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APPROXIMATELY CONVEX SETS

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Abstract. We study two classes of sets whose associated distance functions satisfy properties akin to approximate convexity. Since the class of approximately convex functions is known to enjoy nice properties, one may expect analogous properties for this class of sets. We present characterizations and delineate links with the concept of approximately convex functions through epigraphs and sublevel sets.

1. Introduction

The abundance of constructions in nonsmooth analysis enables to attack various problems by using adapted techniques. However, this abundance of variants is often considered as an inconvenience. Therefore, it is of interest to show that in some classes of sets or functions these variants coincide. This has been done in [32, Thm 3.6], [44], [48], [50] for the family of approximately convex functions, for some favorable classes of functions, and for the class of $\alpha(\cdot)$-paraconvex functions respectively; see also [2], [8], [11, section 5].

In the present paper we study the notion of approximate convexity for sets considered in [11, section 5] under the name of property (ω) and called submoothness in [2]. We also introduce a notion of intrinsically convex set. Both notions are given in terms of the distance function to the set. In ([48]) we study pointwise variants, namely the concepts of approximate starshapedness and intrinsic approximate starshapedness for sets. Since these notions are more general than the ones studied here, we refer the reader to that paper for what concerns regularity properties, i.e. properties ensuring that various concepts of tangent or normal cones to a set coincide at a given point. Such properties are important as they show some unified character of nonsmooth analysis. On the other hand, the class of sets we study here is more general than the class of weakly convex sets considered in [52], [11], [2]. Directional versions of the concepts we study can be introduced, but for the sake of brevity we do not consider them here. Our main goal is to give characterizations of approximate convexity for sets, thus establishing a parallel with the study conducted in [38] for functions.

We adopt a versatile approach which allows one to deal with a large spectrum of notions of normal cones and subdifferentials. That enables one to combine the advantages of these various notions and to use the notion which is the best adapted to a specific problem.

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Finally, we endeavor to relate the concept of approximate convexity of sets to the notion of approximate convexity of functions introduced in [32] and studied in a number of papers ([2], [15], [38]...). It is as follows.

Definition 1.1. ([32]) A function $f : X \to \mathbb{R}_\infty := \mathbb{R}\cup\{+\infty\}$ on a normed vector space X is said to be approximately convex at $\overline{x} \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, y \in B(\overline{x}, \delta)$ and any $t \in [0, 1]$ one has

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t)\left\|x - y\right\|.$$

Such functions are locally Lipschitzian on the interior of their domains by [32, Prop. 3.2]. In ([32, Prop. 3.1]) it is shown that this class of functions is stable under finite sums, under finite suprema and under composition with a continuous affine function; moreover, this class contains the family of functions which are strictly differentiable at $\overline{x}$.

These nice properties incite to guess that the classes of approximately convex sets defined in sections 3 and 4 also enjoy pleasant properties.

2. Preliminaries

In the sequel, X is a Banach space with topological dual space X^*. The open ball with center $\overline{x} \in X$ and radius $\rho > 0$ is denoted by $B(\overline{x}, \rho)$, while $\overline{B}_X$ (resp. $\overline{B}_{X^*}$) stands for the closed unit ball of X (resp. X^*) and S_X stands for the unit sphere. Given a subset E of X, the *distance function* d_E associated with E is given by $d_E(x) := \inf_{e\in E} d(x, e)$ and the *indicator function* ι_E of E is the function defined by $\iota_E(x) = 0$ if $x \in E$, $\iota_E(x) = \infty$ if $x \in X\backslash E$. We write $x \xrightarrow{E} a$ for $x \to a$ and $x \in E$.

Since our study is of geometrical nature, we have to introduce some geometrical concepts. The *tangent cone* to a subset E of X at some $\overline{x} \in \mathrm{cl}(E)$ is the set $T(E, \overline{x})$ of vectors $v \in X$ such that there exist sequences $(t_n) \to 0_+$, $(x_n) \xrightarrow{E} \overline{x}$ (i.e. $(x_n) \to \overline{x}$ and $x_n \in E$ for each $n \in \mathbb{N}$) for which $(t_n^{-1}(x_n - x)) \to v$. The *normal cone* $N(E, \overline{x})$ to E at $\overline{x}$ is the polar cone of $T(E, \overline{x})$. Both play a crucial role in nonlinear analysis and optimization.

The *firm normal cone* (or Fréchet normal cone) to E at $\overline{x}$ is given by

$$x^* \in N^-(E, \overline{x}) \Leftrightarrow \forall\varepsilon > 0\ \exists\delta > 0 : \langle x^*, x - \overline{x}\rangle \leq \varepsilon\left\|x - \overline{x}\right\| \quad \forall x \in E \cap B(\overline{x}, \delta).$$

The *Clarke normal cone* $N^\uparrow(E, \overline{x})$ to E at $\overline{x}$ is defined as the polar cone to the *Clarke tangent cone* $T^\uparrow(E, \overline{x})$, where

$$T^\uparrow(E, \overline{x}) := \{v \in X : \forall(t_n) \to 0_+, \forall(x_n) \xrightarrow{E} \overline{x},\ \exists(v_n) \to v,\ x_n + t_n v_n \in E\ \forall n \in \mathbb{N}\}.$$

To any notion $N^?$ of normal cone one can associate a notion of subdifferential $\partial^?$ by setting

$$\partial^? f(\overline{x}) := \{x^* \in X^* : (x^*, -1) \in N^?(E_f, \overline{x}_f)\},$$

where $E_f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}$ is the epigraph of f and $\overline{x}_f := (\overline{x}, f(\overline{x}))$. The notion of subdifferential we adopt here is as versatile as possible: given a Banach space X and a subset $\mathcal{F}(X)$ of the set $\mathcal{S}(X)$ of lower semicontinuous (l.s.c.) functions $f : X \to \mathbb{R}_\infty := \mathbb{R}\cup\{+\infty\}$) a *subdifferential* on $\mathcal{F}(X)$ will be just a correspondence $\partial : \mathcal{F}(X) \times X \rightrightarrows X^*$ which assigns a subset $\partial f(x)$ of the dual space

X^* of X to any f in $\mathcal{F}(X)$ and any $x \in X$ at which f is finite; we assume it satisfies the following natural property:

(M) $0 \in \partial f(\overline{x})$ when $\overline{x}$ is a minimizer of a Lipschitzian function f.

Conversely, with any subdifferential $\partial^?$ is associated a notion of *normal cone* obtained by setting for a subset E of X and $e \in E$

$$N^?(E, e) := \mathbb{R}_+ \partial^? \iota_E(e),$$

where ι_E is the indicator function of E. In the cases $\partial^? = \partial$, $\partial^? = \partial^{\uparrow}$ and $\partial^? = \partial^-$ we get the *normal cones* $N(E, \overline{x})$, $N^{\uparrow}(E, \overline{x})$ and $N^-(E, \overline{x})$ to E at $\overline{x} \in E$ in the senses of Bouligand, Clarke and Fréchet respectively, as defined above. We refer to [29] and [34] for the definitions of the approximate subdifferential ∂^A and the moderate subdifferential $\partial^{\diamond}$ respectively.

We say that X is a *Lipschitz $\partial^?$-subdifferentiability space* if for any Lipschitz function f on X the domain of $\partial^? f$ is dense in X; this notion is close to the notion of subdifferentiability space introduced in [27]. We say that a subdifferential $\partial^?$ is *Lipschitz-valuable on X*, if for any Lipschitz function f on X and any $a, b \in X$ there exists c in the segment $[a, b]$ joining a to b and $c^* \in \partial^? f(c)$ such that

$$f(b) - f(a) \leq \langle c^*, b - a \rangle.$$

Some of the subdifferentials of current use are related to generalized concepts of directional derivatives (but not all). The *Clarke-Rockafellar* derivative or circa-derivative of a function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ finite at $\overline{x}$ is given by the following formulas in which $E_f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}$, $\overline{x}_f := (\overline{x}, f(\overline{x}))$

$$\begin{aligned} f^{\uparrow}(\overline{x}, v) &:= \inf_{r>0} \limsup_{\substack{(t,y)\to(0_+,\overline{x}) \\ f(y)\to f(\overline{x})}} \inf_{w \in B(v,r)} \frac{1}{t}(f(y + tw) - f(y)) \\ &= \inf\{r \in \mathbb{R} : (v, r) \in T^{\uparrow}(E_f, \overline{x}_f)\}. \end{aligned}$$

When f is locally Lipschitzian, this formula can be simplified into

$$f^{\uparrow}(\overline{x}, v) = \limsup_{(t,y)\to(0_+,\overline{x})} \frac{1}{t}(f(y + tv) - f(y)).$$

The (lower) *directional derivative* (or contingent derivative or lower epiderivative or lower Hadamard derivative) of f at $\overline{x}$ is given by

$$\begin{aligned} f'(\overline{x}, v) &:= \liminf_{(t,w)\to(0_+,v)} \frac{1}{t}(f(\overline{x} + tw) - f(\overline{x})) \\ &= \inf\{r \in \mathbb{R} : (v, r) \in T(E_f, \overline{x}_f)\}. \end{aligned}$$

In particular, one has

$$\begin{aligned} \partial f(\overline{x}) &= \{x^* \in X^* : x^* \leq f'(\overline{x}, \cdot)\}, \\ \partial^{\uparrow} f(\overline{x}) &= \{x^* \in X^* : x^* \leq f^{\uparrow}(\overline{x}, \cdot)\}. \end{aligned}$$

We will need the following results of independent interest. The first one is obtained by an easy argument taken from the proof of [46, Prop. 1].

Lemma 2.1. *Let E be a nonempty subset of a Banach space X and let $x \in E$, $v \in X$. Then, one has*

$$d_E^{\uparrow}(x,v) = \limsup_{t \to 0_+,\, e \xrightarrow{E} x} \frac{1}{t} d_E(e + tv).$$

Proof. By definition of $d_E^{\uparrow}$ there exist some sequences $(t_n) \to 0_+$, (x_n) in X such that

$$d_E^{\uparrow}(x,v) = \lim_n t_n^{-1} \left[d_E(x_n + t_n v) - d_E(x_n) \right].$$

We can find $e_n \in E$ such that $\|e_n - x_n\| \leq d_E(x_n) + t_n^2$. Now $d_E(x_n + t_n v) \leq d_E(e_n + t_n v) + \|e_n - x_n\|$. Thus $(e_n) \xrightarrow{E} x$ and

$$\begin{aligned} d_E^{\uparrow}(x,v) &\leq \limsup_n t_n^{-1} \left[d_E(e_n + t_n v) + \|e_n - x_n\| - (\|e_n - x_n\| - t_n^2) \right] \\ &= \limsup_n t_n^{-1} d_E(e_n + t_n v). \end{aligned}$$

The reverse inequality $d_E^{\uparrow}(x,v) \geq \limsup_{t \to 0_+,\, e \xrightarrow{E} x} \frac{1}{t} d_E(e + tv)$ being always valid, equality is proved. □

The second result we need is close to [29, Lemma 5], [46, Lemma 1], [41, Lemma 3.6] and [2, Lemma 3.7] but it contains a crucial additional information. Recall that an *Asplund space* is a space all of which separable subspaces have a separable dual. Recall also that the norm of X is said to satisfy the *Kadec-Klee property* if for every $x \in X$, a sequence (x_n) of X converges to x whenever it weakly converges to x and $(\|x_n\|) \to \|x\|$.

Lemma 2.2. *Suppose that E is a closed nonempty subset of an Asplund space X and that $w^* \in \partial^- d_E(w)$ with $w \in X \backslash E$. Then $\|w^*\| = 1$ and there exist sequences (x_n), (x_n^*) of E and X^* respectively such that $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$ and*

$$(\|x_n - w\|) \to d_E(w), \quad (\langle x_n^*, w - x_n \rangle) \to d_E(w), \quad (\|x_n^* - w^*\|) \to 0.$$

If moreover X is reflexive and if its norm has the Kadec-Klee property then a subsequence of (x_n) converges to some best approximation x of w in E and one has $\langle w^*, w - x \rangle = \|x - w\| = d_E(w)$.

Proof. The fact that $\|w^*\| = 1$ for each $w^* \in \partial^- d_E(w)$ is well-known (see [4, Prop. 1.4], for instance). By [41, Lemma 3.6] or [2, Lemma 3.7], given a sequence $(\varepsilon_n) \to 0_+$, one can find sequences (x_n), (x_n^*) of E and X^* respectively such that $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$ and $(\|x_n - w\|) \to d_E(w)$, $(\|x_n^* - w^*\|) \to 0$. It remains to apply [41, Lemma 3.6] which asserts that for any sequence (x_n) of E satisfying $(\|x_n - w\|) \to d_E(w)$ one has $(\langle w^*, w - x_n \rangle) \to d_E(w)$. Since $(\|x_n^* - w^*\|) \to 0$ and since $(w - x_n)$ is bounded, one gets $(\langle x_n^*, w - x_n \rangle) \to d_E(w)$.

The last assertion is taken from [6, Lemma 6]. □

Given a subdifferential $\partial^?$ one can associate to it a corresponding *limiting subdifferential* $\overline{\partial^?}$ by setting for a l.s.c. function f and a point x of it domain

$$\overline{\partial^?} f(x) := w^* - \limsup_{(u, f(u) \to (x, f(x))} \partial^? f(u).$$

Similarly, to any notion of normal cone one can associate a corresponding *limiting normal cone* by setting

$$\overline{N^?}(E,x) := w^* - \limsup_{u \overset{E}{\to} x} N^?(E,u).$$

Here, the w^*-limsup of a family $(F_t)_{t\in T}$ of subsets of X^* parametrized by some topological space T with a base point 0 is the set $w^*\text{-}\limsup_t F_t$ of weak* limits of bounded families $(x_t^*)_{t\in S}$ as $t \to 0$, where S is some subset of T containing 0 in its closure $\mathrm{cl}(S)$ and $x_t^* \in F_t$ for each $t \in S$. For the limiting firm normal cone $\overline{N^-}(E,a) := w^*\text{-}\limsup_{x \overset{E}{\to} a} N^-(E,x)$ we have the following result we will use later on.

Corollary 2.3. *Let E be a closed nonempty subset of an Asplund space X, let $\overline{x} \in E$ and $\overline{x}^* \in \overline{\partial^-} d_E(\overline{x})$. Then $\|\overline{x}^*\| \leq 1$ and there exist nets (x_n), (x_n^*) of E and X^* respectively such that $(x_n) \to \overline{x}$, $(x_n^*) \to \overline{x}^*$ weak* and $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$.*

Proof. The inequality $\|\overline{x}^*\| \leq 1$ stems from the fact that $\overline{x}^*$ is a weak* limit point of a net $(w_n^*)_{n\in N}$ of X^* such that $w_n^* \in \partial^- d_E(w_n)$ for each $n \in \mathbb{N}$ with $(w_n) \to \overline{x}$. Let $J := \{n \in N : w_n \in E\}$. For $j \in J$ we take $x_j := w_j$, $x_j^* := w_j^*$. For $k \in K := N \backslash J$, using the preceding lemma, we pick $x_k \in E$ and $x_k^* \in \partial^- d_E(x_k)$ such that $\|x_k - w_k\| \leq 2d_E(w_k)$, $\|x_k^* - w_k^*\| \leq d_E(w_k)$. Then $(x_n) \to \overline{x}$, $(x_n^*) \to \overline{x}^*$ weak*. □

3. Approximate convexity of sets

We observe that using the notion of approximate convexity for the indicator function ι_E of a subset E of X would lead to convexity of E and not to a relaxed form of convexity. Therefore, we rather use the distance function d_E. In the sequel $\overline{x}$ is a point of E.

Definition 3.1. A subset E of X is said to be approximately convex at $\overline{x}$ if its associated distance function d_E is approximately convex at $\overline{x}$.

Example 3.2. The set $E := \{(r,s) \in \mathbb{R}^2 : s \geq \max\left(|r| - r^2, 0\right)\}$ is approximately convex at each of its points (for an appropriate norm) but is nonconvex. This example (and the following one) is a special instance of Proposition 5.4 below

Example 3.3. Let W be an infinite dimensional separable Hilbert space and let $X = W \times \mathbb{R}$ be endowed with the norm given by $\|(w,r)\| = (\|w\|^2 + r^2)^{1/2}$. Then the subset $E := \{(w,r) \in W \times \mathbb{R} : r \geq \max(\|w\| - 2\|w\|^2, -1)\}$ is approximately convex at each of its points; it is closed, but not weakly closed, hence is nonconvex: if (w_n) is an orthonormal base of W, then (x_n) given by $x_n := (w_n, -1)$ weakly converges to $(0,-1)$ but $(0,-1) \notin E$.

Example 3.4. If $\overline{x}$ is an isolated point of E, then E is approximately convex at $\overline{x}$.

It is not obvious to decide whether the preceding definition depends on the choice of the norm in the equivalence class inducing the topology of X; on the contrary, the variant presented in the next section will not depend on the choice of the norm inducing the topology.

In order to look for characterizations, we need the following notion.

Definition 3.5. A multimapping $M : X \rightrightarrows X^*$ is said to be approximately monotone at $\overline{x}$ on $E \subset X$ if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in M(x_1)$, $x_2^* \in M(x_2)$ one has

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\| . \tag{3.1}$$

For $E = X$ one simply says that M is approximately monotone at $\overline{x}$.

When $\partial^? = \partial^\uparrow$, the following result is an easy consequence of [15] applied to d_E. However, since we use here an arbitrary subdifferential contained in the Clarke subdifferential, we have to use [38] with the distance function to get the implications. One can also deduce this result from the proof of Theorem 4.5 below by observing that one can drop the restriction $x \in E$.

Theorem 3.6. *Let $\partial^?$ be a subdifferential on the family $\mathcal{L}(X)$ of Lipschitz functions on X such that $\partial^? f \subset \partial^\uparrow f$ for any $f \in \mathcal{L}(X)$ and let $\overline{x}$ be an element of a subset E of X. Then, among the following assertions, one has the implications (a)⇒(b)⇒(c)⇔(c')⇔(d)⇔(e). If moreover $\partial^?$ is Lipschitz-valuable on X, in particular if $\partial^? = \partial^\uparrow$, $\partial^\diamond$, or the Ioffe subdifferential, all these assertions are equivalent.*

(a) *E is approximately convex at $\overline{x}$;*
(b) *for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(\overline{x}, \rho)$ and any $v \in B(0, \rho)$ one has*

$$d_E^\uparrow(x, v) \leq d_E(x + v) - d_E(x) + \varepsilon \|v\| ; \tag{3.2}$$

(c) *for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(\overline{x}, \rho)$, any $x^* \in \partial^? d_E(x)$ and any $(u, t) \in S_X \times (0, \rho)$ one has*

$$\langle x^*, u \rangle \leq \frac{d_E(x + tu) - d_E(x)}{t} + \varepsilon; \tag{3.3}$$

(c') *for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x \in B(\overline{x}, \rho)$, any $x^* \in \partial^? d_E(x)$ and any $v \in \rho \overline{B}_X$ one has*

$$\langle x^*, v \rangle \leq d_E(x + v) - d_E(x) + \varepsilon \|v\| ; \tag{3.4}$$

(d) *$\partial^? d_E$ is approximately monotone at $\overline{x}$;*
(e) *for any $\varepsilon > 0$ there exists $\sigma > 0$ such that for any $x, y \in B(\overline{y}, \sigma)$, $x^* \in \partial^? d_E(x)$ one has*

$$d_E(x) + \langle x^*, y - x \rangle \leq d_E(y) + \varepsilon \|y - x\| . \tag{3.5}$$

Corollary 3.7. *If E is approximately convex at $\overline{x}$ then d_E is firmly (Clarke) regular at $\overline{x}$, in the sense that for any subdifferential $\partial^?$ such that $\partial^- \subset \partial^? \subset \partial^\uparrow$ one has $\partial^- d_E(\overline{x}) = \partial^? d_E(\overline{x}) = \partial^\uparrow d_E(\overline{x})$.*

For a related assertion about normal cones, we refer to [48]. The result of the preceding corollary will remain valid in a more general class of subsets considered in the next section, so that we omit the proof here.

4. Intrinsic approximate convexity

The terminology of the definition we adopt now is justified by the fact that the notion we introduce is obtained by relaxing the requirement on the distance function to the subset E. Thus this notion is more general than the preceding notion.

Definition 4.1. A subset E of X is said to be intrinsically approximately convex at $\overline{x} \in E$ if for any $\varepsilon > 0$ there exists $\rho > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \rho)$, $t \in [0,1]$, one has

$$d_E((1-t)x_1 + tx_2) \leq \varepsilon t(1-t) \|x_1 - x_2\| . \tag{4.1}$$

It is intrinsically approximately convex if it is intrinsically approximately convex at each of its points.

Let us note that this definition does not depend on the choice of the norm among the ones inducing the same topology. Although it is looser than the notion of approximate convexity, it is sufficiently strong to eliminate pathological subsets.

Example 4.2. Let $X := \mathbb{R}$ and let $E := \{0\} \cup \{x_n\}$, where (x_n) is a decreasing sequence of $(0, +\infty)$ with limit 0. Then E is not intrinsically approximately convex at 0 since for $w \in [x_{n+1}, x_n]$ one has $d_E(w) = \min(x_n - w, w - x_{n+1})$.

Example 4.3. If E is paraconvex around $\overline{x}$ (i.e. locally weakly convex around $\overline{x}$ in the sense of [52]), then it is intrinsically approximately convex since for any given $c > 0$, $\varepsilon > 0$, one has $ct(1-t)\|x_1 - x_2\|^2 \leq \varepsilon t(1-t)\|x_1 - x_2\|$ when $\|x_i - \overline{x}\| \leq c^{-1}\varepsilon/2$ for $i = 1, 2$ (see [52, Prop. 3.4]).

The following example provides a set which is intrinsically approximately convex at some point but not approximately convex at the same point.

Example 4.4 (see [40], Section 8)**.** Given $p \in (1,2)$, let E be the hypograph of the function $f : r \mapsto |r|^p$ from $\mathbb{R}$ to $\mathbb{R}$:

$$E := \{(r,s) \in \mathbb{R}^2 : s \leq |r|^p\}.$$

Let us endow $\mathbb{R}^2$ with the Euclidean norm. It was shown in [40] that E is intrinsically paraconvex around $(0,0)$, hence is intrinsically approximately convex at $(0,0)$. By the same argument as in [40], one can show that E is not approximately convex at $(0,0)$.

Now it is a challenge to show whether the assertions of Theorem 3.6 can be extended to this weaker notion. It appears that most implications can be adapted as follows; they are not as complete as in the preceding theorem. However, we will supplement them in some special cases later on. The equivalence (c)⇔(c') of the next statement is nothing but a reformulation. However, it shows a link with the study made in [11, Prop. 4.2, 4.4]. When one of the assertions (b)-(d) holds, we say that E is $\partial^?$-*intrinsically approximately convex at* $\overline{x}$.

Theorem 4.5. *Let E be a nonempty closed subset of X and let $\partial^?$ be a subdifferential such that $\partial^? f \subset \partial^\uparrow f$ for any Lipschitz function f on X. Then, among the following assertions, one has the implications (a)⇒(b)⇒(c)⇔(c')⇔(d)⇐(e). When X is a Lipschitz $\partial^?$-subdifferentiability space one has (e)⇒(a).*

(a) *E is intrinsically approximately convex at $\overline{x}$;*

(b) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, one has*

$$d_E^{\uparrow}(x, x' - x) \le \varepsilon \left\| x - x' \right\|; \tag{4.2}$$

(c) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial^? d_E(x)$, one has*

$$\langle x^*, x' - x \rangle \le \varepsilon \left\| x - x' \right\|; \tag{4.3}$$

(c') *there exists a function $\alpha : E \times E \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\alpha(x, x') \to 0$ as $x, x' \xrightarrow{E} \overline{x}$ and*

$$\langle x^*, x' - x \rangle \le \alpha(x, x') \left\| x - x' \right\| \quad \text{for any } (x, x') \in E \times E, \ x^* \in \partial^? d_E(x); \tag{4.4}$$

(d) *$\partial^? d_E(\cdot)$ is approximately monotone at $\overline{x}$ on E : for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in \partial^? d_E(x_1)$, $x_2^* \in \partial^? d_E(x_2)$ one has*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \ge -\varepsilon \left\| x_1 - x_2 \right\|; \tag{4.5}$$

(e) *for any $\varepsilon > 0$ there exists $\sigma > 0$ such that for any $w \in B(\overline{x}, \sigma)$, $x \in E \cap B(\overline{x}, \sigma)$, $w^* \in \partial^? d_E(w)$ one has*

$$d_E(w) + \langle w^*, w - x \rangle \le \varepsilon \left\| w - x \right\|. \tag{4.6}$$

Proof. (a)⇒(b) Given $\varepsilon > 0$ let $\rho > 0$ be as in Definition 4.1 and let $x, x' \in E \cap B(\overline{x}, \rho)$. By Lemma 2.1, we have

$$d_E^{\uparrow}(x, x' - x) = \limsup_{t \to 0_+, \ e \xrightarrow{E} x} \frac{1}{t} d_E(e + t(x' - x)).$$

Now, since $d_E(e + t(x' - x)) \le d_E(e + t(x' - e)) + t \left\| e - x \right\| \le \varepsilon t(1 - t) \left\| x' - e \right\| + t \left\| e - x \right\|$, we get

$$d_E^{\uparrow}(x, x' - x) \le \varepsilon \left\| x' - x \right\|.$$

(b)⇒(c) is a consequence of the inclusion $\partial^? d_E(x) \subset \partial^{\uparrow} d_E(x)$ and of the definition of $\partial^{\uparrow} d_E(x)$. The implication (c)⇒(b) also holds when $\partial^? \supset \partial^-$ and X is an Asplund space, since in such a case assertion (c) with $\partial^?$ implies assertion (c) with $\partial^{\uparrow}$ by taking the weak* closure of the convex hull of $\partial^? d_E(x)$.

(c)⇒(c') It suffices to set for $(x, x') \in E \times E$, $\alpha(x, x') := 0$ if $x = x'$ and for $x \ne x'$,

$$\alpha(x, x') := \sup\{\langle x^*, \frac{x' - x}{\left\| x - x' \right\|} \rangle : \ x^* \in \partial^? d_E(x)\}.$$

Then (c) ensures that $\alpha(x, x') \to 0$ as $x, x' \to \overline{x}$.

(c')⇒(c) is obvious.

(c)⇒(d) Given $\varepsilon > 0$ let $\delta > 0$ be as in assertion (c) and let $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in \partial^? d_E(x_1)$, $x_2^* \in \partial^? d_E(x_2)$. Taking $x = x_1$, $x^* = x_1^*$, $x' = x_2$ in inequality (4.3) and adding its sides to the corresponding ones obtained by choosing $x = x_2$, $x^* = x_2^*$, $x' = x_1$, we get relation (4.5) with ε changed into 2ε.

(d)⇒(c) is obtained by taking $x_1 = x$, $x_1^* = x^*$, $x_2 = x'$, $x_2^* = 0$ in assertion (d), using the fact that x_2 is a minimizer of d_E, so that $0 \in \partial^? d_E(x_2)$ by condition (M).

(e)⇒(c) is obvious (change (w,x) into (x,x') in (4.6) with $x \in E$).

(e)⇒(a) when X is a Lipschitz $\partial^?$-subdifferentiability space. Given $\varepsilon > 0$, let $\sigma > 0$ be as in assertion (e) and let $x_1, x_2 \in E \cap B(\overline{x}, \sigma)$, $t \in [0,1]$, $w := (1-t)x_1 + tx_2$. Since X is a Lipschitz $\partial^?$-subdifferentiability space, there exist sequences $(w_n) \to w$, (w_n^*) such that $w_n^* \in \partial^? d_E(w_n)$ for each $n \in \mathbb{N}$. Then, as $\partial^? d_E(w_n) \subset \partial^\uparrow d_E(w_n) \subset \overline{B}_{X^*}$, we have $(\langle w_n^*, w - w_n \rangle) \to 0$. Since by convexity $w \in B(\overline{x}, \sigma)$, we have $w_n \in B(\overline{x}, \sigma)$ for n large enough, hence, by (4.6),

$$(1-t)d_E(w_n) + (1-t)\langle w_n^*, x_1 - w_n \rangle \leq (1-t)\varepsilon \|x_1 - w_n\|,$$
$$t d_E(w_n) + t\langle w_n^*, x_2 - w_n \rangle \leq t\varepsilon \|x_2 - w_n\|.$$

Adding the corresponding sides of these relations, we get

$$d_E(w_n) + \langle w_n^*, w - w_n \rangle \leq (1-t)\varepsilon \|x_1 - w_n\| + t\varepsilon \|x_2 - w_n\|,$$

and, passing to the limit,

$$d_E(w) \leq (1-t)\varepsilon \|x_1 - w\| + t\varepsilon \|x_2 - w\| = 2\varepsilon t(1-t) \|x_1 - x_2\|.$$

□

The preceding implications yield the following regularity result.

Corollary 4.6. *If E is intrinsically approximately convex at $\overline{x}$, then for any subdifferential $\partial^?$ such that $\partial^- \subset \partial^? \subset \partial^\uparrow$ one has $\partial^- d_E(\overline{x}) = \partial^? d_E(\overline{x}) = \partial^\uparrow d_E(\overline{x})$.*

Proof. Let $\overline{x}^* \in \partial^\uparrow d_E(\overline{x})$ and let $\varepsilon > 0$ be given. By (c) with $\partial^? = \partial^\uparrow$, we can find $\delta > 0$ such that for each $x \in B(\overline{x}, \delta)$, setting $x' := x$, $x := \overline{x} \in B(\overline{x}, \delta)$ in (4.3), we have

$$\langle \overline{x}^*, x - \overline{x} \rangle \leq \varepsilon \|x - \overline{x}\|.$$

That shows that $\overline{x}^* \in \partial^- d_E(\overline{x})$. We even have a uniformity property on the elements of $\partial^\uparrow d_E(\overline{x})$. □

Now let us give some specializations to some specific subdifferentials and normal cones. We start with the firm normal cone.

Corollary 4.7. *Suppose that E is a closed subset of an Asplund space X and let $\partial^?$ be the Fréchet subdifferential ∂^-. Then all the assertions of Theorem 4.5 are equivalent to the following assertions:*

(f) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in N^-(E, x)$ one has*

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x^*\| \left\|x - x'\right\|; \tag{4.7}$$

(g) *for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in N^-(E, x_1)$, $x_2^* \in N^-(E, x_2)$ one has*

$$\langle x_1^* - x_2^*, x_2 - x_1 \rangle \leq \varepsilon \max\left(\|x_1^*\|, \|x_2^*\|\right) \|x_1 - x_2\|. \tag{4.8}$$

Proof. The relation $\partial^- d_E(x) = N^-(E, x) \cap \overline{B}_{X^*}$ for each $x \in E$, yields equivalence of assertions (c) and (f) by an homogeneity argument.

(f)⇒(g) by summation, changing ε into $\varepsilon/2$. The reverse implication is obtained by taking $x_1 := x$, $x_2 := x'$, $x_1^* = x^*$, $x_2^* = 0$.

(c)⇒(e) Given $\varepsilon > 0$, let $\delta > 0$ be as in (c). Let $\sigma := \delta/5$ and let $w \in B(\overline{x}, \sigma)$, $x \in E \cap B(\overline{x}, \sigma)$, $w^* \in \partial^- d_E(w)$. By Lemma 2.2 we can find sequences (x_n) in E and (x_n^*) in X^* such that $x_n^* \in \partial^- d_E(x_n)$ for each $n \in \mathbb{N}$ and

$$(4.9) \qquad (||x_n - w\|) \to d_E(w), \qquad (\langle x_n^*, w - x_n\rangle) \to d_E(w), \qquad (||x_n^* - w^*\|) \to 0.$$

Relation (4.6) being trivial if $w = x$, without loss of generality, we may suppose $w \neq x$ and $||x_n - w\| \leq 2||x - w\| \leq 4\sigma$ for each $n \in \mathbb{N}$; then $x_n \in B(\overline{x}, 5\sigma) \subset B(\overline{x}, \delta)$ and $||x_n - x\| \leq ||x_n - w\| + ||w - x\| \leq 3||x - w\|$. Now, for n large enough, relation (4.9) implies the inequality of the first line below, while assertion (c) ensures the passage from the second line to the third one:

$$\begin{aligned} d_E(w) + \langle w^*, x - w\rangle &\leq (\langle x_n^*, w - x_n\rangle + \varepsilon \|x - w\|) \\ &\quad + (\langle x_n^*, x - w\rangle + ||x_n^* - w^*\| \, \|x - w\|) \\ &\leq \langle x_n^*, x - x_n\rangle + \varepsilon \|x - w\| + \varepsilon \|x - w\| \\ &\leq \varepsilon \|x - x_n\| + 2\varepsilon \|x - w\| \leq 5\varepsilon \|x - w\| . \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, assertion (e) holds.

Finally, since X is Asplund, it is a ∂^--subdifferentiability space, so that (e) ensures that E is intrinsically approximately convex by Theorem 4.5. □

The case of the limiting subdifferential can be easily derived from the preceding corollary.

Corollary 4.8. *Suppose that E is a closed subset of an Asplund space X and let $\partial^?$ be the limiting Fréchet subdifferential $\overline{\partial^-}$. Then all the assertions of Theorem 4.5 are equivalent.*

Proof. Using Corollary 2.3, the assertions (*c*), (*d*), (*e*) of Theorem 4.5 with $\overline{\partial^-}$ follow from the corresponding assertions with ∂^- by a passage to the weak* limit for a bounded net; the reverse implications are obvious. □

Now let us turn to the Clarke subdifferential. It would be interesting to know whether one can get rid of the assumption of the last assertion that X is an Asplund space.

Corollary 4.9. *For a closed subset E of a Banach space X and $\partial^? = \partial^\uparrow$, among the assertions of Theorem 4.5, the following implications hold: (e)⇒(a)⇒(b)⇔(c)⇔(c') ⇔(d). If moreover X is an Asplund space, all these assertions are equivalent.*

Proof. (c)⇒(b) is a consequence of the well known fact that for any $x, v \in X$ one has $d_E^\uparrow(x, v) = \max\{\langle x^*, v\rangle : x^* \in \partial^\uparrow d_E(x)\}$, d_E being Lipschitzian. The other implications follow from the choice $\partial^? = \partial^\uparrow$ in Theorem 4.5 since any Banach space is a Lipschitz $\partial^\uparrow$-subdifferentiability space. When X is an Asplund space the equivalences can be deduced from the preceding corollary: since $\partial^\uparrow d_E(x) = \overline{\text{co}}^*(\overline{\partial^-} d_E(x))$ one has the equivalence (e$^-$)⇔($\overline{e^-}$)⇔(e$^\uparrow$) (where (e$^\uparrow$), ($\overline{e^-}$), (e$^-$) are (e) for $\partial^? = \partial^\uparrow$, $\overline{\partial^-}$, ∂^- respectively), hence, with a similar notation, (c$^\uparrow$)⇒(c$^-$)⇒(e$^-$)⇒(e$^\uparrow$)⇒(a). □

5. Approximately convex sets and functions

Some links between geometrical properties and analytical properties are contained in the next statements. Unless otherwise specified, we endow the product space $X := W \times \mathbb{R}$ of a n.v.s. W with $\mathbb{R}$ with a *product norm*, i.e. a norm such that the projections and the insertions $w \mapsto (w, 0)$ and $r \mapsto (0, r)$ are nonexpansive. Then, for each $(w, r) \in W \times \mathbb{R}$ one has

$$\max(\|w\|, |r|) \leq \|(w, r)\| \leq \|w\| + |r|.$$

Proposition 5.1. *Let W be a normed vector space and let $f : W \to \mathbb{R}_\infty$ be a l.s.c. function which is approximately convex at $\overline{w} \in W$. Then, for any $\overline{r} \geq f(\overline{w})$ the epigraph E of f is intrinsically approximately convex at $\overline{x} := (\overline{w}, \overline{r})$.*

Proof. Given $\varepsilon > 0$, let $\rho > 0$ be such that

$$f((1-t)w_1 + tw_2) \leq (1-t)f(w_1) + tf(w_2) + \varepsilon t(1-t)\|w_1 - w_2\|$$

for any $w_1, w_2 \in B(\overline{w}, \rho)$, $t \in [0, 1]$. Let $x_i := (w_i, r_i)$ $(i = 1, 2)$ be elements of the epigraph E of f in $B(\overline{x}, \rho)$ and let $t \in [0, 1]$, $w := (1-t)w_1 + tw_2$, $r := (1-t)r_1 + tr_2$, $x := (w, r)$. Then, as $w_1, w_2 \in B(\overline{w}, \rho)$ and $(w, f(w)) \in E$ one has $d_E(x) = 0$ if $f(w) \leq r$ and $d_E(x) \leq \|(w, r) - (w, f(w))\| \leq f(w) - r$ if $f(w) > r$, so that

$$\begin{aligned} d_E(x) &\leq \max(0, f(w) - r) \\ &\leq \max(0, (1-t)(f(w_1) - r_1) + t(f(w_2) - r_2) + \varepsilon t(1-t)\|w_1 - w_2\|) \\ &\leq \varepsilon t(1-t)\|w_1 - w_2\| \leq \varepsilon t(1-t)\|x_1 - x_2\|. \end{aligned}$$

Thus E is intrinsically approximately convex at $\overline{x}$. □

Let us give a kind of converse to the preceding proposition.

Theorem 5.2. *Let W be a Banach space and let $f : W \to \mathbb{R}$ be a function which is locally Lipschitzian around $\overline{w} \in W$ and such that the epigraph E of f is an intrinsically approximately convex subset of $X := W \times \mathbb{R}$ around $\overline{x} := (\overline{w}, f(\overline{w}))$. Then f is an approximately convex function around $\overline{w}$.*

Proof in the case W is an Asplund space. In view of the characterization of approximate convexity of a function given in [38] it suffices to prove that $\partial^- f$ is approximately monotone at $\overline{w}$. Let c be the Lipschitz rate of f on some ball $B(\overline{w}, \rho_0)$. Given $\varepsilon > 0$ there exists some $\rho \in (0, \rho_0)$ such that for any $x_1, x_2 \in B(\overline{x}, \rho)$ and any $x_1^* \in N^-(E, x_1) \cap (c+1)\overline{B}_{X^*}$, $x_2^* \in N^-(E, x_2) \cap (c+1)\overline{B}_{X^*}$ one has

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\|.$$

Then, for $w_1, w_2 \in B(\overline{w}, \rho/(c+1))$, $w_i^* \in \partial^- f(w_i)$ for $i = 1, 2$, setting $x_i := (w_i, f(w_i))$, $x_i^* := (w_i^*, -1)$ one has $x_i \in B(\overline{x}, \rho)$ and $x_i^* \in N^-(E, x_i) \cap (c+1)\overline{B}_{X^*}$, hence

$$\begin{aligned} \langle w_1^* - w_2^*, w_1 - w_2 \rangle &= \langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \|x_1 - x_2\| \\ &\geq -\varepsilon (\|w_1 - w_2\| + |f(w_1) - f(w_2)|) \geq -\varepsilon(c+1)\|w_1 - w_2\|. \end{aligned}$$

Since ε is arbitrarily small, we get that f is approximately convex at $\overline{w}$. □

The proof in the general case relies on the following lemma extracted from the proof of [46, Prop. 10 c]; it is close to previous results of that kind due to F.H. Clarke [9] and to A.D. Ioffe ([29], [30, Prop. 2.1] for bornological subdifferentials) in the case $x = (\overline{w}, f(\overline{w}))$.

Lemma 5.3. *Let $f : W \to \mathbb{R}$ be a function which is Lipschitzian with rate c on a ball $B(\overline{w}, \rho)$ of W. Then, for $\sigma \in (0, \rho)$ small enough and for any $w \in B(\overline{w}, \sigma)$ and any $w^* \in \partial^{\uparrow} f(w)$ one has $(w^*, -1) \in \partial^{\uparrow} d_E(x)$, where E is the epigraph of f and $x := (w, f(w))$, $X := W \times \mathbb{R}$ being endowed with the norm given by $\|(w, r)\| = c\|w\| + |r|$.*

Proof. By [24, Prop. 2.1], [28] one can find $\sigma \in (0, \rho)$ such that

$$d_E(w, r) = (f(w) - r)_+ \tag{5.1}$$

for $(w, r) \in B(\overline{x}, \sigma)$ with $\overline{x} := (\overline{w}, f(\overline{w}))$, X being endowed with the norm described in the statement; here for $t \in \mathbb{R}$, t_+ stands for $\max(t, 0)$. In fact, $\sigma := (c/2c+1)\rho$ can be chosen: we may suppose $r < f(w)$ and we have $d_E(w, r) \leq \|(w, r) - (w, f(w))\| = f(w) - r \leq \sigma(c+1)$, while for $(u, s) \in E$ with $u \in B(\overline{w}, \rho)$ we have $\|(w, r) - (u, s)\| = c\|w - u\| + |r - s| \geq f(w) - f(u) + s - r \geq f(w) - r$; since for $(u, s) \in E$ with $u \in X \backslash B(\overline{w}, \rho)$ we have $\|(w, r) - (u, s)\| \geq c\rho - c\sigma$, equality (5.1) holds. Let $w \in B(\overline{w}, \sigma)$ and $w^* \in \partial^{\uparrow} f(w)$; we have to prove that for any $(v, s) \in X$ we have

$$\langle (w^*, -1), (v, s) \rangle \leq d_E^{\uparrow}((w, r), (v, s)).$$

Since $w^* \in \partial^{\uparrow} f(w)$ there exist sequences $(\varepsilon_n) \to 0_+$, $(w_n) \to w$, $(t_n) \to 0_+$ such that

$$t_n^{-1}(f(w_n + t_n v) - f(w_n)) > \langle w^*, v \rangle - \varepsilon_n$$

for each n. Setting $r_n := f(w_n)$ and observing that, for n large enough,

$$\begin{aligned} t_n^{-1} d_E(w_n + t_n v, r_n + t_n s) &= t_n^{-1}(f(w_n + t_n v) - r_n - t_n s)_+ \\ &\geq t_n^{-1}(f(w_n + t_n v) - r_n - t_n s) \\ &\geq \langle w^*, v \rangle - \varepsilon_n - s, \end{aligned}$$

using Lemma 2.1 we get the expected inequality:

$$d_E^{\uparrow}((w, r), (v, s)) \geq \limsup_n t_n^{-1} d_E(w_n + t_n v, r_n + t_n s) \geq \langle w^*, v \rangle - s.$$

Proof of the theorem in the general case. Since intrinsic approximate convexity is preserved when using an equivalent norm, we may use the norm described in the lemma and take $\sigma > 0$ as there. We use the implication (a)⇒(c) of Corollary 4.9: for any $\varepsilon > 0$ there exists $\delta \in (0, \sigma)$ such that for any $x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial^{\uparrow} d_E(x)$, one has

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x - x'\|. \tag{5.2}$$

Now, setting $\gamma := \delta/2c$, by the preceding lemma, for every $w \in B(\overline{w}, \gamma)$ and $w^* \in \partial^{\uparrow} f(w)$ we have $(w^*, -1) \in \partial^{\uparrow} d_E(x)$ with $x := (w, f(w)) \in B(\overline{x}, \delta)$. Let $u \in \gamma \overline{B}_X$ be such that $w' := w + u \in B(\overline{w}, \gamma)$. Then we have that $x := (w, f(w))$, $x' := (w', f(w')) \in E \cap B(\overline{x}, \delta)$, $x^* := (w^*, -1) \in \partial^{\uparrow} d_E(x)$, hence, by inequality (5.2),

$$\langle w^*, u \rangle - \big(f(w') - f(w)\big) \leq \varepsilon \left(c\|w' - w\| + |f(w') - f(w)|\right) \leq 2c\varepsilon \|u\|.$$

Thus for any $\varepsilon > 0$ there exists $\gamma > 0$ such that for any $w \in B(\overline{w}, \gamma)$, any $w^* \in \partial^{\uparrow} f(w)$ and any $u \in \overline{B}(0, \gamma)$ with $w + u \in B(\overline{w}, \gamma)$ one has $f(w+u) - f(w) \geq \langle w^*, u \rangle - \varepsilon(c+1) \|u\|$, so that f is approximately convex at $\overline{w}$ by [15, Thm 2] or [38]. □

Let us complete the preceding results with the following one.

Proposition 5.4. *Let $f : W \to \mathbb{R}$ be a function which is Lipschitz with rate $c > 0$ on some ball $B(\overline{w}, \rho)$. Suppose $X := W \times \mathbb{R}$ is endowed with the norm given by $\|(w, r)\| = c \|w\| + |r|$. If f is approximately convex at $\overline{w}$, then, for any $\overline{r} \geq f(\overline{w})$, the epigraph E of f is approximately convex at $\overline{x} := (\overline{w}, \overline{r})$.*

Proof. Let $\overline{x} := (\overline{w}, f(\overline{w}))$ and let us endow X with the norm described in the statement. By [24], [28] we can find $\rho' \in (0, \rho)$ such that, for $(w, r) \in B(\overline{x}, \rho')$ we have

$$d_E(w, r) = (f(w) - r)_+$$

Given $\varepsilon > 0$, let $\delta \in (0, \rho')$ be such that for any $w_1, w_2 \in B(\overline{w}, \delta)$, $t \in [0, 1]$ we have

$$f((1-t)w_1 + tw_2) \leq (1-t)f(w_1) + tf(w_2) + \varepsilon ct(1-t) \|w_1 - w_2\|$$

Let $x_i := (w_i, r_i) \in B(\overline{x}, \delta)$ for $i = 1, 2$ and let $w := (1-t)w_1 + tw_2$, $r := (1-t)r_1 + tr_2$, $x := (w, r) \in B(\overline{x}, \delta)$ by convexity. Then we have

$$f(w) - r \leq (1-t)(f(w_1) - r_1)_+ + t(f(w_2) - r_2)_+ + \varepsilon t(1-t)c \|w_1 - w_2\|, \tag{5.3}$$

hence, since $c \|w_1 - w_2\| \leq \|x_1 - x_2\|$,

$$d_E(x) \leq (1-t)d_E(x_1) + td_E(x_2) + \varepsilon t(1-t) \|x_1 - x_2\|.$$

□

In particular, when f is Lipschitzian with rate 1 around $\overline{w}$, and when X is endowed with the norm given by $\|(w, r)\| = c \|w\| + |r|$, the epigraph E of f is approximately convex at $\overline{x}$ when f is approximately convex at $\overline{w}$.

The preceding results enable us to give a partial answer to the question of the relationships between intrinsic approximate convexity and approximate convexity. We restrict our attention to sets satisfying the cone property (the so-called epi-Lipschitzian sets). Recall that E *satisfies the cone property around* $\overline{x}$ if there exist $r, \rho > 0$ and $u \in S_X$ such that for every $x \in E \cap B(\overline{x}, \rho)$, $v \in B(u, r)$, $t \in (0, r)$ one has $x + tv \in E$. Our argument is close to the one in [2, Thm 4.14], even if intrinsic approximate convexity is not considered there.

Corollary 5.5. *Suppose E satisfies the cone property around $\overline{x}$. Then E is intrinsically approximately convex at $\overline{x}$ if, and only if, it is approximately convex at $\overline{x}$ for some compatible norm on X.*

Proof. It suffices to prove the only if condition. Since E satisfies the cone property around $\overline{x}$ there exist $\rho, \sigma > 0$, some hyperplane W of X and some $u \in S_X$ such that $X = W \oplus \mathbb{R}u$ and a Lipschitzian function $f : B(0, \rho) \cap W \to \mathbb{R}$ with $E \cap B(\overline{x}, \sigma) = \{\overline{x} + w + ru : w \in B(0, \rho), r \geq f(w)\} \cap B(\overline{x}, \sigma)$. Thus, identifying X with $W \times \mathbb{R}u$, locally E is the epigraph of a Lipschitzian function, and by Theorem 5.2, since E is intrinsically approximately convex at $\overline{x}$, f is approximately convex at 0. Then, by Proposition 5.4, we can endow the product $W \times \mathbb{R}u$ with a norm for which E is approximately convex at $\overline{x}$. □

Finally, let us turn to sublevel sets.

Proposition 5.6. *Let X be an Asplund space and let $f : X \to \mathbb{R}$ be a continuous function. Suppose f is approximately convex at $\overline{x} \in S := \{x \in X : f(x) \leq 0\}$ and there exist $c > 0$, $r > 0$ such that $\|x^*\| \geq c$ for each $x \in (X \backslash S) \cap B(\overline{x}, r)$ and each $x^* \in \partial^- f(x)$. Then S is intrinsically approximately convex at $\overline{x}$.*

Proof. Without loss of generality we may suppose f takes the value $+\infty$ on $X \backslash U$, where $U := B(\overline{x}, r)$. Then, by [45, Thm 9.1] with $\varphi = c$, (see also, among several other contributions, [54], [12], [41, Thm 3.2] with various assumptions on X) and we have $f_+(x) \geq c d_S(x)$ for $x \in U'$, where $f_+ := \max(f, 0)$ and $U' := B(\overline{x}, r/2)$. Let $\varepsilon > 0$ be given. Using [38, Thm 7], we can find $\delta \in (0, r/2)$ such that

$$\forall x, x' \in B(\overline{x}, \delta),\ x^* \in \partial^- f(x) \qquad \langle x^*, x' - x\rangle \leq f(x') - f(x) + c\varepsilon \left\|x' - x\right\|.$$

Since f and f_+ are approximately convex at $\overline{x}$, by [32, Thm 3.6] and elementary calculus rules, we have

$$c\partial^- d_S(x) \subset \partial^- f(x) = \partial f(x) = \overline{\text{co}}^*(\partial f(x) \cup \{0\}) = \overline{\text{co}}^*(\partial^- f(x) \cup \{0\}).$$

Given $x, x' \in S \cap B(\overline{x}, \delta)$, $x^* \in \partial^- d_S(x)$, by the preceding inclusion and inequality and a passage to the convex hull and the closure, we get $\langle cx^*, x' - x\rangle \leq c\varepsilon \|x' - x\|$. Thus, assertion (c) of Theorem 4.5 is satisfied for $\partial := \partial^-$ and $E := S$ so that S is intrinsically approximately convex at $\overline{x}$. □

Remark 5.7. The same conclusion holds for a pair $(X, \partial^?)$ which is variational in the sense of [45, Thm 9.1] and such that $\partial^- \subset \partial^? \subset \partial^\uparrow$ (which is the case of $\partial^? = \partial^-$ when X is Asplund).

6. Approximately convex sets and projections

The following result shows that, in the framework of uniformly smooth spaces, approximate convexity of a distance function is equivalent to its continuous differentiability.

First, we need the following lemma which gives the firm regularity of $-d_E$ on uniformly smooth spaces. The result could be deduced from [23, Thm 5.6] or from the fact that an approximately convex function is firmly regular and from the study of marginal functions made in [38]. However, for the reader's convenience, we present a direct proof inspired by [5] where the Gâteaux regularity of $-d_E$ has been established.

Lemma 6.1. *Let X be Fréchet uniformly smooth and let E be an arbitrary nonempty closed subset of X. Then $-d_E$ is firmly (Clarke) regular at any $w \in X \setminus E$ in the sense that $\partial^\uparrow(-d_E) = \partial^-(-d_E)$.*

Proof. Let us denote by j the reduced duality mapping, i.e. the derivative of the function $\|\cdot\|$ on $X \backslash \{0\}$. Let $w \in X \setminus E$. By ([53, Thm 3.7.4]) and the uniform smoothness of X, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{1}{t}\left(||x + tu|| - ||x||\right) - \langle j(x), u\rangle\right| < \varepsilon \qquad \forall x, u \in S_X,\ |t| < \delta. \tag{6.1}$$

Let $u \in S_X$. There exists $(w_n) \to w, (t_n) \to 0_+$, such that

$$(-d_E)^{\uparrow}(w,u) = \lim_{n\to\infty} \frac{1}{t_n}\left(-d_E(w_n + t_n u) + d_E(w_n)\right).$$

For each n, we can find $x_n \in E$ such that

$$-d_E(w_n + t_n u) \leqslant -||w_n + t_n u - x_n|| + t_n^2.$$

Therefore,

$$\frac{1}{t_n}\left(-d_E(w_n + t_n u) + d_E(w_n)\right) \leqslant \frac{1}{t_n}\left(-||w_n + t_n u - x_n|| + ||w_n - x_n||\right) + t_n.$$

Setting $r_n := ||w_n - x_n||$, $u_n := r_n^{-1}(w_n - x_n)$, and observing that $\left(t_n r_n^{-1}\right) \to 0$ as $\left(r_n^{-1}\right) \to 1/d_E(w)$, using (6.1), we have, for n large enough and $t \in (-\delta d_E(w), \delta d_E(w))$,

$$\begin{aligned}
\frac{1}{t_n}\left(-||w_n + t_n u - x_n|| + ||w_n - x_n||\right) &= \frac{1}{t_n r_n^{-1}}\left(-||u_n + t_n r_n^{-1} u|| + \|u_n\|\right) \\
&\leqslant -\langle j(u_n), u\rangle + \varepsilon \\
&\leqslant \frac{1}{t r_n^{-1}}\left(-||u_n + t r_n^{-1} u|| + ||u_n||\right) + 2\varepsilon \\
&\leqslant t^{-1}\left(-||w_n + tu - x_n|| + ||w_n - x_n||\right) + 2\varepsilon \\
&\leqslant t^{-1}\left(-d_E(w_n + tu) + ||w_n - x_n||\right) + 2\varepsilon.
\end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain for any $u \in S_X$, $t \in (-\delta d_E(w), \delta d_E(w))$

$$(-d_E)^{\uparrow}(w,u) \leqslant \frac{1}{t}\left(-d_E(w + tu) + d_E(w)\right) + 2\varepsilon.$$

Since u is arbitrary in S_X, this inequality proves the firm regularity of $-d_E(\cdot)$. □

The following result is reminiscent of [10, Thm 4.1] which takes place in a Hilbert space. However, here U is not a uniform neighborhood of E; it may be small (or large) and far from E.

Theorem 6.2. *Suppose that the norm of X is Fréchet differentiable on $X\backslash\{0\}$. Let E be a closed subset of X and let U be an open subset of X. Consider the following assertions:*

(a) *Each $w \in U$ has a unique metric projection $P_E(w)$ in E and the mapping $P_E(\cdot)$ is continuous on $U \setminus E$.*
(b) *$d_E(\cdot)$ is continuously differentiable on $U \setminus E$.*
(c) *$d_E(\cdot)$ is approximately convex on $U \setminus E$.*

Then, one has (a) ⇒ (b) ⇒ (c). If X is Fréchet uniformly smooth, then (a) ⇒ (b) ⇔ (c).
If, in addition, X is strictly convex and the norm of X has the Kadec-Klee property, then (a)⇔(b)⇔(c).

Proof. (a)⇒(b) For any $w \in U\backslash E, v \in X$, since the norm is differentiable at $w - P_E(w) \neq 0$ and P_E is continuous, we have

$$d_E^{\uparrow}(w,v) = \limsup_{y\to w, t\downarrow 0} \frac{1}{t}\left(d_E(y + tv) - d_E(y)\right)$$

$$\leqslant \limsup_{y\to w, t\downarrow 0} \frac{1}{t} (||y + tv - P_E(y)|| - ||y - P_E(y)||)$$
$$\leqslant \limsup_{y\to w, t\downarrow 0} \sup_{\theta\in[0,1]} \langle j(y - P_E(y) + \theta tv), v\rangle = \langle j(w - P_E(w)), v\rangle.$$

In the last equality we have also used the fact that since the norm is differentiable and convex, the duality mapping $j(.)$ is continuous by [49, p. 20]. Thus $d_E^{\uparrow}(w, \cdot) = \langle j(w - P_E(w)), \cdot\rangle$ and $\partial^{\uparrow} d_E(w) = \{j(w - P_E(w))\}$. Thus $d_E(.)$ is continuously differentiable on $U \setminus E$.

(b)⇒(c) is obvious.

(c)⇒(b) when X is uniformly Fréchet smooth. Assume that $d_E(\cdot)$ is approximately convex on $U \setminus E$. Then, by Corollary 3.7, $d_E(\cdot)$ is firmly regular at all $w \in U \setminus E$. Moreover, by the preceding lemma, $-d_E(\cdot)$ is firmly regular at all $w \in X\backslash E$. Thus $\partial^- d_E(w) \neq \emptyset$ and $\partial^-(-d_E)(w) \neq \emptyset$ at any $w \in U \setminus E$. Therefore, $d_E(\cdot)$ is Fréchet differentiable on $U \setminus E$.

Let us prove that $d'_E(\cdot)$ is continuous on $U \setminus E$. Let $w \in U \setminus E$ and $\varepsilon \in (0, 1)$ be given. There exists $\delta > 0$ such that for all $v \in B(0, \delta)$ we have

$$d_E(w + v) - d_E(w) - \langle d'_E(w), v\rangle \leqslant \varepsilon||v||,$$

On the other hand, by the approximate convexity of $d_E(\cdot)$, there exists $\rho \in (0, \delta)$ such that

$$\langle d'_E(x), v\rangle \leqslant d_E(x + v) - d_E(x) + \varepsilon||v||$$

for all $x \in B(w, \rho)$, $v \in B(0, \rho)$. Thus, for any $x \in B(w, \varepsilon\rho)$, $v \in B(0, \rho)$, we have

$$\langle d'_E(x) - d'_E(w), v\rangle \leqslant 2||x - w|| + 2\varepsilon||v|| \leqslant 4\varepsilon\rho$$

Hence $||d'_E(x) - d'_E(w)|| \leqslant 4\varepsilon$ for $x \in B(w, \varepsilon\rho)$ and $d'_E(\cdot)$ is continuous at w.

(b)⇒(a) when X is strictly convex, uniformly smooth and its norm has the Kadec-Klee property. We follow the argument of [6, Lemma 6]. The uniform smoothness of X ensures that X is reflexive by the Milman-Pettis theorem ([3], [21, Thm 9.12]) and, by Lemma 2.2, for any $w \in U \setminus E$, there exists $x \in E$ such that

$$\langle d'_E(w), w - x\rangle = ||w - x|| = d_E(w). \tag{6.2}$$

Since $\|d'_E(w)\| \leq 1$, we have $\|d'_E(w)\| = 1$ and $j(w - x) = d'_E(w)$. Since the space is strictly convex, j is injective, so that x is the unique point of E satisfying (6.2). In order to prove that $P_E(\cdot)$ is continuous, let us consider a sequence $(w_n) \to w$. Let $x_n := P_E(w_n)$ and let z be a weak limit point of (x_n). Since d'_E is continuous and the norm is weakly lower semicontinuous, passing to the limit in the equality

$$\langle d'_E(w_n), w_n - x_n\rangle = d_E(w_n) = ||w_n - x_n||$$

we get, since $\|d'_E(w)\| \leq 1$,

$$||w - z|| \geq \langle d'_E(w), w - z\rangle = d_E(w) = \lim_n ||w_n - x_n|| \geq ||w - z||.$$

By the Kadec-Klee property we obtain that $(x_n) \to z$, so that $z \in E$ and $z = P_E(w)$. □

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