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EXISTENCE AND UNIQUENESS OF SOME VARIANTS OF NONCONVEX SWEEPING PROCESSES

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Abstract. In this paper we prove the existence and uniqueness of solutions of the following nonconvex variants of the sweeping process:

$$\begin{cases} -\dot{u}(t) \in N(C(t); \dot{u}(t)) \\ u(0) = u_0 \in H, \quad \dot{u}(0) \in C(0), \end{cases}$$

$$\begin{cases} -u(t) \in N(C(t); \dot{u}(t)) \\ u(0) = u_0 \in H, \quad \dot{u}(0) \in C(0), \end{cases}$$

where $C : [0,T] \rightrightarrows H$ is a set valued mapping defined from $[0,T]$ $(T > 0)$ to a Hilbert space H and takes prox-regular values (not necessarily convex).

1. Introduction

In [19], Moreau introduced and studied the following differential inclusion

$$-\dot{u}(t) \in N(C(t); u(t)) \quad \text{a.e. in } [0,T], u(0) = u_0 \in C(0), \tag{1.1}$$

where $C : [0,T] \rightrightarrows H$ is a set valued mapping defined from $[0,T]$ $(T > 0)$ to a Hilbert space H and takes closed **convex** values. $N(C(t); u(t))$ denotes the outward normal cone, in the sense of convex analysis, to the set $C(t)$ at $u(t)$. Thus (1.1) tells us that the velocity $\dot{u}(t)$ of a ball inside a ring has to point inwards to the ring at almost every time $t \in [0,T]$. The initial condition $u(0) \in C(0)$ states that the ball is initially contained in the ring. The differential inclusion (1.1) is known as the sweeping process problem (in French Processus de Rafle). This problem is equivalent to the following evolution variational inequality:
Find $u(t) \in C(t)$ for all $t \in [0,T]$ such that

$$\langle \dot{u}(t), v - u(t) \rangle \geq 0,$$

for all $v \in C(t)$ and for a.e. $t \in [0,T]$. Consequently, the sweeping process includes as a special case the following evolution variational inequality. Find $u(t) \in K$ for all $t \in [0,T]$ such that

$$\langle \dot{u}(t), v - u(t) \rangle \geq \langle f, v - u(t) \rangle, \text{ for a.e. } t \in [0,T] \text{ and for all } v \in K,$$

where K is a closed convex subset of a Hilbert space H, $u : [0,T] \to H$, $f \in L^2_H[0,T]$.

Several extensions of the sweeping process in diverse ways have been done (for details see for example [12, 13, 14, 15, 16, 17, 20, 25, 24, 26]). In [25], the authors have considered the following evolution variational inequality, which is a special

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type of the heat control problem (see for example [2, 3, 7, 9, 10, 11, 18, 22, 23, 24]): Find $u(t)$ such that $\dot{u}(t) \in H^1(\Omega)$ and

$$\langle \dot{u}(t), v - \dot{u}(t) \rangle \geq 0, \text{ for all } v \in H^1(\Omega),$$

which is equivalent, in the convex case, to a special case of the following variant of the sweeping process: Find $u : [0, T] \to H$ such that $\dot{u}(t) \in C(t)$ and

$$\begin{cases} -\dot{u}(t) \in N(C(t); \dot{u}(t)) \\ u(0) = u_0 \in H, \ \ \dot{u}(0) \in C(0). \end{cases} \tag{1.2}$$

They proved the existence and uniqueness of the solution of (1.2), under the convexity assumption of the values of C.

Another different variant is the following: Find $u : [0, T] \to H$ such that $\dot{u}(t) \in C(t)$ and

$$\begin{cases} -u(t) \in N(C(t); \dot{u}(t)) \\ u(0) = u_0 \in H, \ \ \dot{u}(0) \in C(0). \end{cases} \tag{1.3}$$

This variant has been studied in the convex case by some authors (see for instance [16]). It includes as special cases many evolution variational inequalities. We refer the reader to [16] for more details.

The main goal of this paper is to prove the existence and uniqueness of solutions for the two above nonconvex variants of the sweeping process problems, with simple and different proofs than the ones given in the convex case.

2. Preliminaries

Throughout the paper H will denote a *real Hilbert space* .

Let S be a closed subset of H. We denote by $d_S(.)$ or $d(., S)$ the usual distance function to S, i.e., $d_S(x) := \inf_{u \in S} \|x - u\|$. We need first to recall some notation and definitions that will be used in all the paper. Let $C : \mathbb{R} \rightrightarrows H$ be a set-valued mapping from $\mathbb{R}$ to H. We will say that C is Lipschitz continuous with constant $\lambda > 0$ if for any $x \in H$ one has

$$|d_{C(t)}(x) - d_{C(t')}(x')| \leq \|x - x'\| + \lambda |t - t'|, \text{ for any } x, x' \in H \text{ and any } t, t' \in \mathbb{R}.$$

Let $f : H \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous (l.s.c.) function and let x be any point where f is finite. We recall that *the proximal subdifferential* $\partial^P f(x)$ is the set of all $\xi \in H$ for which there exist $\delta, \sigma > 0$ such that for all $x' \in x + \delta\mathbb{B}$

$$\left\langle \xi, x' - x \right\rangle \leq f(x') - f(x) + \sigma \|x' - x\|^2.$$

Here $\mathbb{B}$ denotes the closed unit ball centered at the origin of H.

By convention we set $\partial^P f(x) = \emptyset$ if $f(x)$ is not finite. Note that $\partial^P f(x)$ is always convex but may be non closed.

Let S be a nonempty closed subset of H and x be a point in S. We recall (see [8]) that *the proximal normal cone* of S at x is defined by $N^P(S; x) := \partial^P \psi_S(x)$, where ψ_S denotes the indicator function of S, i.e., $\psi_S(x) = 0$ if $x \in S$ and $+\infty$ otherwise. Note that the proximal normal cone is also given by

$$N^P(S; x) = \{\xi \in H : \exists \alpha > 0 \text{ s.t. } x \in \text{Proj}(x + \alpha\xi, S)\},$$

where

$$\mathrm{Proj}(u, S) := \{y \in S : d_S(u) = \|u - y\|\}.$$

We recall the following proposition needed in the sequel.

Proposition 2.1 ([5]). *Let S be a nonempty closed subset in H. Then*

$$\partial^P d_S(x) = N^P(S; x) \cap \mathbb{B}, \text{ for all } x \in S.$$

Recall now that for a given $r \in]0, +\infty]$ a subset S is uniformly r-prox-regular (see [8, 21]) if and only if every nonzero proximal normal to S can be realized by an r-ball, this means that for all $\bar{x} \in S$ and all $0 \neq \xi \in N^P(S; \bar{x})$ one has

$$\left\langle \frac{\xi}{\|\xi\|}, x - \bar{x} \right\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2,$$

for all $x \in S$. We make the convention $\frac{1}{r} = 0$ for $r = +\infty$. Recall that for $r = +\infty$ the uniform r-prox-regularity of S is equivalent to the convexity of S.

For concrete examples of uniform prox-regular nonconvex sets, we state the followings:

(1) The union of two disjoint intervals $[a, b]$ and $[c, d]$ $(c > b)$ is nonconvex but uniformly r-prox-regular with $r = \frac{c-b}{2}$.
(2) The finite union of disjoint intervals is also nonconvex but uniformly r-prox-regular and the r depends on the distances between the intervals. For more examples we refer the reader to [6].

The following proposition summarizes some important consequences of the uniform prox-regularity needed in the sequel. For the proof of these results we refer the reader to [8, 21] .

Proposition 2.2. *Let S be a nonempty closed subset in H and let $r \in]0, +\infty]$. If the subset S is uniformly r-prox-regular then the following hold:*

i) For all $x \in H$ with $d_S(x) < r$, one has $\mathrm{Proj}(x, S) \neq \emptyset$;

ii) The proximal subdifferential of d_S coincides with all the subdifferentials contained in the Clarke subdifferential at all points $x \in H$ satisfying $d_S(x) < r$. So, in such case, the subdifferential $\partial d_S(x) := \partial^P d_S(x) = \partial^C d_S(x)$ is a closed convex set in H.

As a consequence of (*ii*) we get that for uniformly r-prox-regular sets, the proximal normal cone to S coincides with all the normal cones contained in the Clarke normal cone at all points $x \in S$, i.e., $N^P(S; x) = N^C(S; x)$. In such case, we put $N(S; x) := N^P(S; x) = N^C(S; x)$. Here $\partial^C d_S(x)$ and $N^C(S; x)$ denote respectively the Clarke subdifferential of d_S and the Clarke normal cone to S (see [8] for their definitions and properties).

In [6], the authors established various new characterizations of the uniform prox-regularity in terms of the subdifferential of the distance function. We recall here one of their consequences that will be used in the proof of the next theorem.

Proposition 2.3 ([6]). *Let S be a nonempty closed subset in H and let $r \in (0, +\infty]$. Assume that S is uniformly r-prox-regular. Then for all $x \in S$ and all $\xi \in \partial d_S(x)$*

one has

$$(P_r)\qquad \begin{cases}\text{for all } x \in S \text{ and all } \xi \in \partial d_S(x) \text{ one has}\\ \langle \xi, x' - x\rangle \le \dfrac{2}{r}\|x' - x\|^2 + d_S(x'),\\ \text{for all } x' \in H \text{ with } d_S(x') \le r.\end{cases}$$

We close this section with the following theorem by Bounkhel and Thibault [6]. We give the proof here for the convenience of the reader. It proves a very important closedness property of the subdifferential of the distance function associated with a set-valued mapping. Another version of this result is given in [4] to study some nonconvex economic models.

Theorem 2.1. *Let $r \in]0, +\infty]$, Ω be an open subset in $\mathbb{R}$, and $K : \Omega \rightrightarrows H$ be a Lipschitz continuous set-valued mapping. Assume that $C(t)$ is uniformly r-prox-regular for all t in Ω. Then for a given $0 < \delta < r$ the following holds:*

"for any $\bar{t} \in \Omega$, $\bar{u} \in C(\bar{t})$, $u_n \to \bar{u}$, $t_n \to \bar{t}$ with $t_n \in \Omega$ and with $u_n \in C(t_n)$, and $\xi_n \in \partial d_{C(t_n)}(u_n)$ with $\xi_n \to^w \bar{\xi}$ one has $\bar{\xi} \in \partial d_{C(\bar{t})}(\bar{u})$". Here $\to^w$ means the weak convergence in H.

Proof. Fix $\bar{t} \in \Omega$, and $\bar{u} \in C(\bar{t})$. As $u_n \to \bar{u}$ and $t_n \to \bar{t}$ one gets for n sufficiently large $\|u_n - \bar{u}\| \le \frac{r}{4}$ and $|t_n - \bar{t}| \le \frac{r}{4\lambda}$. So, one can write by the Lipschitz property of C,

$$d_{C(t_n)}(u_n) = d_{C(t_n)}(u_n) - d_{C(\bar{t})}(\bar{u}) \le \lambda |t_n - \bar{t}| + \|u_n - \bar{u}\|,$$

and hence we get for n large enough

$$d_{C(t_n)}(u_n) \le \lambda \frac{r}{4\lambda} + \|u_n - \bar{u}\| \le \frac{r}{4} + \frac{r}{4} = \frac{r}{2} < r.$$

Therefore, for any n large enough, we apply the property (P_r) in Proposition 2.3 with $\xi_n \in \partial d_{C(t_n)}(u_n)$ to get

$$\langle \xi_n, u - u_n\rangle \le \frac{2}{r}\|u - u_n\|^2 + d_{C(t_n)}(u), \tag{2.1}$$

for all $u \in H$ with $d_{C(t_n)}(u) < r$. This inequality still holds for all $u \in \bar{u} + \delta\mathbb{B}$ with $0 < \delta < \frac{r}{4}$ because for such u one has

$$d_{C(t_n)}(u) \le \|u - \bar{u}\| + \|\bar{u} - u_n\| + d_{C(t_n)}(u_n) \le \delta + \frac{r}{4} + \frac{r}{2} < r.$$

Consequently, by the Lipschitz continuity of the distance function with respect to (t, x), the inequality (2.1) gives, by letting $n \to +\infty$,

$$\langle \bar{\xi}, u - \bar{u}\rangle \le \frac{2}{r}\|u - \bar{u}\|^2 + d_{C(\bar{t})}(u) - d_{C(\bar{t})}(\bar{u}) \text{ for all } u \in \bar{u} + \delta'\mathbb{B}.$$

This ensures that $\bar{\xi} \in \partial d_{C(\bar{t})}(\bar{u})$ and so the proof of the theorem is complete. □

Remark 2.1. As a direct consequence of this theorem we have the upper semicontinuity of the set-valued mapping $(t, x) \mapsto \partial d_{C(t)}(x)$ from $gph\, C \subset \mathbb{R} \times H$ to H endowed with the weak topology, which is equivalent (see for example Proposition 1.4.1 and Theorem 1.4.2 in [1]) to the u.s.c. of the function $(t, x) \mapsto \sigma(\partial d_{C(t)}(x), p)$

for any $p \in H$. Here $\sigma(S,p)$ denotes the support function associated with S, i.e., $\sigma(S,p) := \sup_{s\in S} \langle s, p\rangle$.

3. Main results

We state our main results that will be proved in the next section.

Theorem 3.1. *Let $r \in (0,+\infty]$. Assume that $C : [0,T] \rightrightarrows H$ is a Lipschitz set-valued mapping with r-prox-regular values and assume that $C(t) \subset \mathcal{K}$ for any $t \in [0,T]$, for some convex compact set $\mathcal{K}$. Then (1.3) has at least one solution.*

Theorem 3.2. *Let $r \in (0,+\infty]$. Assume that $C : [0,T] \rightrightarrows H$ is a Lipschitz set-valued mapping with r-prox-regular values and assume that $C(t) \subset \mathcal{K}$ for any $t \in [0,T]$, for some convex compact set $\mathcal{K}$. Assume also that C satisfies $0 \in C(t) + r\mathbb{B}$, for any $t \in [0,T]$. Then (1.2) has at least one solution.*

Remark 3.1. Note that the assumption $0 \in C(t)+r\mathbb{B}$, for any $t \in [0,T]$ in Theorem 3.2 is always satisfied in the convex case. Indeed, if C has convex values, then $r = +\infty$ and so $C(t)+r\mathbb{B}$ is the whole space H. For the nonconvex case, we can take the following example for which this assumption is satisfied. Take $T = 1$, $H = \mathbb{R}^2$ with the Euclidean norm, $C(t) = \{(x_1, x_2) \in H : \frac{1}{2} \leq \|(x_1, x_2) - (0, 2)\| \leq \frac{7}{2} + t\}$. This set is not convex but it is r-prox-regular with $r = \frac{1}{4}$. It is easy to see that $C(t) + r\mathbb{B} = \{(x_1, x_2) \in H : \frac{1}{4} \leq \|(x_1, x_2) - (0, 2)\| \leq 2 + t\}$. It is also clear that $(0, 0) \in C(t) + r\mathbb{B}$, for any $t \in [0,T]$. It is interesting to point out that we do not need 0 to be in $C(t)$ for all $[0,T]$.

4. Proofs of main results

Proof of Theorem 3.1. Put $I := [0,T]$. Fix $n_0 \geq 1$ satisfying

$$\frac{\lambda T}{2^{n_0}} \leq \frac{r}{2}.$$

For every $n \geq n_0$, we put

$$\mu_n := \frac{T}{2^n},$$

and we consider the following partition of I:

$$\begin{cases} t_{n,i} := i\mu_n, \quad (\text{ for } 0 \leq i \leq 2^n,) \text{ and} \\ I_{n,i+1} :=]t_{n,i}, t_{n,i+1}], (\text{ for } 0 \leq i \leq 2^n - 1.) \text{ and } I_{n,0} := \{t_{n,0}\}. \end{cases}$$

For every $n \geq n_0$, we choose by induction

Algorithm 4.1.

- •) $z_{n,0} := u_0 \in C(0)$ and $u_{n,0} := z_{n,0}$;
- •) $\underline{i \geq 0}$: $z_{n,i+1} = Proj_{C(t_{n,i+1})}(z_{n,i})$ and $u_{n,i+1} := u_{n,i} + \mu_n z_{n,i}$.

This algorithm is well defined. Indeed, for $i = 0$, we have by the Lipschitz property of C

$$\begin{aligned} d(u_{n,0}, C(t_{n,1})) &= d(u_{n,0}, C(t_{n,1})) - d(u_{n,0}, C(t_{n,0})) \\ &\leq \lambda|t_{n,1} - t_{n,0}| = \lambda\mu_n = \frac{\lambda T}{2^n} \leq \frac{\lambda T}{2^{n_0}} \leq \frac{r}{2} < r. \end{aligned}$$

The prox-regularity of the set $C(t_{n,1})$ ensures by Proposition 2.2 part (i), the existence and the uniqueness of the projection $Proj_{C(t_{n,1})}(z_{n,0})$ and then we can take $z_{n,1} = Proj_{C(t_{n,1})}(z_{n,0})$ and $u_{n,1} := u_{n,0} + \mu_n z_{n,0}$. Assume now that $i \geq 1$. We have by Algorithm 4.1, $z_{n,i} \in C(t_{n,i})$ and so by the Lipschitz property of C we get

$$\begin{aligned} d(z_{n,i}, C(t_{n,i+1})) &= d(z_{n,i}, C(t_{n,i+1})) - d(z_{n,i}, C(t_{n,i})) \\ &\leq \lambda |t_{n,i+1} - t_{n,i}| = \lambda \mu_n = \frac{\lambda T}{2^n} \leq \frac{\lambda T}{2^{n_0}} \leq \frac{r}{2} < r, \end{aligned}$$

which ensures by the prox-regularity of the set $C(t_{n,i+1})$ and Proposition 2.2 part (i), the existence and the uniqueness of the projection $Proj_{C(t_{n,i+1})}(z_{n,i})$ and hence we can take $z_{n,i+1} = Proj_{C(t_{n,i+1})}(z_{n,i})$ and $u_{n,i+1} := u_{n,i} + \mu_n z_{n,i}$.
Now we use the sequence to construct a sequence of mappings u_n from I to H by defining their restrictions to each interval $I_{n,i}$ as follows:

- For $t \in I_{n,0}$, set $u_n(t) = u_{n,0} = u_0$;
- For $t \in I_{n,i+1}$, $(0 \leq i \leq 2^n - 1)$ set

$$u_n(t) = \frac{z_{n,i+1} - z_{n,i}}{\mu_n} + z_{n,i+1}(t - t_{n,i+1}). \tag{4.1}$$

It is clear by construction that u_n is absolutely continuous on I and differentiable a.e. on I with

$$\dot{u}_n(t) = z_{n,i+1}, \text{ a.e. on } I. \tag{4.2}$$

By Algorithm 4.1 and the definition of the proximal normal cone we have

$$z_{n,i+1} - z_{n,i} \in -N^P(C(t_{n,i+1}); z_{n,i+1})$$

and by (4.1) and (4.2) we obtain

$$u_n(t_{n,i+1}) = \frac{z_{n,i+1} - z_{n,i}}{\mu_n} \in -N^P(C(t_{n,i+1}); \dot{u}_n(t)) \tag{4.3}$$

Now let us define the step function from I to I by

$$\theta_n(t) = t_{n,i+1}, t \in I_{n,i+1}. \tag{4.4}$$

Then we get by (4.3) and (4.4)

$$u_n(\theta_n(t)) \in -N^P(C(\theta_n(t)); \dot{u}_n(t)), \text{ a.e. on } I.$$

On the other hand, we have

$$\begin{aligned} \|u_n(\theta_n(t))\| &= \left\| \frac{z_{n,i+1} - z_{n,i}}{\mu_n} \right\| = \frac{1}{\mu_n} d_{C(t_{n,i+1})}(z_{n,i}) \\ &= \frac{1}{\mu_n} \left[d_{C(t_{n,i+1})}(z_{n,i}) - d_{C(t_{n,i})}(z_{n,i}) \right] \\ &\leq \frac{1}{\mu_n} \lambda |t_{n,i+1} - t_{n,i}| = \lambda, \end{aligned} \tag{4.5}$$

and so by Proposition 2.1 we obtain

$$u_n(\theta_n(t)) \in -\lambda \partial^P d_{C(\theta_n(t))}(\dot{u}_n(t)), \text{ a.e. on } I. \tag{4.6}$$

We define now the piecewise affine mapping:

$$v_n(t) = z_{n,i+1} + \frac{t - t_{n,i+1}}{\mu_n}(z_{n,i+1} - z_{n,i}), \quad t \in I_{n,i+1}. \tag{4.7}$$

Observe that

$$v_n(\theta_n(t)) = z_{n,i+1} \in C(\theta_n(t)) \subset \mathcal{K} \subset l\mathbb{B}.$$

Now, we show that the mappings v_n are equilipschitz with ratio λ. Indeed, for any $t, s \in I_{n,i+1}$ we have (4.7) and (4.5)

$$\|v_n(t) - v_n(s)\| = \left\| \frac{z_{n,i+1} - z_{n,i}}{\mu_n} \right\| |t - s| \leq \lambda \|t - s|.$$

It is also clear, by the construction of v_n and u_n that

$$\|v_n(t) - \dot{u}_n(t)\| = \left\| \frac{z_{n,i+1} - z_{n,i}}{\mu_n} \right\| |t - t_{n,i+1}| \leq \lambda |t - t_{n,i+1}| \leq \lambda \mu_n, \tag{4.8}$$

and hence $\|v_n - \dot{u}_n\|_\infty \to 0$, as $n \to +\infty$.
Uniform convergence of v_n. Observe that

$$\|\dot{v}_n(t)\| = \left\| \frac{z_{n,i+1} - z_{n,i}}{\mu_n} \right\| \leq \lambda, \tag{4.9}$$

and

$$v_n(t) = \left(\frac{t_{n,i+1} - t}{\mu_n} \right) z_{n,i} + \left(1 - \frac{t_{n,i+1} - 1}{\mu_n} \right) z_{n,i+1} \in \mathcal{K}.$$

Thus, for every $t \in I$, the set $\{v_n(t) : \ n \geq 1\}$ is relatively strongly compact in H. Therefore, the estimate (4.9) and Theorem 0.4.4 in [1] ensure that there exists a Lipschitz mapping $v : I \to H$ with ratio λ such that $v_n \to v$ uniformly on I. This with (4.8) prove the uniform convergence of $\dot{u}_n$ to v on I.
Now, we define the Lipschitz mapping $u : I \to H$ as follows

$$u(t) = u_0 + \int_0^t v(s) ds, \qquad \text{for all } t \in I.$$

Then $\dot{u}(t) = v(t)$ a.e. on I. By the definition of u_n and u we obtain for all $t \in I$

$$\|u_n(t) - u(t)\| = \| \int_0^t (\dot{u}_n(s) - v(s)) ds\| \leq T \|\dot{u}_n - v\|_\infty,$$

and so by (4.8) we get

$$\|u_n - u\|_\infty \leq T \|\dot{u}_n - v\|_\infty \leq T \|\dot{u}_n - v_n\|_\infty + T \|v_n - v\|_\infty \to 0 (\text{ as } n \to +\infty).$$

This proves the uniform convergence of u_n to u on I. Since $|\theta_n(t) - t| < \mu_n$ on I, then $\theta_n(t) \to t$ uniformly on I and so $u_n(\theta_n(\cdot))$ converges uniformly to u on I. Now, as $\dot{u}_n(t) \in C(\theta_n(t))$, a.e. on I, we get for a.e. $t \in I$

$$\begin{aligned} d(\dot{u}(t); C(t)) &= d(\dot{u}(t); C(t)) - d(\dot{u}_n(t); C(\theta_n(t))) \\ &\leq \|\dot{u}_n(t) - \dot{u}(t)\| + \lambda |\theta_n(t) - t| \to 0, \text{ as } n \to +\infty \end{aligned}$$

and since $C(t)$ is closed we get

$$\dot{u}(t) \in C(t), \text{ a.e. on } I. \tag{4.10}$$

Fix now any $\zeta \in H$ and any t for which $\dot{u}$ and $\dot{u}_n$ exist and for which (4.6) is satisfied. Then the uniform convergence of $u_n(\theta_n(\cdot))$ to u entails

$$u(t) \in \bigcap_n \{u_k(\theta_k(t)) : \ k \geq n\},$$

and so

$$\langle u(t), \zeta \rangle \leq \inf_{n \geq 1} \sup_{k \geq n} \langle u_k(\theta_k(t)), \zeta \rangle = \limsup_n \langle u_n(\theta_n(t)), \zeta \rangle.$$

Hence by (4.6), one obtains

$$\langle u(t), \zeta \rangle \leq \limsup_n \sigma \left(-\lambda \partial^P d_{C(\theta_n(t))}(\dot{u}_n(t)), \zeta \right).$$

It follows then by Remark 2.1 and Theorem 2.1 that

$$\langle u(t), \zeta \rangle \leq \sigma \left(-\lambda \partial^P d_{C(t)}(\dot{u}(t)), \zeta \right).$$

Finally, by Proposition 2.2 part (ii), we have $\partial^P d_{C(t)}(\dot{u}(t))$ is a closed convex set in H and so the last inequality entails

$$u(t) \in -\lambda \partial^P d_{C(t)}(\dot{u}(t)). \text{ a.e. on } I,$$

Finally, by (4.10) and Proposition 2.1 we get

$$u(t) \in -\lambda \partial^P d_{C(t)}(\dot{u}(t)) \subset -N^P(C(t); \dot{u}(t)) = -N(C(t); \dot{u}(t)), \text{ a.e. on } I.$$

This completes the proof of the existence. □

Proof of Theorem 3.2. Put $I := [0, T]$. Fix $n_0 \geq 1$ such that $n_0 > T$. For every $n > n_0$, we put $\mu_n := \frac{T}{n} < 1$, and we consider the following partition of I:

$$\begin{cases} t_{n,i} := i\mu_n, & (\text{ for } 0 \leq i \leq n,) \text{ and} \\ I_{n,i+1} :=]t_{n,i}, t_{n,i+1}], (\text{ for } 0 \leq i \leq n-1.) \text{ and } I_{n,0} := \{t_{n,0}\} \end{cases}$$

For every $n > n_0$, we choose by induction $u_{n,0} := u_0$, and $u_{n,i+1} = u_{n,i} + \mu_n Proj_{C(t_{n,i+1})}(0)$. This induction is well defined by using the assumption $0 \in C(t) + r\mathbb{B}$, for all $t \in I$. The sequence $\{u_{n,i}\}_i$ is used to construct a sequence of mappings $\{u_n\}_n$ from I to H by defining their restrictions to each interval $\{I_{n,i}\}$ as follows:

$$u_n(t) = u_{n,i} + \frac{t - t_{n,i}}{\mu_n}(u_{n,i+1} - u_{n,i}), \quad t \in I_{n,i+1}.$$

It is clear by construction that

$$\dot{u}_n(t) = \frac{u_{n,i+1} - u_{n,i}}{\mu_n} = Proj_{C(t_{n,i+1})}(0), \quad \text{a. e. } t \in I.$$

Thus by the definition of the proximal normal cone we get

$$-\dot{u}_n(t) \in N^P(C(t_{n,i+1}); \dot{u}_n(t)), \quad \text{a. e. } t \in I. \tag{4.11}$$

Let $\theta_n : I \to I$ by $\theta_n(t) = t_{n,i+1}$, for all $t \in I_{n,i+1}$. Then (4.11) is equivalent to

$$-\dot{u}_n(t) \in N^P(C(\theta_n(t)); \dot{u}_n(t)), \quad \text{a. e. } t \in I. \tag{4.12}$$

On the other hand, since $\dot{u}_n(t) \in C(\theta_n(t)) \subset \mathcal{K} \subset l\mathbb{B}$ we obtain $\|\dot{u}_n(t)\| \leq l$ and so by Proposition 2.1 we get

$$-\dot{u}_n(t) \in l\partial^P d_{C(\theta_n(t))}(\dot{u}_n(t)), \quad \text{a. e. } t \in I. \tag{4.13}$$

We define now the piecewise affine mapping from I to H by defining their restrictions to each interval $\{I_{n,i}\}$ as follows:

$$v_n(t) = \frac{u_{n,i+1} - u_{n,i}}{\mu_n} + \mu_n(t - t_{n,i+1})(u_{n,i+1} - u_{n,i}), \quad t \in I_{n,i+1}.$$

Observe that

$$v_n(\theta_n(t)) = \frac{u_{n,i+1} - u_{n,i}}{\mu_n},$$

and so

$$v_n(\theta_n(t)) = \frac{u_{n,i+1} - u_{n,i}}{\mu_n} \in C(\theta_n(t)) \subset \mathcal{K} \subset l\mathbb{B}.$$

Now, we show that the mappings v_n are equilipschitz with ratio l. Indeed, for any $t, s \in I_{n,i+1}$ we have by construction

$$\begin{aligned}\|v_n(t) - v_n(s)\| &= \mu_n|t - s|\|u_{n,i+1} - u_{n,i}\| \\ &= \mu_n\|\mu_n Proj_{C(t_{n,i+1})}(0)\||t - s| \leq l\mu_n^2|t - s| \leq l|t - s|.\end{aligned}$$

It is also clear, by the construction of v_n and u_n that

$$\|v_n(t) - \dot{u}_n(t)\| = \mu_n|t - t_{n,i+1}|\|u_{n,i+1} - u_{n,i}\| \leq l\mu_n^3, \tag{4.14}$$

and hence $\|v_n - \dot{u}_n\|_\infty \to 0$, as $n \to +\infty$.
Uniform convergence of v_n. Observe that

$$\|\dot{v}_n(t)\| = \mu_n\|u_{n,i+1} - u_{n,i}\| \leq l\mu_n^2 \leq l, \tag{4.15}$$

and

$$\begin{aligned}v_n(t) &= \left[1 + \mu_n^2(t - t_{n,i+1})\right] \frac{u_{n,i+1} - u_{n,i}}{\mu_n} \\ &\in \left[1 + \mu_n^2(t - t_{n,i+1})\right] C(t_{n,i+1}) \\ &\subset \left[1 + \mu_n^2(t - t_{n,i+1})\right] \mathcal{K} \\ &\subset 2\mathcal{K}.\end{aligned}$$

Thus, for every $t \in I$, the set $\{v_n(t) : \ n \geq 1\}$ is relatively strongly compact in H. Therefore, the estimate (4.15) and Theorem 0.4.4 in [1] ensure the existence of a Lipschitz mapping $v : I \to H$ with ratio l such that $v_n \to v$ uniformly on I. This with (4.14) ensure the uniform convergence of $\dot{u}_n$ to v on I.

Now, we define the Lipschitz mapping $u : I \to H$ as follows

$$u(t) = u_0 + \int_0^t v(s)ds, \quad \text{for all } t \in I.$$

Then $\dot{u}(t) = v(t)$ a.e. on I. By the definition of u_n and u we obtain for all $t \in I$

$$\|u_n(t) - u(t)\| = \|\int_0^t (\dot{u}_n(s) - v(s))ds\| \leq T\|\dot{u}_n - v\|_\infty,$$

and so by (4.14) we get

$$\|u_n - u\|_\infty \le T\|\dot u_n - v\|_\infty \le T\|\dot u_n - v_n\|_\infty + T\|\dot v_n - v\|_\infty \to 0(\text{ as } n \to +\infty).$$

This proves the uniform convergence of u_n to u and v on I.

Fix now any $\zeta \in H$ and any t for which (4.13) is satisfied. Then

$$\dot u(t) \in \bigcap_n \{\dot u_k(t) : \ k \ge n\},$$

and so

$$\langle \dot u(t), \zeta\rangle \le \inf_{n\ge 1}\sup_{k\ge n}\langle \dot u_k(t), \zeta\rangle = \limsup_n \langle \dot u_n(t), \zeta\rangle.$$

Hence by (4.13), one obtains

$$\langle \dot u(t), \zeta\rangle \le \limsup_n \sigma\left(-l\partial^P d_{C(\theta_n(t))}(\dot u_n(t)), \zeta\right),$$

Since $|\theta_n(t) - t| < \mu_n$ on I, then $\theta_n(t) \to t$ uniformly on I. It follows then by Remark 2.1 and Theorem 2.1 that

$$\langle \dot u(t), \zeta\rangle \le \sigma\left(-l\partial^P d_{C(t)}(\dot u(t)), \zeta\right).$$

Finally, by Proposition 2.2 part (ii) we have $\partial^P d_{C(t)}(\dot u(t))$ is a closed convex set in H and so the last inequality entails

$$-\dot u(t) \in l\partial^P d_{C(t)}(\dot u(t)). \text{ a.e. on } I.$$

On the other hand, by (4.12) we have $\dot u_n(t) \in C(\theta_n(t))$ a.e. $t \in I$ and so

$$\begin{aligned} d(\dot u(t), C(t)) &= d(\dot u(t), C(t)) - d(\dot u_n(t), C(\theta_n(t))) \\ &\le \|\dot u_n(t) - \dot u(t)\| + \lambda|t - \theta_n(t)| \to 0 \text{ (as } n \to +\infty) \end{aligned}$$

and since $C(t)$ is closed we get $\dot u(t) \in C(t)$. Therefore,

$$-\dot u(t) \in l\partial^P d_{C(t)}(\dot u(t)) \subset N^P(C(t); \dot u(t)), \text{ a.e. on } I.$$

Thus completing the proof of Theorem 3.2. □

5. Uniqueness of Solutions

We start with the uniqueness of solutions for the problem (1.2).

Theorem 5.1. *Under the same assumptions of Theorem 3.2, the problem (1.2) has a unique solution, whenever* $0 < l < r$. *Here the positive scalar* l *is as in the proof of Theorem 3.2, that is* $\mathcal{K} \subset l\mathbb{B}$.

Note that the additional assumption $0 < l < r$ is always satisfied in the convex case, since in such case we have $r = +\infty$.

Proof. Let u_1 and u_2 be two solutions of (1.2) with the same initial value $u_1(0) = u_2(0) = u_0$. Then

$$-\dot u_1(t) \in N(C(t); \dot u_1(t)) \text{ and } -\dot u_2(t) \in N(C(t); \dot u_2(t)).$$

Observe from the proof of Theorem 3.2, that the derivative of any solution of (1.2) is bounded by l almost everywhere. Then the above relations with Proposition 2.1 yield

$$-\dot u_1(t) \in l\partial d_{C(t)}(\dot u_1(t)) \text{ and } -\dot u_2(t) \in l\partial d_{C(t)}(\dot u_2(t)).$$

Using now Proposition 2.3, we obtain for any $v \in C(t)$

$$\langle -\dot{u}_1(t), v - \dot{u}_1(t)\rangle \leq \frac{l}{2r}\|v - \dot{u}_1(t)\|^2 \tag{5.1}$$

and

$$\langle -\dot{u}_2(t), v - \dot{u}_2(t)\rangle \leq \frac{l}{2r}\|v - \dot{u}_2(t)\|^2. \tag{5.2}$$

Put $v = \dot{u}_2(t)$ and $v = \dot{u}_1(t)$ respectively in (5.1) and (5.2) then we get

$$\langle -\dot{u}_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle \leq \frac{l}{2r}\|\dot{u}_2(t) - \dot{u}_1(t)\|^2 \tag{5.3}$$

$$\langle -\dot{u}_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \leq \frac{l}{2r}\|\dot{u}_1(t) - \dot{u}_2(t)\|^2. \tag{5.4}$$

From (5.3) and (5.4) we get

$$\left(1 - \frac{l}{r}\right)\langle \dot{u}_1(t) - \dot{u}_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \leq 0.$$

By the assumption $r > l > 0$, we get $\dot{u}_1(t) = \dot{u}_2(t)$, a.e. on I. Therefore,

$$u_1(t) = u_0 + \int_0^t \dot{u}_1(s)ds = u_0 + \int_0^t \dot{u}_2(s)ds = u_2(t),$$

for any $t \in I$. The uniqueness then is proved. □

Now, we prove the uniqueness of solutions for the problem (1.3).

Theorem 5.2. *Under the same assumptions of Theorem 3.1, the problem (1.3) has a unique solution, whenever C has convex values, that is, $r = +\infty$.*

Proof. Let u_1 and u_2 be two solutions of (1.3) with the same initial value $u_1(0) = u_2(0) = u_0$. Then

$$-u_1(t) \in N(C(t); \dot{u}_1(t)) \text{ and } -u_2(t) \in N(C(t); \dot{u}_2(t)).$$

Observe from the proof of Theorem 3.1, that the derivative of any solution of (1.3) is bounded by l almost everywhere and then any solution is bounded by $\beta := u_0 + lT$. Then the above relations with Proposition 2.1 yield

$$-u_1(t) \in \beta\partial d_{C(t)}(\dot{u}_1(t)) \text{ and } -u_2(t) \in \beta\partial d_{C(t)}(\dot{u}_2(t)).$$

Using now Proposition 2.3 and the assumption C has convex values, we obtain for any $v \in C(t)$

$$\langle -u_1(t), v - \dot{u}_1(t)\rangle \leq \frac{l}{2r}\|v - \dot{u}_1(t)\|^2 = 0 \tag{5.5}$$

and

$$\langle -\dot{u}_2(t), v - \dot{u}_2(t)\rangle \leq \frac{l}{2r}\|v - \dot{u}_2(t)\|^2 = 0. \tag{5.6}$$

Put $v = \dot{u}_2(t)$ and $v = \dot{u}_1(t)$ respectively in (5.5) and (5.6) then we get

$$\langle -u_1(t), \dot{u}_2(t) - \dot{u}_1(t)\rangle \leq 0 \tag{5.7}$$

$$\langle -u_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \leq 0. \tag{5.8}$$

From (5.7) and (5.8) we get

$$\langle u_1(t) - u_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \leq 0.$$

Put $w(t) = \|u_1(t) - u_2(t)\|^2$. Then,

$$\dot{w}(t) = 2\langle u_1(t) - u_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \leq 0, \text{ a.e. on } I.$$

Integrating over I, yields for any $t \in I$,

$$w(t) = w(0) + \int_0^t \dot{w}(s)ds \leq w(0) = \|u_1(0) - u_2(0)\|^2 = 0.$$

The uniqueness then is proved. □

Remark 5.1. The uniqueness of solutions for the problem is not true in the noconvex case, even in the uniform prox regular case. Take for example $T = 1$, $H = \mathbb{R}$, $C(t) = [0, 1] \cup [2, t + 3]$, and $u_0 = 2$. Using the fact that the union of two disjoint intervals is r-prox-regular mentioned in Section 2, we obtain that $C(t)$ is r-prox-regular with $r = \frac{1}{2}$, for all $t \in I = [0, 1]$. It is not difficult to check that C is a Lipschitz set-valued mapping with ratio $\lambda = 1$. Also, it is clear that all the other assumptions of Theorem 3.1 are satisfied. Consider the mapping $u_1 : I \to \mathbb{R}$ defined by $u_1(t) = 2$. Easily, we have $\dot{u}_1(t) = 0 \in C(t)$, $\forall t \in I$, $N(C(t); \dot{u}_1(t)) = N(C(t); 0) = (-\infty, 0]$, $-u_1(t) = -2 \in N(C(t); \dot{u}_1(t))$, $\forall t \in I$, and so u_1 is a solution of (1.3) with the initial value $u_1(0) = 2$. Consider now the mapping $u_2 : I \to \mathbb{R}$ defined by $u_2(t) = 2 + 2t$. Easily, we have $\dot{u}_2(t) = 2 \in C(t)$, $\forall t \in I$, $N(C(t); \dot{u}_2(t)) = N(C(t); 2) = (-\infty, 0]$, $-u_2(t) = -2(1 + t) \in N(C(t); \dot{u}_1(t))$, $\forall t \in I$, and so u_2 is another solution of (1.3) with the same initial value $u_2(0) = 2$. This proves that there is no uniqueness in the nonconvex case even in the prox-regular case.

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