Journal of Nonlinear and Convex Analysis
Volume 8, Number 3, 2007, 357–366

Yokohama Publishers
ISSN 1880-5221 ONLINE JOURNAL

ON REGULARITY OF CONSTRAINT SYSTEMS WITH CONTINUOUS OPERATORS

YOSHIYUKI SEKIGUCHI AND WATARU TAKAHASHI

Abstract. In the work, we consider a constraint system of a finite dimensional optimization problem defined with an operator and closed sets. The operator is assumed to be merely continuous. Under a constraint qualification, we give a straightforward proof for a representation of normal vectors to the feasible region. Then metric regularity of the constraint system are obtained by the formula. Moreover the rate of metric regularity is estimated.

1. Introduction

We consider constraint systems of minimization problems in Euclidean spaces. If a constraint system is defined by smooth functions, constraint qualifications at a point, like linear independence of their gradients, are often imposed to ensure existence of proper Lagrange multipliers. This can be proved through a representation of normal vectors to the feasible region. Constraint qualifications actually guarantee validity of the representation and in addition, some stability of feasible sets under perturbations.

Let us consider the following parameterized constraint system:

$$C_y = \{x \in A : F(x) \in y + B\},$$

where X, Y are Euclidean spaces, $y \in Y$, F is a smooth operator from X into Y and A, B are closed sets in X, Y respectively. Then if the Jacobian of F at $\bar{x} \in C_0$ is surjective, there exists $K \geq 0$ such that

$$d(x, C_y) \leq K d(F(x), y + B)$$

for $(x, y) \in A \times Y$ close to $(\bar{x}, 0)$; see e.g. [4], [11].

This inequality says that C_y is nonempty for all y close to 0 and that the distance from a given point x to C_y is bounded by the residual $d(F(x), y+B)$, which is often treated more easily. The former guarantees the consistency of constraint systems under data perturbations and the latter gives, in particular, an error bound of the distance from a feasible point $\bar{x}$ to feasible regions of perturbed constraint systems. Such regularity is called metric regularity. To have metric regularity of constraint systems, surjectivity of the Jacobian can be replaced by other constraint qualifications. In addition, smoothness of the constraint operator is not required to obtain above regularity and a representation of normal vectors to the feasible region.

2000 *Mathematics Subject Classification.* Primary 90C30, 49J53.
Key words and phrases. Mathematical programming, metric regularity, normal cones, optimality conditions.

In this paper, we argue normal vectors to a constraint system with a continuous operator and closed sets, and its regularity. Tangent vectors are discussed in [12]. The treatise that an operator and sets are used to define constraint systems gives a unifying framework concerning multiplier rules for various constraints. Several authors have worked on the subject. For example, smooth and finite dimensional cases are given in [11]. Constraints with Lipschitz continuous operators are discussed in [8] for finite dimensional cases and in [7] for infinite dimensional cases.

We consider constraint systems in Euclidean spaces, where constraint operators are merely continuous. Our main tool is a sum rule for subdifferentials. A representation of normal vectors to such a region is essentially obtained in [10] . However it is not explicitly given and the proof is complicated since they show it in Asplund spaces with several additional conditions. Thus we give a straightforward proof for the formula in Euclidean spaces, which is also in an easy accessible format.

In addition, a constraint qualification with the coderivative of the operator is shown to be sufficient for regularity of such constraint system. Moreover the rate of regularity, which is a quantitative measure of regularity is estimated through the formula. With these results, we present a necessary optimality condition for a minimization problem.

2. Preliminaries

Throughout the paper, let X and Y be Euclidean spaces. Our notation and constructions follow [11]. Let C be a closed subset of X and $\bar{x} \in C$. A vector $x^* \in X$ is said to be *normal* to C at $\bar{x}$ *in the regular sense* if it satisfies

$$\langle x^*, x - \bar{x} \rangle \leq o(\|x - \bar{x}\|)$$

for $x \in C$. It can also be written by

$$\limsup_{\substack{x \xrightarrow{C} \bar{x} \\ x \neq \bar{x}}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0.$$

Such vectors form a closed convex cone. The set is called the *regular normal cone* to C at $\bar{x}$ and denoted by $\widehat{N}_C(\bar{x})$. If C is convex, an easy calculation gives $\widehat{N}_C(\bar{x}) = \{x^* \in X : \langle x^*, x - \bar{x} \rangle \leq 0,\ x \in C\}$, which is the standard normal cone of convex analysis. For calculus of regular normal vectors, see e.g. [11], [12]. A vector x^* is said to be *normal* to C at $\bar{x}$ (*in the general sense*) if there exists $x_k \xrightarrow{C} \bar{x}$ and $x_k^* \to x^*$ with $x_k^* \in \widehat{N}_C(x_k)$. The whole set is called the *normal cone* to C at $\bar{x}$ and denoted by $N_C(\bar{x})$. If $\widehat{N}_C(\bar{x}) = N_C(\bar{x})$, we say C is *Clarke regular* at $\bar{x}$. When C is convex, it is Clarke regular at every its point.

For a lower semicontinuous function f from X into $(-\infty, \infty]$ and $\bar{x} \in \mathrm{dom} f$, three types of subdifferentials are defined as follows:

The *Fréchet subdifferential* $\widehat{\partial} f(\bar{x})$ of f at $\bar{x}$;

$$\widehat{\partial} f(\bar{x}) := \{x^* \in X : (x^*, -1) \in \widehat{N}_{\mathrm{epi} f}((\bar{x}, f(\bar{x})))\},$$

the *subdifferential*;

$$\partial f(\bar{x}) := \{x^* \in X : (x^*, -1) \in N_{\mathrm{epi} f}((\bar{x}, f(\bar{x})))\},$$

the *singular subdifferential*;

$$\partial^\infty f(\bar{x}) := \{x^* \in X : (x^*, 0) \in N_{\mathrm{epi} f}((\bar{x}, f(\bar{x})))\}.$$

If f is convex, the Fréchet subdifferential and the subdifferential are equal to the subdifferential of convex analysis;

$$\partial f(\bar{x}) = \widehat{\partial} f(\bar{x}) = \{x^* \in X : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle,\ x \in X\}.$$

We have the following formulas.

Proposition 2.1 ([10]). *Let f be a lower semicontinuous function from X into $(-\infty, \infty]$ and $\bar{x} \in \mathrm{dom}\, f$. Then*

$$\widehat{\partial} f(\bar{x}) = \left\{ x^* \in X : \liminf_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \right\};$$

$$\partial f(\bar{x}) = \limsup_{x \xrightarrow{f} \bar{x}} \widehat{\partial} f(x) = \limsup_{\substack{x \xrightarrow{f} \bar{x} \\ \epsilon \searrow 0}} \widehat{\partial}_\epsilon f(x),$$

where

$$\widehat{\partial}_\epsilon f(\tilde{x}) = \left\{ x^* \in X : \liminf_{\substack{x \to \tilde{x} \\ x \neq \tilde{x}}} \frac{f(x) - f(\tilde{x}) - \langle x^*, x - \tilde{x}\rangle}{\|x - \tilde{x}\|} \geq -\epsilon \right\}$$

for $\tilde{x} \in \mathrm{dom}\, f$ and $\epsilon > 0$.

Note that for the *indicator function*

$$I_C(x) = \begin{cases} 0, & x \in C; \\ \infty, & x \notin C, \end{cases}$$

we have the relation that $\widehat{\partial} I_C(\bar{x}) = \widehat{N}_C(\bar{x})$ and $\partial I_C(\bar{x}) = \partial^\infty I_C(\bar{x}) = N_C(\bar{x})$ for all $\bar{x} \in C$.

Let S be a set-valued mapping from X into Y and $\mathrm{gph}\, S = \{(x, y) \in X \times Y : y \in S(x)\}$. For $(\bar{x}, \bar{y}) \in \mathrm{gph}\, S$, the *coderivative* of S at $\bar{x}$ for $\bar{y}$ is the set-valued mapping $D^*S(\bar{x}, \bar{y})$ from X into Y defined by

$$D^*S(\bar{x}, \bar{y})(y^*) = \{x^* \in X : (x^*, -y^*) \in N_{\mathrm{gph} S}(\bar{x}, \bar{y})\}$$

for $y^* \in Y$. If S is single-valued, we write $D^*S(\bar{x}, S(\bar{x})) = D^*S(\bar{x})$. We present some useful calculus rules. For the proofs, see [11] and the references therein.

Proposition 2.2 (normals to product sets). *Let $C = C_1 \times C_2$ for closed sets $C_1 \in X$ and $C_2 \in Y$. Then for $\bar{z} = (\bar{x}, \bar{y})$ with $\bar{x} \in C_1$ and $\bar{y} \in C_2$,*

$$\widehat{N}_C(\bar{z}) = \widehat{N}_{C_1}(\bar{x}) \times \widehat{N}_{C_2}(\bar{y});$$

$$N_C(\bar{z}) = N_{C_1}(\bar{x}) \times N_{C_2}(\bar{y}).$$

Theorem 2.3 (addition of functions)**.** *Let f_i be proper, lower semicontinuous functions from X into $(-\infty, \infty]$ and $\bar{x} \in \mathrm{dom} f_1 \cap \cdots \cap \mathrm{dom} f_n$. Suppose for $x_i^* \in \partial^\infty f_i(\bar{x})$,*

$$x_1^* + \cdots + x_n^* = 0 \Rightarrow x_1^* = \cdots = x_n^* = 0.$$

Then the following inclusions hold:

$$\partial(f_1 + \cdots + f_n)(\bar{x}) \subset \partial f_1(\bar{x}) + \cdots + \partial f_n(\bar{x}),$$
$$\partial^\infty(f_1 + \cdots + f_n)(\bar{x}) \subset \partial^\infty f_1(\bar{x}) + \cdots + \partial^\infty f_n(\bar{x}).$$

Moreover if each epi f_i *is Clarke regular, equalities hold.*

3. Normal cones to continuous constraints

Let F be an operator from X into Y and A, B closed sets. We consider the following minimization problem:

$$\mathcal{P} \qquad \text{minimize } f(x) \text{ subject to } x \in C,$$

where

$$C = \{x \in A : F(x) \in B\}. \tag{3.1}$$

We show a formula for a normal cone to C. If F is locally Lipschitz, it is found in [8]. The formula for continuous operators is, in fact, essentially obtained in [10]. However since it is not explicitly given and the proof is complicated, we reformulate it in an easy accessible format and give a straightforward proof.

Theorem 3.1. *Let F be a continuous operator from X into Y and C be defined in* (3.1) *for closed sets A, B. Suppose the following constraint qualification is satisfied at $\bar{x}$:*

$$(*) \quad \begin{cases} \text{for } z^* \in N_A(\bar{x}), y^* \in N_B(F(\bar{x})), \\ 0 \in D^*F(\bar{x})(y^*) + z^* \;\; \Rightarrow \;\; y^* = 0, z^* = 0. \end{cases}$$

Then

$$N_C(\bar{x}) \subset \{D^*F(\bar{x})(y^*) + z^* : z^* \in N_A(\bar{x}), y^* \in N_B(F(\bar{x}))\}.$$

Moreover if A, B are Clarke regular at $\bar{x}$, $F(\bar{x})$ respectively and F is continuously Fréchet differentiable at $\bar{x}$, equality holds

Proof. We define $\|(x, y)\| = \|x\| + \|y\|$ for $(x, y) \in X \times Y$. For all $x \in X$ and $y \in Y$, we have

$$I_C(x) = I_A(x) + I_B(F(x)) \leq I_A(x) + I_B(y) + I_{\mathrm{gph}F}((x, y)).$$

Let $f(x, y) = I_A(x) + I_B(y) + I_{\mathrm{gph}F}((x, y))$. Then $I_C(x) \leq f(x, y)$ and $I_C(x) = f(x, F(x))$.

We will show the following inclusion:

$$\partial I_C(\bar{x}) \subset \{x^* \in X : (x^*, 0) \in \partial f(\bar{x}, F(\bar{x}))\}. \tag{3.2}$$

Suppose $x^* \in \partial I_C(\bar{x})$. Then there exist $x_k \xrightarrow{I_C} \bar{x}$ and $x_k^* \to x^*$ with $x_k^* \in \widehat{\partial} I_C(x_k)$ by the definition. Since $I_C(\bar{x}) = 0$, we have $I_C(x_k) = 0$ for large k and hence we may assume $x_k \in C$ for all k. It follows from Proposition 2.1 that for arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that

$$I_C(x) \geq I_C(x_k) + \langle x_k^*, x - x_k \rangle - \epsilon \|x - x_k\|,$$

whenever $\|x - x_k\| \le \delta$. This implies

$$f(x,y) \ge f(x_k, F(x_k)) + \langle x_k^*, x - x_k\rangle - \epsilon\|x - x_k\| - \epsilon\|y - F(x_k)\|$$

for $y \in Y$. Since ϵ is arbitrary, we obtain $(x_k^*, 0) \in \widehat{\partial} f(x_k, F(x_k))$. Now we have $f(x_k, F(x_k)) = I_C(x_k) = 0 = f(\bar{x}, F(\bar{x}))$ for all k. Thus continuity of F gives that $(x_k, F(x_k)) \xrightarrow{f} (\bar{x}, F(\bar{x}))$. Therefore $(x^*, 0) \in \partial f(\bar{x}, F(\bar{x}))$ and hence (3.2) holds.

Next we describe the set $\partial f(\bar{x}, F(\bar{x}))$. We can write

$$f(x,y) = I_{A\times Y}((x,y)) + I_{X\times B}((x,y)) + I_{\mathrm{gph}F}((x,y))$$

for all $(x,y) \in X \times Y$. To apply Theorem 2.3, we suppose that $(x_1^*, y_1^*) \in \partial I_{A\times Y}((\bar{x}, F(\bar{x})))$, $(x_2^*, y_2^*) \in \partial I_{X\times B}((\bar{x}, F(\bar{x})))$, $(x_3^*, y_3^*) \in \partial I_{\mathrm{gph}F}((\bar{x}, F(\bar{x})))$ and that $(x_1^*, y_1^*) + (x_2^*, y_2^*) + (x_3^*, y_3^*) = (0,0)$. Here we have

$$\partial I_{A\times Y}((\bar{x}, F(\bar{x}))) = N_{A\times Y}((\bar{x}, F(\bar{x}))) = N_A(\bar{x}) \times \{0\},$$
$$\partial I_{X\times B}((\bar{x}, F(\bar{x}))) = N_{X\times B}((\bar{x}, F(\bar{x}))) = \{0\} \times N_B(F(\bar{x})),$$

by Proposition 2.2. Thus $y_1^* = 0$, $x_2^* = 0$ and

$$(x_1^* + x_3^*, y_2^* + y_3^*) = (x_1^*, 0) + (0, y_2^*) + (x_3^*, y_3^*) = (0,0).$$

Since $x_3^* \in D^*F(\bar{x})(-y_3^*)$, one has

$$(0,0) \in \{x_1^* + D^*F(\bar{x})(-y_3^*)\} \times \{y_2^* + y_3^*\},$$

and then $0 \in x_1^* + D^*F(\bar{x})(y_2^*)$. Since $x_1^* \in N_A(\bar{x})$ and $y_2^* \in N_B(F(\bar{x}))$, the constraint qualification guarantees that $x_1^* = 0$, $y_2^* = 0$ and hence we have $x_3^* = 0$, $y_3^* = 0$. Therefore we can apply Theorem 2.3 and obtain

$$\partial f(\bar{x}, F(\bar{x})) \subset N_A(\bar{x}) \times \{0\} + \{0\} \times N_B(F(\bar{x})) + N_{\mathrm{gph}F}((\bar{x}, F(\bar{x}))). \tag{3.3}$$

For any $x^* \in N_C(\bar{x})$, we have $(x^*, 0) \in \partial f(\bar{x}, F(\bar{x}))$ by the inclusion (3.2). Thus there exist $x_1^* \in N_A(\bar{x})$, $y_2^* \in N_B(F(\bar{x}))$ and $(x_3^*, y_3^*) \in N_{\mathrm{gph}F}((\bar{x}, F(\bar{x})))$ such that

$$(x^*, 0) = (x_1^*, 0) + (0, y_2^*) + (x_3^*, y_3^*) = (x_1^* + x_3^*, y_2^* + y_3^*).$$

Then $x^* = x_1^* + x_3^*$ and $y_2^* + y_3^* = 0$. Since $x_3^* \in D^*F(\bar{x})(-y_3^*)$, we conclude $x^* \in x_1^* + D^*F(\bar{x})(y_2^*)$. This shows the first part of the statement.

Next we suppose that the operator F is strictly differentiable at $\bar{x}$ and A, B are Clarke regular at $\bar{x}$, $F(\bar{x})$ respectively. If we show that the inclusions in (3.2), equalities in (3.3) hold. Then the proof is completed.

Suppose $(x^*, 0) \in \partial f(\bar{x}, F(\bar{x}))$. Then there exists $(x_k, y_k) \xrightarrow{f} (\bar{x}, \bar{y})$ and $(x_k^*, y_k^*) \to (x^*, 0)$ with $(x_k^*, y_k^*) \in \widehat{\partial} f(x_k, y_k)$. The similar argument above tells us that $x_k \in C$ and $(x_k, y_k) \in \mathrm{gph}\, F$. For $\eta_k \searrow 0$, we can find $\delta_0 > 0$ such that

$$f(x,y) \ge f(x_k, y_k) + \langle (x_k^*, y_k^*), (x - x_k, y - y_k)\rangle - \eta_k\{\|x - x_k\| + \|y - y_k\|\},$$

whenever $\|x - x_k\| + \|y - y_k\| < \delta_0$. Since F is continuous, there exists $\delta > 0$ such that $\|x - x_k\| \le \delta$ implies $\|x - x_k\| + \|F(x) - F(x_k)\| < \delta_0$. Thus if $\|x - x_k\| < \delta$, we have

$$\begin{aligned} f(x, F(x)) &\ge f(x_k, F(x_k)) + \langle (x_k^*, y_k^*), (x - x_k, F(x) - F(x_k))\rangle \\ &\qquad\qquad - \eta_k\{\|x - x_k\| + \|F(x) - F(x_k)\|\}, \\ &\ge f(x_k, F(x_k)) + \langle x_k^*, x - x_k\rangle - \|y_k^*\|\|F(x) - F(x_k)\| \end{aligned}$$

$$- \eta_k\{\|x - x_k\| + \|F(x) - F(x_k)\|\}.$$

Now $f(x, F(x)) = I_C(x)$ and strict differentiability at $\bar{x}$ implies that $\|F(x) - F(x_k)\| \le \|\nabla F(\bar{x})\|\|x - x_k\| + \|x - x_k\|$ for x, x_k close to $\bar{x}$. Thus we have for large k,

$$\liminf_{x \to x_k} \frac{I_C(x) - I_C(x_k) - \langle x_k^*, x - x_k \rangle}{\|x - x_k\|} \ge -\epsilon_k,$$

where $\epsilon_k = \max\{\eta_k, \|y_k^*\|\}(\|\nabla F(\bar{x})\| + 2)$. This implies that $x_k^* \in \widehat{\partial}_{\epsilon_k} I_C(x_k)$ and we obtain $x^* \in \partial I_C(\bar{x})$ by Proposition 2.1. Since $\operatorname{gph} F$ is Clarke regular at every its point [11], the regularity of A, B and $\operatorname{gph} F$ ensures the equality of (3.3). □

4. Constraint qualification and Metric regularity

Metric regularity is one of the central concepts in theoretical studies of optimization problems. A set-valued mapping S from X into Y is said to be *metrically regular* around $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ if there exists $K \ge 0$ such that

$$d(x, S^{-1}(y)) \le K d(y, S(x))$$

for (x, y) close to $(\bar{x}, \bar{y})$. The infimum of such K is called the *rate of (metric) regularity* at $(\bar{x}, \bar{y})$ and denoted by $\operatorname{reg} S(\bar{x}, \bar{y})$. In particular, we consider the following set-valued mapping from X into Y:

$$\Omega(x) = \begin{cases} F(x) - B, & x \in A; \\ \emptyset, & x \notin A. \end{cases} \tag{4.1}$$

Metric regularity of Ω around $(\bar{x}, \bar{y}) \in \operatorname{gph} \Omega$ implies the inequality

$$d(x, C_y) \le K d(F(x), B)$$

for $C_y = \{x \in A : F(x) \in y + B\}$ and $(x, y) \in A \times Y$ close to $(\bar{x}, \bar{y})$. This coincides the regularity mentioned in the introduction and expresses consistency and stability of constraint systems with perturbations as explained there. In addition metric regularity of Ω gives a representation of tangent vectors to the feasible region [12].

The rate of regularity is a quantitative measure of stability and related to a condition number in linear programming; see [1] and references therein. We use the normal formula obtained in the previous section and a coderivative criterion to show that the constraint qualification is sufficient for metric regularity of the mapping Ω. Moreover we obtain an upper bound for the rate of regularity. The estimate below has been studied in [2], [9] and a short proof is given in [5].

Theorem 4.1 ([9]). *Let S be a set-valued mapping from X into Y with a closed graph and $(\bar{x}, \bar{y}) \in \operatorname{gph} S$. Then*

$$\operatorname{reg} S(\bar{x}|\bar{y}) = [\inf\{\|x^*\| : x^* \in D^*S(\bar{x}, \bar{y})(y^*), \|y^*\| = 1\}]^{-1}.$$

and the following conditions are equivalent:

(i) *S is metrically regular around $(\bar{x}, \bar{y})$;*
(ii) $\ker D^*S(\bar{x}|\bar{y}) = \{0\}$.

Lemma 4.2. *Let F be a continuous operator from X into Y. Set $F_0(x,u) = F(x) - u$. Then*

$$D^*F_0(\bar{x},\bar{u})(y^*) = D^*F(\bar{x})(y^*) \times \{-y^*\}$$

for all $y^* \in Y$.

Proof. Let $\|(x,y)\| = \|x\| + \|y\|$ for $(x,y) \in X \times Y \times Y$. It is sufficient to show that $(x^*,u^*,-y^*) \in \widehat{N}_{\mathrm{gph}\, F_0}(x_0,u_0,F(x_0)-u_0)$ if and only if $u^* = -y^*$ and $(x^*,-y^*) \in \widehat{N}_{\mathrm{gph}\, F}(x_0,F(x_0))$, for $(x_0,u_0,F(x_0)-u_0)$ close to $(\bar{x},\bar{u},F(\bar{x})-\bar{u})$.

Let $y_0 = F(x_0) - u_0$. Suppose $(x^*,u^*) \in \widehat{D}^*F_0(x_0,u_0)(y^*)$. The definition says that for arbitrary $\epsilon > 0$, there exists $\delta > 0$ such that

$$\langle x^*, x - x_0\rangle + \langle u^*, u - u_0\rangle + \langle -y^*, y - y_0\rangle \le \epsilon\{\|x - x_0\| + \|u - u_0\| + \|y - y_0\|\},$$

whenever $\|x-x_0\|+\|u-u_0\|+\|y-y_0\| < \delta$ and $(x,u,y) \in \mathrm{gph} F_0$. Since $y = F(x)-u$, the above inequality means

$$\begin{aligned}(4.2)\quad \langle x^*, x - x_0\rangle + \langle u^*, u - u_0\rangle &+ \langle -y^*, F(x) - u - (F(x_0) - u_0)\rangle \\ &\le \epsilon\{\|x - x_0\| + \|u - u_0\| + \|F(x) - u - (F(x_0) - u_0)\|\}.\end{aligned}$$

Setting $x = x_0$, we have $\langle u^*, u - u_0\rangle + \langle -y^*, -u + u_0\rangle \le \epsilon\{\|u - u_0\| + \|u - u_0\|\}$ and hence $\langle u^* + y^*, u - u_0\rangle \le 2\epsilon\|u - u_0\|$. Since this holds for any u with $\|u - u_0\| < \delta/2$ and ϵ is arbitrary, we obtain $u^* = -y^*$.

Now set $u = u_0$ in (4.2). We have

$$\langle x^*, x - x_0\rangle + \langle -y^*, F(x) - F(x_0)\rangle \le \epsilon\{\|x - x_0\| + \|F(x) - F(x_0)\|\}.$$

Continuity of F guarantees the inequality

$$\limsup_{(x,z)\overset{\mathrm{gph}F}{\to}(x_0,F(x_0))} \frac{\langle (x^*,-y^*),(x,z) - (x_0,F(x_0))\rangle}{\|x - x_0\| + \|z - F(x_0)\|} \le \epsilon.$$

Letting ϵ to 0, this implies $(x^*,-y^*) \in \widehat{N}_{\mathrm{gph}F}(x_0,F(x_0))$. Thus $x^* \in \widehat{D}^*F(x_0)(y^*)$. Therefore we obtain $\widehat{D}^*F_0(x_0,u_0)(y^*) \subset \widehat{D}^*F(x_0)(y^*) \times \{-y^*\}$.

Next we show the reverse inclusion. Suppose $x^* \in \widehat{D}^*F(x_0)(y^*)$. Then for any $\epsilon > 0$, there exits $\delta > 0$ such that

$$\langle x^*, x - x_0\rangle + \langle -y^*, y - F(x_0)\rangle \le \epsilon\{\|x - x_0\| + \|y - F(x_0)\|\},$$

whenever $\|x - x_0\| + \|y - F(x_0)\| < \delta$ and $(x,y) \in \mathrm{gph}\, F$. Let us define $u_0 = F(x_0) - y_0$ and take any point $(x,u,y) \in \mathrm{gph}\, F_0$ which satisfies $\|x - x_0\| + \|u - u_0\| + \|y - y_0\| < \delta$. Then we have $(x, y + u) \in \mathrm{gph}\, F$ and

$$\begin{aligned}\|x - x_0\| + \|y + u - F(x_0)\| &= \|x - x_0\| + \|(y + u) - (y_0 + u_0)\| \\ &\le \|x - x_0\| + \|y - y_0\| + \|u - u_0\| < \delta.\end{aligned}$$

Thus we have

$$\langle x^*, x - x_0\rangle + \langle -y^*, (y + u) - F(x_0)\rangle \le \epsilon\{\|x - x_0\| + \|(y + u) - F(x_0)\|\},$$

and hence

$$\langle (x^*,-y^*,-y^*),(x,u,y) - (x_0,u_0,y_0)\rangle \le \epsilon\{\|x - x_0\| + \|u - u_0\| + \|y + y_0\|\}.$$

This means $(x^*,-y^*) \in \widehat{D}^*F_0(x_0,u_0)(y^*)$ and the proof is completed. □

Theorem 4.3. *Let F be a continuous operator and C defined in* (3.1) *for closed sets A, B. Suppose the constraint qualification* $(*)$ *holds at $\bar{x}$. Then Ω defined in* (4.1) *is metrically regular around $(\bar{x}, \bar{y})$ and*

$$\operatorname{reg}\Omega(\bar{x}|0) \leq \big[\inf\{\|x^* + z^*\| : x^* \in D^*F(\bar{x})(y^*),\\ y^* \in N_B(F(\bar{x})), z^* \in N_A(\bar{x}), \|y^*\| = 1\}\big]^{-1}.$$

Moreover if A, B are Clarke regular at $\bar{x}$, $F(\bar{x})$ respectively, equality holds.

Proof. Let $F_0(x,u) = F(x) - u$ and $D = \{(x,u) \in X \times Y : x \in A, F_0(x,u) \in B\}$. Then $\operatorname{gph}\Omega = D$. Suppose $z^* \in N_A(\bar{x})$, $y^* \in N_B(F_0(\bar{x},0))$ and $0 \in D^*F_0(\bar{x},0)(y^*) + (z^*, 0)$. Then by Lemma 4.2, $0 \in D^*F(\bar{x})(y^*) \times \{-y^*\} + (z^*, 0)$. Since $z^* \in N_A(\bar{x})$, $y^* \in N_B(F(\bar{x}))$ and $0 \in D^*F(\bar{x})(y^*) + z^*$, we have $y^* = 0$ and $z^* = 0$ by the constraint qualification. Applying Theorem 3.1 and Lemma 4.2, we obtain

$$\begin{aligned} N_D(\bar{x},0) &\subset \{D^*F_0(\bar{x},0)(y^*) + (z^*,0) : z^* \in N_A(\bar{x}), y^* \in N_B(F_0(\bar{x},0))\} \\ &= \{\{D^*F(\bar{x})(y^*) + z^*\} \times \{-y^*\} : z^* \in N_A(\bar{x}), y^* \in N_B(F(\bar{x}))\}. \end{aligned}$$

Thus the relation $w^* \in D^*\Omega(\bar{x},0)(y^*)$ implies that $y^* \in N_B(F(\bar{x}))$ and that there exist $x^* \in D^*F(\bar{x})(y^*)$ and $z^* \in N_A(\bar{x})$ such that $w^* = x^* + z^*$. We are to apply Theorem 4.1. If we take $w^* = 0$ in the relation, we have $0 \in D^*F(\bar{x})(y^*) + z^*$ and hence $y^* = 0$, $z^* = 0$ by the constraint qualification. Therefore metric regularity of Ω around $(\bar{x}, 0)$ and the desired inequality are obtained. □

Finally we obtain a necessary condition for local optimals of optimization problems. We need the following lemma, which is shown by the argument in [3]. For $\bar{x} \in X$ and a locally Lipschitz function f on X, let $B_r(\bar{x})$ be the open ball with the center $\bar{x}$ and the radius r, and

$$\operatorname{Lip} f(\bar{x}) = \limsup_{\substack{x,x' \to \bar{x} \\ x \neq x'}} \frac{|f(x) - f(x')|}{\|x - x'\|}.$$

Lemma 4.4. *Let f be a locally Lipschitz function on X, S a set-valued mapping from X into Y. Consider the problem*

$$\textit{minimize } f(x) \textit{ s.t. } 0 \in S(x).$$

Suppose $\bar{x}$ is a local optimal of the problem and S is metrically regular around $(\bar{x}, 0)$. Then $\bar{x}$ is a local optimal for an unconstrained problem

$$\textit{minimize } f(x) + LKd(0, S(x)),$$

whenever $L > \operatorname{Lip} f(\bar{x})$ and $K > \operatorname{reg} S(\bar{x}|0)$.

Proof. Let $L > \operatorname{Lip} f(\bar{x})$ and $K > \operatorname{reg} S(\bar{x}|0)$. Since $\bar{x}$ is a local optimal, there exists $r > 0$ such that $f(\bar{x}) \leq f(x)$ for all $x \in B_r(\bar{x}) \cap S^{-1}(0)$, f is L-Lipschitz on $B_r(\bar{x})$ and $d(x, S^{-1}(0)) \leq Kd(0, S(x))$ for $x \in B_r(\bar{x})$. Let $x \in B_{r/3}(\bar{x})$ and $x' \in S^{-1}(0)$. By Lipschitz continuity of f, we have $f(\bar{x}) - f(x) \leq L\|\bar{x} - x\| \leq Lr/3$. If $x' \in B_r(\bar{x})$,

$$f(\bar{x}) - f(x) \leq f(x') - f(x) \leq L\|x' - x\|.$$

If $x' \notin B_r(\bar{x})$,

$$\|x' - x\| \geq \|x' - \bar{x}\| - \|\bar{x} - x\| \geq r - \frac{r}{3} = \frac{2}{3}r,$$

and hence $f(\bar{x}) - f(x) \leq Lr/3 \leq L\|x' - x\|$. Thus regularity of F gives

$$f(\bar{x}) - f(x) \leq Ld(x, S^{-1}(0)) \leq LKd(0, S(x)).$$

Therefore we obtain $f(\bar{x}) \leq f(x) + LKd(0, S(x))$ for all $x \in B_{r/3}(\bar{x})$. □

Corollary 4.5. *Let f be a locally Lipschitz function, F a continuous operator from X into Y and A, B are closed sets in X, Y respectively. Suppose $\bar{x}$ is a local optimal of the minimization problem*

$$\textit{minimize } f(x) \textit{ subject to } x \in A,\ F(x) \in B$$

and the constraint qualification $(*)$ holds at $\bar{x}$. Then there exist $y^* \in N_B(F(\bar{x}))$ and $z^* \in N_A(\bar{x})$ such that

$$0 \in \partial f(\bar{x}) + D^*F(\bar{x})(y^*) + z^*$$

and $\bar{x}$ is a local optimal to the problem

$$\textit{minimize } f(x) + LKd(F(x), B) \textit{ subject to } x \in A$$

for $L > \operatorname{Lip} f(\bar{x})$ and $K > \operatorname{reg} \Omega(\bar{x}|0)$, where Ω is defined by (4.1).

Proof. Let $f_0(x) = f(x) + I_C(x)$. It follows from Proposition 2.1 that $0 \in \partial f_0(0)$. A lower semicontinuous function f from X into R is locally Lipschitz at x if and only if $\partial^\infty f(x) = \{0\}$ [11]. Thus by Theorem 2.3, we have $\partial f_0(\bar{x}) \subset \partial f(\bar{x}) + N_C(\bar{x})$. Since the constraint qualification is satisfied, we obtain the desired inclusion and metric regularity of Ω around $(\bar{x}, 0)$ from Theorem 3.1 and Theorem 4.3 respectively. In addition the remain of the statement is followed by Lemma 4.4 and the equality $d(0, \Omega(x)) = Kd(F(x), B)$ for $x \in A$.

□

References

[1] A. L. Dontchev, A. S. Lewis and R. T. Rockafellar, *The radius of metric regularity*, Trans. Amer. Math. Soc. **355** (2002), 493-517.

[2] A. D. Ioffe, *On the local surjection property*, Nonlinear Anal. **11** (1987), 565-592.

[3] A. D. Ioffe, *Variational methods in local and global non-smooth analysis*, Nonlinear analysis, differential equations and control, NATO Sci. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, **528** (1999), 447-502.

[4] A. D. Ioffe, Metric regularity and subdifferential calculus, Russian Math. Surveys **55** (2000), 501-558.

[5] A. D. Ioffe and Y. Sekiguchi, *Exact formulae for regularity estimates*, in Proceedings of the Fourth International Conference on Nonlinear Analysis and Convex Analysis, W. Takahashi, T. Tanaka (eds.), Yokohama Publisher, Yokohama, 2007, pp. 185-198.

[6] A. D. Ioffe and Y. Sekiguchi, *Regularity estimates for convex multifunctions*, Math. Program. to appear.

[7] A. Jourani and L. Thibault, *Metric regularity and subdifferential calculus in Banach spaces*, Set-valued Anal. **3** (1995), 87-100.

[8] P. D. Loewen, Optimal Control via Nonsmooth Analysis, American Mathematical Society, Providence, RI, 1993.

[9] B. S. Mordukhovich, *Complete characterization of openness, metric regularity and Lipschitzian properties of multifunctions*, Trans. Amer. Math. Soc. **340** (1993), 1-35.

[10] B. S. Mordukhovich and Y. Shao, *Nonsmooth sequential analysis in Asplund spaces*, Trans. Amer. Math. Soc. **348** (1996), 1235-1280.

[11] R. T. Rockafellar and R. J.-B. Wets, Variational Analysis, Springer-Verlag, Berlin 1997.

[12] Y. Sekiguchi and W. Takahashi, *Tangent and normal vectors to feasible regions with geometrical derivability*, Sci. Math. Jpn. **64** (2006), 61-71.

Manuscript received December 4, 2006
revised May 9, 2007

Y. Sekiguchi
Department of Informatics, Kwansei Gakuin University
2-1 Gakuen, Sanda, Hyogo, 669-1337, Japan
E-mail address: ys@kwansei.ac.jp

W. Takahashi
Department of Mathematical and Computing Sciences
Graduate School of Information Sciences and Engineering
Tokyo Institute of Technology
2-12-1, Oh-okayama, Meguro, Tokyo 152-8552, Japan
E-mail address: wataru@is.titech.ac.jp