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A NOTE ON THE MULTIPLE-SET SPLIT CONVEX FEASIBILITY PROBLEM IN HILBERT SPACE

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Abstract. We prove weak and strong convergence theorems for an algorithm that solves the multiple-set split convex feasibility problem in Hilbert space.

Let H_1 and H_2 be two real Hilbert spaces, and let r and p be two natural numbers. For each $1 \leq i \leq p$, let C_i be a closed convex subset of H_1, and for each $1 \leq j \leq r$, let Q_j be a closed convex subset of H_2. Further, for each $1 \leq j \leq r$, let $T_j : H_1 \to H_2$ be a bounded linear operator, and let Ω be an additional closed convex subset of H_1.

The (constrained) *multiple-set split convex feasibility problem* (MSCFP) is finding a point $x^* \in \Omega$ such that

$$x^* \in C := \bigcap_{i=1}^{p} C_i \quad \text{and} \quad T_j x^* \in Q_j, \ \ 1 \leq j \leq r. \tag{1}$$

This problem extends the well-known (and by now classical) convex feasibility problem (CFP) and the recent MSCFP proposed and studied in [7]. That paper also contains many relevant references, as well as the real-world application to the inverse problem of intensity-modulated radiation therapy (IMRT) which inspired it. We replace the Euclidean spaces in [7] with Hilbert spaces and the single matrix considered there with r bounded linear operators.

Following [7], we propose to solve this problem by employing the following algorithm.

For each $1 \leq i \leq p$ and $1 \leq j \leq r$, let α_i and β_j be positive numbers, and for each closed convex subset K of a Hilbert space H, let $P_K : H \to K$ denote the nearest point projection of H onto K.

Let $G : H_1 \to H_1$ be the gradient ∇f of the convex and continuously differentiable functional $f : H_1 \to \mathbb{R}$ defined by

$$f(x) := \frac{1}{2} \sum_{i=1}^{p} \alpha_i \left| x - P_{C_i} x \right|^2 + \frac{1}{2} \sum_{j=1}^{r} \beta_j \left| T_j x - P_{Q_j} T_j x \right|^2 \tag{2}$$

for all $x \in H_1$, where $| \cdot |$ denotes the norms of both H_1 and H_2.

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It follows, for example, from [5, page 940] that

$$Gx = \sum_{i=1}^{p} \alpha_i (I - P_{C_i})x + \sum_{j=1}^{r} \beta_j T_j^* (I - P_{Q_j}) T_j x \tag{3}$$

for all $x \in H_1$, where I denotes the identity operator and T_j^* is the adjoint operator of T_j, $j = 1, 2, \dots, r$. Since the operator $I - P_K$ is nonexpansive (see, for example, [9, p. 17]), we see that G is Lipschitz with Lipschitz constant

$$L := \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{r} \beta_j \left\| T_j \right\|^2 . \tag{4}$$

We are now ready to present our iterative algorithm for solving the MSCFP and to prove our main convergence theorem.

Theorem 1. *If the MSCFP has a solution, then, given a point $x_0 \in H_1$ and a number $s \in (0, 2/L)$, the sequence $(x_n)_{n=1}^{\infty} \subset H_1$ defined by*

$$x_{n+1} = P_\Omega [x_n - sGx_n], \; n = 0, 1, 2, \dots, \tag{5}$$

converges weakly to a solution of the MSCFP.

We precede the proof of this theorem with several lemmata.

Lemma 2. *Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $|\cdot|$, and let $\varphi : H \to \mathbb{R}$ be a continuously differentiable and convex functional.*
If G, the gradient of φ, is L-Lipschitz, then

$$\langle Gx - Gy, x - y \rangle \geq \frac{1}{L} \left| Gx - Gy \right|^2 \tag{6}$$

for all points $x, y \in H$.

This lemma is a special case of [2, Corollaire 10].

Let X be a Banach space. A mapping $U : X \to X$ is said to be *averaged* ([4], [1], [19]) if there exist a nonexpansive mapping $S : X \to X$ and a number $c \in (0, 1)$ such that $U = (1 - c)I + cS$.

Lemma 3. *If a mapping $G : H \to H$ satisfies (6) and $0 < s < 2/L$, then the mapping $U = I - sG$ is averaged.*

Proof. Choose a number $c \in (0, 1)$ such that $c \geq Ls/2$, and set $S := I - \frac{s}{c}G$. Then $U = (1 - c)I + cS$ and S is nonexpansive:

$$\begin{aligned} &|x - y|^2 - |Sx - Sy|^2 = \\ &= |x - y|^2 - \Big(|x - y|^2 - 2\frac{s}{c} \langle Gx - Gy, x - y \rangle + \Big(\frac{s}{c}\Big)^2 |Gx - Gy|^2 \Big) \geq \\ &\geq \frac{s}{c} \Big(\frac{2}{L} - \frac{s}{c} \Big) |Gx - Gy|^2 \geq 0. \end{aligned}$$

□

A mapping $S : X \to X$ is called *strongly nonexpansive* [4] if it is nonexpansive and whenever $(x_n - y_n)_{n=1}^{\infty}$ is bounded and $|x_n - y_n| - |Sx_n - Sy_n| \longrightarrow 0$, it follows that $(x_n - y_n) - (Sx_n - Sy_n) \longrightarrow 0$.

We now quote Proposition 2.4 in [18].

Lemma 4. *Let X be a Banach space and let $S : X \to X$ be a strongly nonexpansive mapping. Assume that both X and its dual X^* are uniformly convex.*
If S has a fixed point, then for each x in X, the sequence of iterates $(S^n x)_{n=1}^{\infty}$ converges weakly to a fixed point of S.

We continue by quoting special cases of Proposition 1.3 and Lemma 2.1 in [4], respectively. We denote the fixed point set of S by $F(S)$.

Lemma 5. *In a uniformly convex Banach space, each averaged mapping is strongly nonexpansive.*

Lemma 6. *If S_1 and S_2 are strongly nonexpansive mappings, and $F(S_1) \cap F(S_2) \neq \emptyset$, then $F(S_1) \cap F(S_2) = F(S_2 S_1) = F(S_1 S_2)$.*

Proof of Theorem 1. Consider the mapping $S : H_1 \to H_1$ defined by $S := P_\Omega(I - sG)$.

Since G is L-Lipschitz and $0 < s < 2/L$, Lemma 3, when combined with Lemma 2, shows that the mapping $U = I - sG$ is averaged. Since the nearest point mapping P_Ω is also averaged (see, for example, [9, page 17]), so is their composition S. It is also strongly nonexpansive by Lemma 5.

Every solution of the MSCFP is a null point of G and a fixed point of U, P_Ω and S. Thus, the sequence $(x_n)_{n=1}^{\infty}$ defined by (5) converges weakly to a fixed point of S by Lemma 4. This fixed point x^* of S is also a fixed point of both P_Ω and U by Lemma 6. Hence it is a null point of G which belongs to Ω.

In other words, the point x^* is a minimum point of the functional f and a solution of the MSCFP (because $f(x^*) = 0$), as asserted. □

This theorem contains the finite dimensional Theorem 3 of [7]. However, in the infinite dimensional case, strong convergence cannot be guaranteed. To see this, consider the special case of the convex feasibility problem (CFP), where there are no $Q_1, \dots, Q_r$, $p = 2$, $\alpha_1 = \alpha_2 = 1$, $s = \frac{1}{2}$, and $\Omega = H_1$. Then the iterative scheme (5) takes the form

$$x_{n+1} = \frac{1}{2}(P_{C_1} x_n + P_{C_2} x_n), \tag{7}$$

and it is known [3, Theorem 5.1] that the sequence $(x_n)_{n=1}^{\infty}$ generated by (7), although weakly convergent, need not converge in norm.

We now present an interesting case where strong convergence is assured. Recall first that a mapping $S : X \to X$ is said to be *asymptotically regular* if

$\lim_{n\to\infty}(S^n x - S^{n+1}x) = 0$ for each x in X. Next, we quote Theorem 1.1 of [1]. We say that $S : X \to X$ is *odd* if $S(-x) = -Sx$ for all $x \in X$.

Lemma 7. *If the Banach space X is uniformly convex, and the mapping $S : X \to X$ is nonexpansive, odd and asymptotically regular, then for each $x \in X$, the sequence of iterates $(S^n x)_{n=1}^{\infty}$ converges strongly to a fixed point of S.*

We say that a subset $D \subset X$ is *symmetric* (with respect to the origin) if $-D = D$.

Theorem 8. *If the closed and convex sets $\{C_i \mid 1 \le i \le p\}$, $\{Q_j \mid 1 \le j \le r\}$ and Ω are all symmetric, then the sequence $(x_n)_{n=1}^{\infty}$ generated by (5) converges strongly to a solution of the MSCFP.*

Proof. Since the nearest point projections onto the closed and convex sets $C_1, \dots, C_p$, $Q_1, \dots, Q_r$ and Ω are all odd (cf. [18, Lemma 2.2]), so are G and the composition $S = P_\Omega(I - sG)$. This composition is also strongly nonexpansive and asymptotically regular by [4, Corollary 1.1]. Therefore the result follows from Lemma 7. □

Another instance of strong convergence occurs when the solution set of the MSCFP has a nonempty interior (cf. [14] and [15, Section 6]). In this case the parameter s may even equal $2/L$.

We remark in passing that the sequence $(x_n)_{n=1}^{\infty}$ generated by (5) may converge (albeit weakly) even if the MSCFP has no solution (cf., for example, [13, Corollary 4.10] and [11, Theorem 4.2]). This will happen when the mapping $S = P_\Omega(I - sG)$ has a fixed point, or equivalently, when the functional $f : H_1 \to \mathbb{R}$ defined by (2) attains its minimum over Ω. In this case the limit of $(x_n)_{n=1}^{\infty}$ may be considered a generalized solution of the MSCFP. More information regarding the MSCFP and its solutions may be found in the recent paper [8] and the references mentioned therein. When the functional f does not attain its infimum over Ω, then the mapping S is fixed point free and $|x_n| \to \infty$ as $n \to \infty$. This follows from either [1, Corollary 2.2] or [4, Corollary 1.4].

It may be of interest to note that when the sequence $(x_n)_{n=1}^{\infty}$ generated by our algorithm does converge (either weakly or strongly), it will continue to converge even in the presence of summable computational errors. This follows from Theorem 4.1 and 4.2 in [6] (see also [4, Theorem 2.5] and [15, Theorem 2]).

Finally, we also observe that other algorithms for solving the MSCFP can be based on [16, Theorem 2], [10, Theorem 3], [12, Théorème 1], [17, Corollary 2], and on their more recent counterparts (see, for example, the papers [21] and [20], as well as their references).

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