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AN INEQUALITY CONCERNING THE JAMES CONSTANT AND THE WEAKLY CONVERGENT SEQUENCE COEFFICIENT

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Abstract. Every Banach space X satisfies the inequality $WCS(X) \geq \dfrac{J(X)+1}{|J(X)|^2}$. As a consequence, we obtain a fixed point theorem for asymptotically regular mappings.

1. Introduction

As it is well-known, the notions of normal structure and uniform normal structure play important role in metric fixed point theory (see Goebel and Kirk [20]). Some parameters and constants defined on Banach spaces can be used to verify whether a specific Banach space enjoys uniform normal structure. These constants include the James constants and the Jordan-von Neumann constants, which are introduced by Gao and Lau [16] and Clarkson [7], respectively.

For a Banach space X, we show that the James constant $J(X)$ is related to, as an inequality, the weakly convergent sequence coefficient $WSC(X)$ defined by Bynum [5]. As a consequence, we obtain the the latest upper bound of the James constant $J(X)$ at $\frac{1+\sqrt{5}}{2}$ for X to have uniform normal structure [9, Dhompongsa et. al]. By applying Domínguez and Xu's theorem [15, Theorem 3.2], we also obtain fixed point results for asymptotically regular mappings.

2. Preliminaries and notations

Throughout the paper we let X and X^* stand for a Banach space and its dual space, respectively. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. Let A be a nonempty bounded set in X. The number $r(A) = \inf\{\sup_{y\in A} \|x-y\| : x \in A\}$ is called the Chebyshev radius of A. The number $\operatorname{diam}(A) = \sup\{\|x-y\| : x, y \in A\}$ is called the diameter of A. A Banach space X has normal structure (resp. weak normal structure) if

$$r(A) < \operatorname{diam}(A)$$

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for every bounded closed (resp. weakly compact) convex subset A of X with $\operatorname{diam}(A) > 0$. The normal structure coefficient $N(X)$ of X [5, Bynum] is the number

$$N(X) = \inf\left\{\frac{\operatorname{diam}(A)}{r(A)}\right\},$$

where the infimum is taken over all bounded closed convex subsets A of X with $\operatorname{diam}(A) > 0$. The weakly convergent sequence coefficient $WCS(X)$ [5] of X is the number

$$WCS(X) = \inf\left\{\frac{A(\{x_n\})}{r_a(\{x_n\})}\right\},$$

where the infimum is taken over all sequences $\{x_n\}$ in X which are weakly (not strongly) convergent, $A(\{x_n\}) = \limsup_n\{\|x_i - x_j\| : i, j \geq n\}$ is the asymptotic diameter of $\{x_n\}$, and $r_a(\{x_n\}) = \inf\{\limsup_n \|x_n - y\| : y \in \overline{\operatorname{co}}\{x_n\}\}$ is the asymptotic radius of $\{x_n\}$. A space X with $N(X) > 1$ (resp. $WCS(X) > 1$) is said to have uniform (resp. weak uniform) normal structure. For a Banach space X, the James constant, or the nonsquare constant is defined by Gao and Lau [16] as

$$J(X) = \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}.$$

It is known that $J(X) < 2$ if and only if X is uniformly nonsquare. Dhompongsa et. al [9, Theorem 3.1] showed that if $J(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure. The Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ of X, which is introduced by Clarkson [7], is defined by

$$C_{\mathrm{NJ}}(X) = \sup\left\{\frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ not both zero}\right\}.$$

A relation between these two constants is

$$\frac{(J(X))^2}{2} \leq C_{\mathrm{NJ}}(X) \leq \frac{(J(X))^2}{(J(X) - 1)^2 + 1} \text{([23, Kato et. al]).}$$

From this relation, it is easy to conclude that $C_{\mathrm{NJ}}(X) < 2$ is equivalent to $J(X) < 2$. Recently, Dhompongsa and Kaewkhao [10, Theorem 3.16] obtained the latest upper bound of the Jordan-von Neumann constant $C_{\mathrm{NJ}}(X)$ at $\frac{1+\sqrt{3}}{2}$ for X to have uniform normal structure. However, it is still not clear that if the upper bounds of the James constants and of the Jordan-von Neumann constants are sharp for having uniform normal structure (see a conjecture in [9]). The constant $R(a, X)$, which is a generalized Garcí a-Falset coefficient [18], is introduced by Domínguez [12] : for a given positive real number a

$$R(a, X) := \sup\{\liminf_n \|x + x_n\|\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequence $\{x_n\}$ in the unit ball of X such that

$$D(x_n) = \limsup_n \left(\limsup_m \|x_n - x_m\|\right) \leq 1.$$

Concerning with this coefficient, Domínguez obtained a fixed point theorem which states that if X is a Banach space with $R(a, X) < 1 + a$ for some a, then X has the weak fixed point property (for details about the (weak) fixed point property, the readers are referred to Goebel and Kirk [21]). In [28], Mazcuňán-Navarro showed that

$$R(1, X) \leq J(X),$$

and then combined it with the fixed point theorem of Domínguez to solve a long stand open problem. Indeed, it was proved that the uniform nonsquareness implies the weak fixed point property. A mapping $T : X \to X$ is called asymptotically regular if

$$\lim_n \|T^n x - T^{n+1} x\| = 0 \quad \text{for all } x \in X.$$

The concept of asymptotically regular mappings is due to Browder and Petryshyn [2]. We set

$$s(T) = \liminf_n |T^n|,$$

where $|T^n| = \sup\left\{\frac{\|T^n x - T^n y\|}{\|x-y\|} : x, y \in C,\ x \neq y\right\}$. Fixed point results for asymptotically regular mappings can be found in [3, 4, 13, 14, 15, 22, 25, 26]. Most of these results are related to geometric coefficients in Banach spaces. We state here the one using the weak convergent sequence coefficients.

Theorem 2.1. [15, Theorem 3.2] *Suppose X is a Banach space with $WCS(X) > 1$, C is a nonempty weakly compact convex subset of X, and $T : C \to C$ is a uniformly Lipschitzian mapping such that $s(T) < \sqrt{WCS(X)}$. Suppose in addition that T is asymptotically regular on C. Then T has a fixed point.*

One last concept we need to mention is ultrapowers of Banach spaces. We recall some basic facts about the ultrapowers. Let $\mathcal{F}$ be a filter on an index set I and let $\{x_i\}_{i\in I}$ be a family of points in a Hausdorff topological space X. $\{x_i\}_{i\in I}$ is said to converge to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an ultrafilter if it is maximal with respect to the set inclusion. An ultrafilter is called trivial if it is of the form $\{A : A \subset I, i_0 \in A\}$ for some fixed $i_0 \in I$, otherwise, it is called nontrivial. We will use the fact that

(i) $\mathcal{U}$ is an ultrafilter if and only if for any subset $A \subset I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$, and

(ii) if X is compact, then the $\lim_{\mathcal{U}} x_i$ of a family $\{x_i\}$ in X always exists and is unique.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space $\Pi_{i\in I} X_i$ equipped with the norm $\|\{x_i\}\| := \sup_{i\in I} \|x_i\| < \infty$.

Let $\mathcal{U}$ be an ultrafilter on I and let

$$N_{\mathcal{U}} = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}.$$

The ultraproduct of $\{X_i\}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm. Write $(x_i)_{\mathcal{U}}$ to denote the elements of the ultraproduct. It follows from (ii) above and the definition of the quotient norm that

$$\|\{x_i\}_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|.$$

In the following, we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X$, $i \in \mathbb{N}$, for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we write $\widetilde{X}$ to denote the ultraproduct which will be called an ultrapower of X. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically (for more details see Aksoy and Khamsi [1] or Sims [30]).

3. The James constant

Before we present the first result, we need another equivalent definition of the weakly convergent sequence coefficient $WCS(X)$ of X which is shown in [11, Theorem 1.1] as follows :

Definition 3.1. For a Banach space X,

$$WCS(X) = \inf\Big\{\frac{1}{r_a(x_n)} : x_n \xrightarrow{w} 0, \lim_{n\neq m} \|x_n - x_m\| = 1\Big\}.$$

Now we can state the following.

Theorem 3.2. *For a Banach space X,*

$$WCS(X) \geq \frac{J(X)+1}{[J(X)]^2}.$$

In particular, if $J(X) < \frac{1+\sqrt{5}}{2}$, then $WCS(X) > 1$.

Proof. For a sake of convenience we put $J(X) = \alpha$. Let $\{x_n\}$ be a weakly null sequence in X such that

$$\lim_{n\neq m} \|x_n - x_m\| = 1. \tag{3.1}$$

Put $C = \overline{\text{co}}\{x_n\}$ and $r = r_a\{x_n\}$. Since $0 \in C$, we obtain

$$r \leq \limsup_n \|x_n\|. \tag{3.2}$$

Fix $\varepsilon > 0$. By (3.1) there exists $K \in \mathbb{N}$ such that

$$\limsup_n \|x_n - x_m\| \leq 1 + \varepsilon, \quad \forall m \geq K, \tag{3.3}$$

and it follows from the weak lower semicontinuity of the norm that

$$\|x_m\| \leq 1 + \varepsilon, \quad \forall m \geq K.$$

We have, for all $m \geq K$,

$$\limsup_n \Big\|\frac{x_n}{1+\varepsilon} + \frac{x_m}{1+\varepsilon}\Big\| = \limsup_n \Big\|\frac{x_{(n+K)}}{1+\varepsilon} + \frac{x_m}{1+\varepsilon}\Big\| \leq R(1, X) \leq J(X) = \alpha. \tag{3.4}$$

From (3.2), we can find an integer $M \geq K$ such that

$$r(1-\varepsilon) \leq \|x_M\|. \tag{3.5}$$

By definition of r and convexity of C, we must have

$$r \leq \limsup_n \Big\|x_n - \Big(\frac{\alpha-1}{\alpha+1}\Big)x_M\Big\|. \tag{3.6}$$

We can assume, by passing through a subsequence if necessary, that "lim sup" in (3.6) can be replaced by "lim". Now let $\widetilde{X}$ be a Banach space ultrapower of X over an ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Set

$$\tilde{x} = \Big\{\frac{x_n - x_M}{1+\varepsilon}\Big\}_{\mathcal{U}} \text{ and } \tilde{y} = \Big\{\frac{x_n + x_M}{(1+\varepsilon)\alpha}\Big\}_{\mathcal{U}}.$$

(3.3) and (3.4) guarantee that $\tilde{x}$, $\tilde{y} \in B_{\widetilde{X}}$. Consider, by using (3.6),

$$\begin{aligned}
\|\tilde{x}+\tilde{y}\| &= \lim_{\mathcal{U}} \Big\|\frac{x_n - x_M}{1+\varepsilon} + \frac{x_n + x_M}{(1+\varepsilon)\alpha}\Big\| \\
&= \Big(\frac{1}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) \lim_{\mathcal{U}} \Big\|x_n - \Big(\frac{\alpha-1}{\alpha+1}\Big)x_M\Big\| \\
&= \Big(\frac{1}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) \lim_{n} \Big\|x_n - \Big(\frac{\alpha-1}{\alpha+1}\Big)x_M\Big\| \\
&\geq \Big(\frac{1}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) r.
\end{aligned}$$

On the other hand, by using the weak lower semicontinuity of $\|\cdot\|$ and (3.5),

$$\begin{aligned}
\|\tilde{x}-\tilde{y}\| &= \lim_{\mathcal{U}} \Big\|\frac{x_n - x_M}{1+\varepsilon} - \frac{x_n + x_M}{(1+\varepsilon)\alpha}\Big\| \\
&= \Big(\frac{1}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) \lim_{\mathcal{U}} \Big\|\Big(\frac{\alpha-1}{\alpha+1}\Big)x_n - x_M\Big\| \\
&\geq \Big(\frac{1}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) \liminf_{n} \Big\|\Big(\frac{\alpha-1}{\alpha+1}\Big)x_n - x_M\Big\| \\
&\geq \Big(\frac{1}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) \|x_M\| \\
&\geq \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) r.
\end{aligned}$$

It follows from the definition of $J(\widetilde{X})$ that

$$\begin{aligned}
J(\widetilde{X}) &\geq \|\tilde{x}+\tilde{y}\| \wedge \|\tilde{x}-\tilde{y}\| \\
&\geq \Big(\frac{1}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) r \wedge \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) r \\
&= \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)\Big(\frac{\alpha+1}{\alpha}\Big) r.
\end{aligned}$$

According to the fact that $J(X) = J(\widetilde{X})$ ([17, Gao and Lau]) and $\alpha = J(X)$, we have

$$J(X) \geq \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)\Big(\frac{J(X)+1}{J(X)}\Big) r.$$

Thus,

$$\frac{1}{r} \geq \Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)\Big(\frac{J(X)+1}{[J(X)]^2}\Big).$$

Since ε is arbitrary, it follows that

$$\frac{1}{r} \geq \frac{J(X)+1}{[J(X)]^2}.$$

Hence, by Definition 3.1 we conclude that

$$WCS(X) \geq \frac{J(X)+1}{[J(X)]^2}$$

as desired. □

As a consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.3. [9, Theorem 3.1] *Let X be a Banach space. If $J(X) < \frac{1+\sqrt{5}}{2}$, then X has uniform normal structure.*

Proof. Let $\widetilde{X}$ be a Banach space ultrapower of X over an ultrafilter. Since $J(\widetilde{X}) = J(X)$, Theorem 3.2 can be applied to $\widetilde{X}$ and then $WCS(\widetilde{X}) > 1$. Since $WCS(\widetilde{X}) > 1$, $\widetilde{X}$ has weak normal structure [5] and since $\widetilde{X}$ is reflexive, it must be the case that $\widetilde{X}$ has normal structure. By [17, Theorem 5.2], X has uniform normal structure as desired. □

In view of Theorem 2.1, we obtain a fixed point result about asymptotically regular mappings concerning the James constants.

Corollary 3.4. *Suppose X is a Banach space such that $J(X) < \frac{1+\sqrt{5}}{2}$, C is a nonempty closed bounded convex subset of X, and $T : C \to C$ is a uniformly Lipschitzian mapping such that*

$$s(T) < \frac{\sqrt{J(X)+1}}{J(X)}.$$

Suppose in addition that T is asymptotically regular on C. Then T has a fixed point.

Proof. This follows immediately from Theorem 2.1 and Theorem 3.2. □

4. Conclusions and open problems

The objective of this section is to examine what is known, and not known, about fixed point results for several kinds of mappings related to the two constants. In the notion of geometric properties in Banach spaces especially the notions of normal and uniform normal structure, four important kinds of mappings are involved. Let recall their definitions. Let C be a subset of a Banach space X and $T : C \to C$ be a mapping. Firstly, T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\}$ of positive real numbers satisfying $\lim_n k_n = 1$ such that $\|T^n x - T^n y\| \leq k_n \|x - y\|\ \forall x, y \in C, \forall n \in \mathbb{N}$ [19, Goebel and Kirk]. Secondly, if $k_n \equiv 1, \forall n \in \mathbb{N}$, then T is called a nonexpansive mapping. Thirdly, if there exists a constant k such that $k_n \equiv k, \forall n \in \mathbb{N}$, then T is said to be uniformly Lipschitzian. The final one is an asymptotically regular mapping which has already been defined in Section 2. Now we collect fixed point results for such mappings concerning the two constants.

In the following, let C be a closed bounded convex subset of a Banach space X.

Fact 4.1. [28, Mazcuñán-Navarro] *If $J(X) < 2$, equivalently $C_{NJ}(X) < 2$, then every nonexpansive mapping $T : C \to C$ has a fixed point.*

In [24, Theorem 1], Kim and Xu proved that if a Banach space X has uniform normal structure, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point. By combining this theorem with Corollary 3.3, we obtain the following

Fact 4.2. *If $J(X) < \frac{1+\sqrt{5}}{2}$, or $C_{NJ}(X) < \frac{1+\sqrt{3}}{2}$, then every asymptotically nonexpansive mapping $T : C \to C$ has a fixed point.*

In [6], Casini and Maluta proved the existence of fixed points of a uniformly $k-$Lipschitzian mapping T with $k < \sqrt{N(X)}$ in a space X with uniform normal structure. (As before, $N(X)$ is the normal structure coefficient of X.) Prus showed in [29] that $N(X) \geq J(X) + 1 - \sqrt{(J(X)+1)^2 - 4}$ (see also Llorens-Fuster [27]). On the other hand, Kato et al. [23] showed that $N(X) \geq \frac{1}{\sqrt{C_{\mathrm{NJ}}(X) - \frac{1}{4}}}$ (see also [27]). By using the results just mentioned, we now conclude the following results.

Fact 4.3. (1) *Suppose $J(X) < \frac{3}{2}$, and $T : C \to C$ is a uniformly $k-$Lipschitzian mapping such that*

$$k < \sqrt{J(X) + 1 - \sqrt{(J(X)+1)^2 - 4}}.$$

Then T has a fixed point.
(2) *Suppose $C_{NJ}(X) < \frac{5}{4}$, and $T : C \to C$ is a uniformly $k-$Lipschitzian mapping such that*

$$k < \frac{1}{\sqrt{\sqrt{C_{NJ}(X) - \frac{1}{4}}}}.$$

Then T has a fixed point.

We end this paper by posing some open questions about these concepts.

Problem 4.4. Are the upper bounds of the James constants and of the Jordan-von Neumann constants sharp for a space to have uniform normal structure?

Problem 4.5. Does the asymptotically regularity of T in Corollary 3.4 can be dropped?

Problem 4.6. Can the upper bounds of $J(X)$ and $C_{NJ}(X)$ appearing in Fact 4.3 be improved?

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