n^0(t))dt\,,$$

so using (3.27) we deduce that

$$\begin{aligned} &c_{16}\|y_n\|^2 + c_{17}\|y_n\| + c_{18}\|x_n^0\|^\theta\|y_n\| + c_{19}\|y_n\|^{\theta+1} \\ &\geq -M_1 + \int_0^b j(t,x_n^0(t))dt \text{ for some } c_{16},\, c_{17}\, c_{18} > 0, \end{aligned}$$

hence

$$\begin{aligned} &c_{16}\left(\frac{\|y_n\|}{\|x_n^0\|^\theta}\right)^2 + c_{17}\frac{\|y_n\|}{\|x_n^0\|^{2\theta}} + c_{18}\frac{\|y_n\|}{\|x_n^0\|^\theta} + c_{19}\frac{\|y_n\|}{\|x_n^0\|^\theta}\left(\frac{\|y_n\|}{\|x_n^0\|}\right)^\theta \\ &\geq \frac{-M_1}{\|x_n^0\|^{2\theta}} + \frac{1}{\|x_n^0\|^{2\theta}}\int_0^b j(t,x_n^0(t))dt\,. \end{aligned} \tag{3.28}$$

If $\{x_n^0\} \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)$ is bounded, then so is $\{y_n\} \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)$. If $\|x_n^0\| \to \infty$, then by passing to the limit $n \to \infty$ in (3.28) we reach a contradiction, since from (3.26) the left hand side has a finite limsup, while the right hand side goes to $+\infty$ (see hypothesis $H(j)_2(iv)$). This proves that $\{x_n^0\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)$ is bounded, hence $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b),\mathbb{R}^N)$ is bounded (see (3.26)). Therefore we may assume that $x_n \rightharpoonup x$ in $W^{1,2}_{per}((0,b),\mathbb{R}^N)$ and $x_n \to x$ in $C(T,\mathbb{R}^N)$. We have

$$\left|\langle V(x_n), x_n - x\rangle - \int_0^b (A(t)x_n(t),(x_n-x)(t))_{\mathbb{R}^N}\,dt - \int_0^b (u_n^*(t),(x_n-x)(t))_{\mathbb{R}^N}dt\right| \leq \varepsilon_n\|x_n - x\|\,. \tag{3.29}$$

Evidently

$$\int_0^b (A(t)x_n(t),(x_n-x)(t))_{\mathbb{R}^N}\,dt \to 0 \text{ and } \int_0^b (u_n^*(t),(x_n-x)(t))_{\mathbb{R}^N}dt \to 0 \text{ as } n\to\infty\,.$$

So, from (3.29) it follows that

$$\lim_{n\to\infty}\langle V(x_n), x_n - x\rangle = 0\,. \tag{3.30}$$

But $V \in \mathcal{L}\left(W^{1,2}_{per}((0,b),\mathbb{R}^N), W^{1,2}_{per}((0,b),\mathbb{R}^N)^*\right)$ and so $V(x_n) \rightharpoonup V(x)$ in $W^{1,2}_{per}((0,b),\mathbb{R}^N)^*$. Hence, from (3.30) we have $\langle V(x_n),x_n\rangle \to \langle V(x),x\rangle$, so

$$\|x_n'\|_2 \to \|x'\|_2 \text{ as } n\to\infty\,. \tag{3.31}$$

Also we have

$$x_n' \rightharpoonup x' \text{ in } L^2(T,\mathbb{R}^N) \text{ as } n\to\infty\,. \tag{3.32}$$

From (3.31), (3.32) and the Kadec-Klee property of Hilbert spaces, we infer that $x_n' \to x'$ in $L^2(T,\mathbb{R}^N)$. Therefore $x_n \to x$ in $W^{1,2}_{per}((0,b),\mathbb{R}^N)$ and so φ satisfies the nonsmooth PS-condition. □

In what follows let

$$W = H_- \oplus H_0, \text{ and } Z = H_0 \oplus H_+\,.$$

Then $W = H_+^\perp$, $Z = H_-^\perp$ and W is finite dimensional.

Proposition 3.7. *(a) If hypotheses $H(A)$ and $H(j)_2$ (with the $+\infty$ limit in $H(j)_2(iv)$) hold, then $\varphi_{|H_+}$ is coercive.*
(b) If hypotheses $H(A)$ and $H(j)_2$ (with the $-\infty$ limit in $H(j)_2(iv)$) hold, then $\varphi_{|Z}$ is coercive.

Proof. (a) For every $x \in H_+$, using Lemma 3.2 (a), we have

$$\varphi(x) \geq \frac{\xi_+}{2}\|x\|^2 - \int_0^b j(t,x(t))dt\,. \tag{3.33}$$

By virtue of hypothesis $H(j)_2(iii)$ and the mean value theorem for locally Lipschitz functions (see Clarke [3], p.41), we have

$$|j(t,x)| \leq a_0(t)\|x\| + c_0(t)\|x\|^{\theta+1} \text{ for a.a. } t\in T, \text{ all } x\in\mathbb{R}^N\,.$$

Using this in (3.33), we obtain

$$\varphi(x) \geq \frac{\xi_+}{2}\|x\|^2 - c_{20}\|x\| - c_{21}\|x\|^{\theta+1} \text{ for some } c_{20},\ c_{21}>0, \text{ all } x\in H_+\,. \tag{3.34}$$

Since $\theta+1<2$, from (3.34) we infer that $\varphi_{|H_+}$ is coercive.
(b) For every $x\in Z$, we write $x=x^0+\widehat{x}$ with $x^0\in H_0$ and $\widehat{x}\in H_+$. Then exploiting the orthogonality of the component spaces, and using Lemma 3.2 (a), we have

$$\varphi(x)\geq\frac{\xi_+}{2}\|\widehat{x}\|^2-\int_0^b j(t,x(t))dt$$

$$=\frac{\xi_+}{2}\|\widehat{x}\|^2-\int_0^b (j(t,x(t))-j(t,x^0(t)))dt-\int_0^b j(t,x^0(t))dt\,. \tag{3.35}$$

As in the proof of Proposition 3.6, using the mean value theorem for locally Lipschitz functions, we obtain

$$\left|\int_0^b (j(t,x(t))-j(t,x^0(t)))dt\right|\leq c_{20}\|\widehat{x}\|+c_{21}\|x^0\|^\theta\|\widehat{x}\|+c_{22}\|\widehat{x}\|^{\theta+1}$$
$$\text{for some } c_{20},\ c_{21},\ c_{22}>0\,.$$

Using this estimate in (3.35) together with Young's inequality with $\varepsilon>0$ small, we obtain

$$\varphi(x)\geq c_{23}\|\widehat{x}\|^2-c_{24}\|x^0\|^{2\theta}-c_{25}-\int_0^b j(t,x^0(t))dt \text{ for some } c_{23},\ c_{24},\ c_{25}>0\,,$$

so

$$\frac{\varphi(x)}{\|x^0\|^{2\theta}}\geq c_{23}\frac{\|\widehat{x}\|^2}{\|x^0\|^{2\theta}}-c_{24}-\frac{c_{25}}{\|x^0\|^{2\theta}}-\frac{1}{\|x^0\|^{2\theta}}\int_0^b j(t,x^0(t))dt\,. \tag{3.36}$$

If $\|x\|\to\infty$, then $\|\widehat{x}\|\to\infty$ and/or $\|x^0\|\to\infty$. Therefore from (3.36) and hypothesis $H(j)_2(iv)$ (the $-\infty$ option), we have that $\varphi(x)\to+\infty$ as $\|x\|\to\infty$, so $\varphi_{|Z}$ is coercive. □

Proposition 3.8. (a) *If hypotheses $H(A)$ and $H(j)_2$ (with the $+\infty$ limit in $H(j)_2(iv)$) hold, then $\varphi_{|W}$ is anticoercive (i.e. $\varphi(x)\to-\infty$ as $\|x\|\to\infty$, $x\in W$).*
(b) *If hypotheses $H(A)$ and $H(j)_2$ (with the $-\infty$ limit in $H(j)_2(iv)$) hold, then $\varphi_{|H_-}$ is anticoercive (i.e. $\varphi(x)\to-\infty$ as $\|x\|\to\infty$, $x\in H_-$).*

Proof. (a) For $x\in W$, we write $x=\overline{x}+x^0$ with $\overline{x}\in H_-$, $x^0\in H_0$. Then using Lemma 3.2 (b), we have

$$\varphi(x)\leq-\frac{\xi_-}{2}\|\overline{x}\|^2-\int_0^b j(t,x(t))dt$$
$$=-\frac{\xi_-}{2}\|\overline{x}\|^2-\int_0^b (j(t,x(t))-j(t,x^0(t)))dt-\int_0^b j(t,x^0(t))dt\,.$$

As in the proof of Proposition 3.6, we check that

$$\left|\int_0^b (j(t,x(t))-j(t,x^0(t)))dt\right|\leq c_{26}\|\overline{x}\|+c_{27}\|x^0\|^\theta\|\overline{x}\|+c_{28}\|\overline{x}\|^{\theta+1}$$
$$\text{for some } c_{26},\ c_{27},\ c_{28}>0\,.$$

Hence it follows that

$$\varphi(x) \le -c_{29}\|\overline{x}\|^2 + c_{30}\|x^0\|^{2\theta} + c_{31} - \int_0^b j(t, x^0(t))dt \text{ for some } c_{29},\ c_{30},\ c_{31} > 0\,,$$

so

$$\frac{\varphi(x)}{\|x^0\|^{2\theta}} \le -c_{29}\frac{\|\overline{x}\|^2}{\|x^0\|^{2\theta}} + c_{30} + \frac{c_{31}}{\|x^0\|^{2\theta}} - \frac{1}{\|x^0\|^{2\theta}}\int_0^b j(t, x^0(t))dt\,. \tag{3.37}$$

Then as in the proof of Proposition 3.6, from (3.37) we infer that $\varphi_{|W}$ is anticoercive.
(b) Let $x \in H_-$. Then using Lemma 3.2 (b), we have

$$\varphi(x) \le -\frac{\xi_-}{2}\|x\|^2 - \int_0^b j(t, x(t))dt\,. \tag{3.38}$$

Recall that

$$|j(t,x)| \le a_0(t)\|x\| + c_0(t)\|x\|^{\theta+1} \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N\,.$$

Using this growth estimate in (3.38), we have

$$\varphi(x) \le -\frac{\xi_-}{2}\|x\|^2 + c_{32}\|x\| + c_{33}\|x\|^{\theta+1} \text{ for some } c_{32},\ c_{33} > 0\,.$$

Since $\theta + 1 < 2$, from the above inequality we infer that $\varphi_{|H_-}$ is anticoercive. □

Now we are ready for the second existence theorem

Theorem 3.9. *If hypotheses $H(A)$ and $H(j)_2$ hold, then problem (1.1) has a solution $x_0 \in C^1(T, \mathbb{R}^N) \cap W^{2,1}((0,b), \mathbb{R}^N)$.*

Proof. First we assume that in hypothesis $H(j)_2(iv)$ the limit is $+\infty$. By virtue of Proposition 3.7 (a) we can find $\beta_0 \in \mathbb{R}$ such that

$$\varphi_{|H_+} \ge \beta_0\,. \tag{3.39}$$

On the other hand from Proposition 3.8 (a), we see that we can find $r > 0$ large such that

$$\varphi_{|\partial B_r \cap W} < \beta_0 \text{ where } \partial B_r = \{x \in W^{1,2}_{per}((0,b), \mathbb{R}^N) : \|x\| = r\}\,. \tag{3.40}$$

Because of (3.39), (3.40) and Proposition 3.6 we can apply Theorem 2.1 and obtain $x_0 \in W^{1,2}_{per}((0,b), \mathbb{R}^N)$ such that $0 \in \partial\varphi(x_0)$, that is $V(x_0) - \widehat{A}(x_0) = u_0^*$ with $u_0^* \in L^1(T, \mathbb{R}^N)$, $u_0^*(t) \in \partial j(t, x_0(t))$ a.e. on T. From this, as in the proof of Theorem 3.4, we conclude that $x_0 \in C^1(T, \mathbb{R}^N) \cap W^{2,1}((0,b), \mathbb{R}^N)$ and it solves problem (1.1).
Next suppose that in hypothesis $H(j)_2(iv)$ the limit is $-\infty$. In this case Proposition 3.7 (b) implies that we can find $\beta_1 \in \mathbb{R}$ such that

$$\varphi_{|Z} \ge \beta_1\,. \tag{3.41}$$

Also from Proposition 3.8 (b), we see that we can find $\rho > 0$ large such that

$$\varphi_{|\partial B_\rho \cap H_-} < \beta_1\,. \tag{3.42}$$

Note that (3.41), (3.42) and Proposition 3.6, permit the use of Theorem 2.1, which gives $x_0 \in W^{1,2}_{per}((0,b), \mathbb{R}^N)$ such that $0 \in \partial\varphi(x_0)$, that is $V(x_0) - \widehat{A}(x_0) = u_0^*$ with $u_0^* \in L^1(T, \mathbb{R}^N)$ $u_0^*(t) \in \partial j(t, x_0(t))$ a.e. on T. From this as before we conclude that $x_0 \in C^1(T, \mathbb{R}^N) \cap W^{2,1}((0,b), \mathbb{R}^N)$ and it solves problem (1.1). □

4. Multiplicity result

In this section, using Theorem 2.2 we prove a multiplicity result for problem (1.1). The hypotheses on the nonsmooth potential $j(t,x)$ are the following:
$\underline{H(j)_3}$: $j : T \times \mathbb{R}^N \to \mathbb{R}$ is a function such that $j(t,0) = 0$ a.e. on T, hypotheses $H(j)_3(i)$, (ii) and (iv) are the same as hypotheses $H(j)_1(i)$, (ii) and (iv), and

(iii) for almost all $t \in T$, all $x \in \mathbb{R}^N$ and all $u \in \partial j(t,x)$, we have

$$\|u\| \leq a(t) + c(t)\|x\|^r \text{ with } a,\, c \in L^1(T)_+, \text{ and } 2 \leq r < \infty\,;$$

(v) if $\lambda_m > 0$ is the first positive eigenvalue of $x \to -x'' - \widehat{A}(x)$, then there exist $\theta \in L^\infty(T)_+$ and $\delta_0 > 0$ such that

$$\theta(t) \leq \lambda_m \quad \text{a.e. on } T,\ \theta \neq \lambda_m$$

$$\text{and } 0 \leq j(t,x(t)) \leq \frac{1}{2}\theta(t)\|x\|^2 \quad \text{for a.a. } t \in T \text{ and all } \|x\| \leq \delta_0\,.$$

Examples : The following functions satisfies hypotheses $H(j)_3$:

$$j_1(t,x) = \begin{cases} \frac{\theta(t)}{2}\|x\|^2 & \text{if } \|x\| \leq 1 \\ \frac{1}{r}\chi_C(t)(1 - \|x\|^r) + \frac{\theta(t)}{2} & \text{if } \|x\| > 1 \end{cases}$$

and

$$j_2(t,x) = \frac{\theta(t)}{2}\|x\|^2 - c_0\|x\|^r$$

with $\theta \in L^\infty(T)_+$ as in hypothesis $H(j)_3(v)$, $2 < r$, $c_0 > 0$ and $C \subseteq T$ measurable with $|C| > 0$.

We will need the following lemma.

Lemma 4.1. *If $\theta \in L^\infty(T)_+$, $\theta(t) \leq \lambda_m$ a.e. on T and $\theta \neq \lambda_m$, then there exists $\widehat{\xi} > 0$ such that for all $x \in H_+$*

$$\widehat{\psi}(x) = \|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt - \int_0^b \theta(t)\|x(t)\|^2 dt \geq \widehat{\xi}\|x\|^2\,.$$

Proof. Clearly $\widehat{\psi} \geq 0$ on H_+. We argue indirectly. So suppose that the lemma is not true. Exploiting the 2-homogeneity of the functional $\widehat{\psi}$, we can find a sequence $\{x_n\}_{n\geq 1} \subseteq H_+$ with $\|x_n\| = 1$ for all $n \geq 1$, such that $\widehat{\psi}(x_n) \downarrow 0$. By passing to a suitable subsequence if necessary, we may assume that

$$x_n \rightharpoonup x \text{ in } W^{1,2}_{per}((0,b), \mathbb{R}^N) \text{ and } x_n \to x \text{ in } C(T, \mathbb{R}^N)\,.$$

Note that $\widehat{\psi}$ is weakly lower semicontinuous on $W^{1,2}_{per}((0,b), \mathbb{R}^N)$. So we obtain

$$\widehat{\psi}(x) \leq \liminf_{n\to\infty} \widehat{\psi}(x_n) = 0\,,$$

hence

$$\|x'\|_2^2 - \int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt = \int_0^b \theta(t)\|x(t)\|^2 dt \geq \lambda_m\|x\|_2^2\,. \tag{4.1}$$

Since $x \in H_+$ and λ_m is the first positive eigenvalue, from (4.1) we infer that $x \in C^1(T, \mathbb{R}^N)$ is an eigenfunction corresponding to the eigenvalue $\lambda_m > 0$. Hence

$x(t) \neq 0$ for a.a. $t \in T$. Therefore from (4.1) and the hypothesis on θ, we have a contradiction. $\square$

Now we are ready for the multiplicity result. We require that $dim\, H_- = 0$ but the kernel of the differential operator $x \to -x'' - \widehat{A}(x)$ can be nontrivial.

Theorem 4.2. *If hypotheses $H(j)_3$ hold, and $dim\, H_- = 0$, then problem (1.1) admits at least two nontrivial solutions $x_0, \widehat{x} \in C^1(T, \mathbb{R}^N) \cap W^{2,1}((0,b), \mathbb{R}^N)$.*

Proof. We consider the orthogonal direct sum decomposition

$$W^{1,2}_{per}((0,b), \mathbb{R}^N) = H_0 \oplus H_+ .$$

Recall that $H_0 \subseteq C(T, \mathbb{R}^N) \subseteq W^{1,2}_{per}((0,b), \mathbb{R}^N)$ is a finite dimensional subspace, so all norms are equivalent. This means that if we choose $\delta_0 > 0$ as in hypothesis $H(j)_3(v)$, then, if $\|x^0\| \leq \delta_0$, $x^0 \in H_0$, then $\|x^0(t)\| \leq \delta_0$ for all $t \in T$. Then for every such $x^0 \in H_0$, we have

$$0 \leq j(t, x^0(t)) \text{ a.e. on } T . \tag{4.2}$$

Hence, using (4.2) and the fact that $x^0 \in H_0$

$$\varphi(x^0) = -\int_0^b j(t, x^0(t))dt \leq 0 . \tag{4.3}$$

From hypotheses $H(j)_3(iii)$ and (v), we see that

$$j(t, x(t)) \leq \frac{\theta(t)}{2}\|x\|^2 + \widehat{c}(t)\|x\|^r \text{ for a.a. } t \in T . \tag{4.4}$$

Then, using Lemma 4.1 and (4.4), we can find $c_{34} > 0$ such that, for every $x \in H_+$, we have

$$\begin{aligned} \varphi(x) &\geq \frac{1}{2}\|x'\|_2^2 - \frac{1}{2}\int_0^b (A(t)x(t), x(t))_{\mathbb{R}^N}\, dt - \frac{1}{2}\int_0^b \theta(t)\|x(t)\|^2 dt - c_{34}\|x\|^r \\ &\geq \frac{\widehat{\xi}}{2}\|x\|^2 - c_{34}\|x\|^r . \end{aligned} \tag{4.5}$$

Because $r > 2$, we can find $\widehat{\delta} \leq \delta_0$ such that

$$\varphi(x) \geq 0 \text{ for all } x \in H_+ \text{ with } \|x\| \leq \delta_0 . \tag{4.6}$$

Arguing as in the proof of Claim 2 of Theorem 3.4, we can check that the Euler functional φ is coercive. This implies that φ satisfies the nonsmooth PS-condition. Indeed, let $\{x_n\}_{n\geq 1} \in W^{1,2}_{per}((0,b), \mathbb{R}^N)$ be a sequence such that

$$|\varphi(x_n)| \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1 \text{ and } m(x_n) \to 0 \text{ as } n \to \infty .$$

As before, we can find $x_n^* \in \partial\varphi(x_n)$ such that $m(x_n) = \|x_n^*\|$ for all $n \geq 1$. Since φ is coercive and $\{\varphi(x_n)\}_{n\geq 1}$ is bounded, we infer that $\{x_n\}_{n\geq 1} \subseteq W^{1,2}_{per}((0,b), \mathbb{R}^N$ is bounded. From this fact, as in the last part of the proof of Proposition 3.6, we can check that φ satisfies the nonsmooth PS-condition. Also φ is bounded from below. If $inf\, \varphi = 0 = \varphi(0)$, then, by virtue of (4.3), all $x^0 \in H_0$, with $0 < \|x^0\| \leq \delta_0$ are nontrivial minimizers of φ, hence nontrivial solutions of (1.1). Therefore we have a continuum of solutions belonging in $C^1(T, \mathbb{R}^N) \cap W^{2,1}((0,b), \mathbb{R}^N)$.

If $inf\,\varphi < 0$, then by virtue of Theorem 2.2, we know that φ has at least two non-trivial critical points x_0, $\widehat{x} \in W_0^{1,2}(Z)$. As before, we have that x_0, $\widehat{x} \in C^1(T, \mathbb{R}^N) \cap W^{2,1}((0,b), \mathbb{R}^N)$ and solve problem (1.1). □

Remark 4.3. *Theorem 4.2 above improves Theorem 8 of Motreanu-Motreanu- Papageorgiou* [11]*, where the potential function $j(t,x)$ is necessarily quadratic near infinity. The following potential function satisfies hypotheses $H(j)_3$ but not those of Theorem 8 in* [11]*. For simplicity we drop the t-dependence.*

$$j(x) = \begin{cases} \frac{\theta}{2}\|x\|^2 & \text{if } \|x\| \leq 1 \\ -c\ln\|x\| + \frac{\theta}{2} & \text{if } \|x\| > 1 \end{cases}$$

with $\theta < \lambda_m$ and $c > 0$. Note that, if $c = \theta$, then $j \in C^1(\mathbb{R})$.

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Giuseppina Barletta
Dipartimento Patrimonio Architettonico e Urbanistico
Facoltà di Architettura, Università di Reggio Calabria
Salita Melissari, 89124 Reggio Calabria, Italy
E-mail address: giuseppina.barletta@unirc.it

Nikolaos S. Papageorgiou
National Technical University, Departement of Mathematics
Zagrafou Campus, Athens 15780, Greece
E-mail address: npapg@math.ntua.gr