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A GENERIC WELL-POSEDNESS RESULT FOR A CLASS OF NONCONVEX OPTIMAL CONTROL PROBLEMS

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Abstract. The Tonelli existence theorem in the calculus of variations and its subsequent modifications were established for integrands f which satisfy convexity and growth conditions. In our previous work a generic existence and uniqueness result (with respect to variations of the integrand of the integral functional) without the convexity condition was established for a class of optimal control problems satisfying the Cesari growth condition. In this paper we extend this generic existence and uniqueness result to a class of optimal control problems in which the right-hand side of differential equations and constraint maps are also subject to variations.

1. Introduction

The Tonelli existence theorem in the calculus of variations [20] and its subsequent generalizations and extensions (e.g. [6, 7, 12, 16]) are based on two fundamental hypotheses concerning the behavior of the integrand as a function of the last argument (derivative): one that the integrand should grow superlinearly at infinity and the other that it should be convex (or exhibit a more special convexity property in case of a multiple integral with vector-valued functions) with respect to the last variable. Moreover, certain convexity assumptions are also necessary for properties of lower semicontinuity of integral functionals which are crucial in most of the existence proofs, although there are some interesting theorems without convexity (see [6, Ch. 16] and [3, 5, 8, 14, 15, 19, 21]).

In [22-27] it was shown that the convexity condition is not needed generically, and not only for the existence but also for the uniqueness of a solution and even for well-posedness of the problem (with respect to some natural topology in the space of integrands). Instead of considering the existence of a solution for a single integrand f, we investigated it for a space of integrands and showed that a unique solution exists for most of the integrands in the space. Such approach is often used in many situations when a certain property is studied for the whole space rather than for a single element of the space. See, for example, [1, 17, 28] and the references mentioned there. Interesting generic existence results were obtained for particular cases of variational problems [4, 13]. Important generic existence and uniqueness result for a class of nonconvex Mayer type optimal control problems with smooth cost functions was obtained in [11].

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In [23] this approach allowed us to establish the generic existence of solutions for a large class of optimal control problems without convexity assumptions. More precisely, in [23] we considered a class of optimal control problems (with the same system of differential equations, the same functional constraints and the same boundary conditions) which is identified with the corresponding complete metric space of cost functions (integrands), say $\mathcal{F}$. We did not impose any convexity assumptions. These integrands are only assumed to satisfy the Cesari growth condition. The main result in [23] establishes the existence of an everywhere dense G_δ-set $\mathcal{F}' \subset \mathcal{F}$ such that for each integrand in $\mathcal{F}'$ the corresponding optimal control problem has a unique solution.

The next steps in this area of research were done in [10, 22, 24-27]. In [10] we introduced a general variational principle having its prototype in the variational principle of Deville, Godefroy and Zizler [9]. A generic existence result in the calculus of variations without convexity assumptions was then obtained as a realization of this variational principle. It was also shown in [10] that some other generic well-posedness results in optimization theory known in the literature and their modifications are obtained as a realization of this variational principle. Note that the generic existence result in [10] was established for variational problems but not for optimal control problems and that the topologies in the spaces of integrands in [23] and [10] are different.

In [22] we suggested a modification of the variational principle in [10] and applied it to classes of optimal control problems with various topologies in the corresponding spaces of integrands. As a realization of this principle we established a generic existence result for a class of optimal control problems in which constraint maps are also subject to variations as well as the cost functions [22].

The variational principle in [22] asserts that a generic well-posedness result is true if some basic hypotheses hold. These hypotheses (H1) and (H2) introduced in [22] are stated in Section 2 of the present paper. Proofs of applications of the variational principle of [22] consist in verification of hypotheses (H1) and (H2) for classes of optimization problems.

In [27] using the variational principle of [22] we established generic well-posedness results for classes of nonconvex optimal control problems in which the right-hand side of differential equations is also subject to variations as well as the integrands.

Note that the methods and techniques in [22] and [27] are different. In this paper combining the methods of these two papers we extend the results of [22] and [27] and establish generic well-posedness results for two classes of nonconvex optimal control problems in which the right-hand side of differential equations and constraint maps are also subject to variations as well as the integrands.

We obtain our main results as realizations of the general variational principle of [22]. The verification of the hypothesis (H1) for our classes of optimal control problems is highly complicated. To simplify the verification of (H1) in Section 3 we suggest a concretization of the hypothesis (H1). We introduce new assumptions (A1)-(A7) and show that they imply (H1) (see Proposition 3.1). Thus to verify (H1) we need to show that the assumptions (A1)-(A7) are valid. This approach allows us to simplify the problem.

The main results of the paper (Theorems 4.1 and 4.2) are presented in Section 4. Section 5 contains auxiliary results for Theorems 4.1 and 4.2 which are proved in Section 6. Extensions of Theorems 4.1 and 4.2 are obtained in Section 7.

In this paper we use the following notations and definitions. We denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset R^1$. For each function $f : X \to [-\infty, \infty]$, where X is nonempty, we set

$$\inf(f) = \inf\{f(x) : \ x \in X\}.$$

We use the convention that $\infty - \infty = 0$ and $\infty/\infty = 1$ and the notation $\exp(t) = e^t$, $t \in R^1$.

Assume that (X_i, ρ_i), $i = 1, 2$ are metric spaces. For each mapping $f : X_1 \to X_2$ we set

$$\mathrm{Lip}(f) = \sup\{\rho_1(y, z)^{-1}\rho_2(f(y), f(z)) : \ y, z \in X_1 \text{ and } y \neq z\}.$$

Assume that $g : X \times Y \to Z$, where X, Y and Z are nonempty sets. For each $x \in X$ the function $y \to g(x, y)$, $y \in Y$ is denoted by $g(x, \cdot)$. For each $y \in Y$ the function $x \to g(x, y)$, $x \in X$ is denoted by $g(\cdot, y)$.

In this paper we usually consider topological spaces with two topologies where one is weaker than the other. (Note that they can coincide.) We refer to them as the weak and the strong topologies, respectively. If (X, d) is a metric space with a metric d and $Y \subset X$, then usually Y is also endowed with the metric d (unless another metric is introduced in Y). Assume that X_1 and X_2 are topological spaces and that each of them is endowed with a weak and a strong topologies. Then for the product $X_1 \times X_2$ we also introduce a pair of topologies: a weak topology which is the product of the weak topologies on X_1 and X_2 and a strong topology which is the product of the strong topologies of X_1 and X_2. If $Y \subset X_1$, then we consider the topological subspace Y with the relative weak and strong topologies (unless other topologies are introduced). If (X_i, d_i), $i = 1, 2$ are metric spaces with the metric d_1 and d_2, respectively, then the space $X_1 \times X_2$ is endowed with the metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2), \ (x_i, y_i) \in X \times Y, \ i = 1, 2.$$

2. A variational principle

We consider a metric space (X, ρ) which is called the domain space and a complete metric space $(\mathcal{A}, d)$ which is called the data space. We always consider the set X with the topology generated by the metric ρ. For the space $\mathcal{A}$ we consider the topology generated by the metric d. This topology will be called the strong topology. In addition to the strong topology we also consider a weaker topology on $\mathcal{A}$ which is not necessarily Hausdorff. This topology will be called the weak topology. (Note that these topologies can coincide.) We assume that with every $a \in \mathcal{A}$ a lower semicontinuous function f_a on X is associated with values in $\bar{R} = [-\infty, \infty]$.

Let $a \in \mathcal{A}$. We say that the minimization problem for f_a on (X, ρ) is strongly well-posed with respect to $\mathcal{A}$ [29] if $\inf(f_a)$ is finite and attained at a unique point $x_a \in X$ and the following assertion holds:

For each $\epsilon > 0$ there exist a neighborhood $\mathcal{V}$ of a in $\mathcal{A}$ with the weak topology and $\delta > 0$ such that for each $b \in \mathcal{V}$, $\inf(f_b)$ is finite and if $z \in X$ satisfies $f_b(z) \le \inf(f_b) + \delta$, then $\rho(x_a, z) \le \epsilon$ and $|f_b(z) - f_a(x_a)| \le \epsilon$.

In our study we use the following basic hypotheses about the functions.

(H1) For any $a \in \mathcal{A}$, any $\epsilon > 0$ and any $\gamma > 0$ there exist a nonempty open set $\mathcal{W}$ in $\mathcal{A}$ with the weak topology, $x \in X$, $\alpha \in R^1$ and $\eta > 0$ such that

$$\mathcal{W} \cap \{b \in \mathcal{A} : \ d(a, b) < \epsilon\} \neq \emptyset,$$

and for any $b \in \mathcal{W}$
 (i) $\inf(f_b)$ is finite;
 (ii) if $z \in X$ is such that $f_b(z) \le \inf(f_b) + \eta$, then $\rho(z, x) \le \gamma$ and $|f_b(z) - \alpha| \le \gamma$.

(H2) if $a \in \mathcal{A}$, $\inf(f_a)$ is finite, $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence and the sequence $\{f_a(x_n)\}_{n=1}^{\infty}$ is bounded, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges in (X, ρ).

The following result was obtained in [22, Theorem 2.1].

Theorem 2.1. *Assume that (H1) and (H2) hold. Then there exists an everywhere dense (in the strong topology) set $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for any $a \in \mathcal{B}$ the minimization problem for f_a on (X, ρ) is strongly well-posed with respect to $\mathcal{A}$.*

3. Concretization of the hypothesis (H1)

Let (X, ρ) be a metric space with the topology generated by the metric ρ and let $(\mathcal{A}_i, d_i)$ $(i = 1, 2, 3)$ be metric spaces. For the space $\mathcal{A}_i$ $(i = 1, 2, 3)$ we consider the topology generated by the metric d_i. This topology is called the strong topology. In addition to the strong topology we consider a weak topology on $\mathcal{A}_i$, $i = 1, 2, 3$ which is weaker than the strong topology.

We assume that X is also equipped with a metric ρ_s such that the following property holds:

(P1) For each $\epsilon > 0$ there is $\delta > 0$ such that $\rho(x_1, x_2) \le \epsilon$ for each $x_1, x_2 \in X$ satisfying $\rho_s(x_1, x_2) \le \delta$.

We equip the space $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ with a metric d defined by

$$d((a_1, a_2, a_3), (b_1, b_2, b_3)) = d_1(a_1, b_1) + d_2(a_2, b_2) + d_3(a_3, b_3),$$

$$(a_1, a_2, a_3), (b_1, b_2, b_3) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3.$$

The strong topology of $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ is the product of the strong topologies of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ and the weak topology of $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ is the product of the weak topologies of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$.

For each function $g : X \to R^1 \cup \{\infty\}$ and each $Y \subset X$ set

$$\inf(g; Y) = \inf\{g(z) : \ z \in Y\}.$$

Assume that with every $a \in \mathcal{A}_1$ a function $\phi_a : X \to R^1 \cup \{\infty\}$ is associated, with every $a \in \mathcal{A}_2$ a set $S_a \subset X$ is associated and with every $a \in \mathcal{A}_3$ a set $Q_a \subset X$ is associated.

For each $a = (a_1, a_2, a_3) \in \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ define $f_a : X \to R^1 \cup \{\infty\}$ by

$$f_a(x) = \phi_{a_1}(x) \text{ for all } x \in S_{a_2} \cap Q_{a_3}, \tag{3.1}$$

$$f_a(x) = \infty \text{ for all } x \in X \setminus (S_{a_2} \cap Q_{a_3}). \tag{3.2}$$

Let $\mathcal{D}$ be a nonempty closed open subset of $\mathcal{A}_1 \times \mathcal{A}_2$ with the weak topology. Denote by $\mathcal{A}$ the closure of the set $\{a \in \mathcal{D} \times \mathcal{A}_3 : \inf(f_a) < \infty\}$ in the space $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ with the strong topology. We assume that $\mathcal{A} \neq \emptyset$ and use the following hypotheses:

(A1) For each $a_1 \in \mathcal{A}_1$, $\inf(\phi_{a_1}) > -\infty$ and for each $a \in \mathcal{D} \times \mathcal{A}_3$ the function f_a is lower semicontinuous on (X, ρ).

(A2) For each $a = (a_1, a_2) \in \mathcal{D}$, $\inf\{\phi_{a_1}(x) : x \in S_{a_2}\}$ if finite.

(A3) For each $a_1 \in \mathcal{A}_1$, each $\epsilon > 0$ and each $D > 0$ there exists a neighborhood $\mathcal{V}$ of a_1 in $\mathcal{A}_1$ with the weak topology such that for each $b \in \mathcal{V}$ and each $x \in X$ satisfying $\min\{\phi_{a_1}(x), \phi_b(x)\} \le D$ the inequality $|\phi_{a_1}(x) - \phi_b(x)| \le \epsilon$ holds.

(A4) For each $\gamma \in (0, 1)$ there exists $\delta(\gamma) \in (0, 1)$ such that for each $a = (a_1, a_2) \in \mathcal{D}$, each nonempty set $Y \subset X$ satisfying $\inf(\phi_{a_1}; Y) < \infty$, each $r \in (0, 1)$ and each $\bar{x} \in Y$ satisfying

$$\phi_{a_1}(\bar{x}) \le \inf(\phi_{a_1}; Y) + r\delta(\gamma)/2$$

there exists $\bar{a}_1 \in \mathcal{A}_1$ such that

$$(\bar{a}_1, a_2) \in \mathcal{D},\ d_1(a_1, \bar{a}_1) \le r,$$

$$\phi_{\bar{a}_1}(z) \ge \phi_{a_1}(z) \text{ for all } z \in X,\ \phi_{\bar{a}_1}(\bar{x}) \le \phi_{a_1}(\bar{x}) + r\delta(\gamma)$$

and the following property holds:

For each $y \in Y$ satisfying $\phi_{\bar{a}_1}(y) \le \inf(\phi_{\bar{a}_1}; Y) + 2\delta(\gamma)r$ the inequality $\rho(y, \bar{x}) \le \gamma$ is valid.

(A5) For each $a = (a_1, a_2) \in \mathcal{D}$ and each $M, \epsilon > 0$ there exist a number $\delta > 0$ and a neighborhood $\mathcal{V}$ of a_2 in $\mathcal{A}_2$ with the weak topology such that the following property holds:

For each $x \in \cup\{S_b : b \in \mathcal{V}\}$ satisfying $\phi_{a_1}(x) \le M$ and each $y \in X$ satisfying $\rho_s(x, y) \le \delta$ the inequality $|\phi_{a_1}(x) - \phi_{a_1}(y)| \le \epsilon$ is true.

(A6) For each $a = (a_1, a_2) \in \mathcal{D}$ and each $\epsilon, M > 0$ there exists a neighborhood $\mathcal{V}$ of a_2 in $\mathcal{A}_2$ with the weak topology such that the following property holds:

For each $b_1, b_2 \in \mathcal{V}$ and each $x \in S_{b_1}$ satisfying $\phi_{a_1}(x) \le M$ there exists $y \in S_{b_2}$ such that $\rho_s(x, y) \le \epsilon$.

(A7) For each $a = (a_1, a_2, a_3) \in \mathcal{D} \times \mathcal{A}_3$ satisfying $\inf(f_a) < \infty$ and each $\epsilon, \delta > 0$ there exist $\gamma > 0$, $\bar{a}_3 \in \mathcal{A}_3$, $\bar{x} \in S_{a_2} \cap Q_{\bar{a}_3}$, an open set $\mathcal{U}$ in $\mathcal{A}_3$ with the weak topology such that

$$d_3(a_3, \bar{a}_3) < \epsilon,\ \mathcal{U} \cap \{b \in \mathcal{A}_3 : d_3(b, a_3) < \epsilon\} \neq \emptyset,$$

$$\phi_{a_1}(\bar{x}) \le \inf\{\phi_{a_1}(z) : z \in S_{a_2} \cap Q_{\bar{a}_3}\} + \delta < \infty,$$

$$\{x \in X : \rho_s(x, \bar{x}) \le \gamma\} \subset Q_b \subset Q_{\bar{a}_3} \text{ for all } b \in \mathcal{U},$$

$$\cup_{x \in Q_b}\{z \in X : \rho_s(z, x) \le \gamma\} \subset Q_{\bar{a}_3} \text{ for all } b \in \mathcal{U}.$$

In this section we will prove the following result.

Proposition 3.1. *Assume that* (A1)–(A7) *hold. Then* (H1) *holds.*

Proof. Let

$$a = (a_1, a_2, a_3) \in \mathcal{A},\ \epsilon, \gamma \in (0,1).$$

The construction of the set $\mathcal{W}$ (see (H1)) is rather complicated. We will construct $\mathcal{W}$ as the product $\mathcal{V}_1 \times [\mathcal{V}_{21} \cap \mathcal{V}_{22}] \times \mathcal{U}$ where $\mathcal{V}_1$ is an open subset of $\mathcal{A}_1$, $\mathcal{V}_{21}$ and $\mathcal{V}_{22}$ are open subsets of $\mathcal{A}_2$ and $\mathcal{U}$ is an open subset of $\mathcal{A}_3$.

By the definition of $\mathcal{A}$ we may assume without loss of generality that $\inf(f_a) < \infty$. Clearly, $(a_1, a_2) \in \mathcal{D}$. Choose a positive number

$$\gamma_0 \in (0, 8^{-1} \min\{\epsilon, \gamma\}). \tag{3.3}$$

Let $\delta(\gamma_0) \in (0,1)$ be as guaranteed by (A4) (namely (A4) is true with $\gamma = \gamma_0,\ \delta(\gamma) = \delta(\gamma_0)$). Choose

$$r \in (0, \gamma_0/4),\ \delta_1 \in (0, 4^{-1}\delta(\gamma_0)r). \tag{3.4}$$

By Property (P1) there is

$$\gamma_1 \in (0, \gamma_0/16) \tag{3.5}$$

such that

$$\rho(z_1, z_2) \le \gamma_0/16 \text{ for each } z_1, z_2 \in X \text{ satisfying } \rho_s(z_1, z_2) \le \gamma_1. \tag{3.6}$$

First we define the open subset $\mathcal{U}$ of $\mathcal{A}_3$. By (A7) there are

$$\bar{a}_3 \in \mathcal{A}_3,\ \bar{x} \in S_{a_2} \cap Q_{\bar{a}_3},\ \Delta_0 \in (0,1), \tag{3.7}$$

and an open set $\mathcal{U}$ in $\mathcal{A}_3$ with the weak topology such that

$$d_3(a_3, \bar{a}_3) < \delta_1,\ \mathcal{U} \cap \{b \in \mathcal{A}_3 :\ d_3(b, a_3) < \delta_1\} \ne \emptyset, \tag{3.8}$$

$$\phi_{a_1}(\bar{x}) \le \inf\{\phi_{a_1}(z) :\ z \in S_{a_2} \cap Q_{\bar{a}_3}\} + \delta_1 < \infty, \tag{3.9}$$

$$\{x \in X :\ \rho_s(x, \bar{x}) \le \Delta_0\} \subset Q_b \subset Q_{\bar{a}_3} \text{ for all } b \in \mathcal{U}, \tag{3.10}$$

$$\cup_{x \in Q_b}\{z \in X :\ \rho_s(z, x) \le \Delta_0\} \subset Q_{\bar{a}_3} \text{ for all } b \in \mathcal{U}. \tag{3.11}$$

It follows from the choice of $\delta(\gamma_0)$, (A4) which holds with $\gamma = \gamma_0,\ \delta(\gamma) = \delta(\gamma_0)$, $Y = S_{a_2} \cap Q_{\bar{a}_3}$, (3.7), (3.9) and (3.4) that there exists $\bar{a}_1 \in \mathcal{A}_1$ such that

$$d_1(a_1, \bar{a}_1) \le r,\ (\bar{a}_1, a_2) \in \mathcal{D}, \tag{3.12}$$

$$\phi_{\bar{a}_1}(z) \ge \phi_{a_1}(z) \text{ for all } z \in X,\ \phi_{\bar{a}_1}(\bar{x}) \le \phi_{a_1}(\bar{x}) + r\delta(\gamma_0) \tag{3.13}$$

and that the following property holds:

(Pi) For each $y \in S_{a_2} \cap Q_{\bar{a}_3}$ satisfying $\phi_{\bar{a}_1}(y) \le \inf(\phi_{\bar{a}_1}; S_{a_2} \cap Q_{\bar{a}_3}) + 2\delta(\gamma_0)r$ the inequality $\rho(y, \bar{x}) \le \gamma_0$ is valid.

Let us now define an open subset $\mathcal{V}_1$ of $\mathcal{A}_1$. Choose a number

$$D > |\phi_{\bar{a}_1}(\bar{x})| + 4. \tag{3.14}$$

In view of (A3) there exists an open neighborhood $\mathcal{V}_1$ of $\bar{a}_1$ in $\mathcal{A}_1$ with the weak topology such that the following property holds:

(Pii) For each $h \in \mathcal{V}_1$ and each $x \in X$ satisfying $\min\{\phi_{\bar{a}_1}(x), \phi_h(x)\} \le D$ the inequality $|\phi_{\bar{a}_1}(x) - \phi_h(x)| \le \Delta_0\delta_1/4$ holds.

In our next step we construct the open subsets $\mathcal{V}_{21}$ and $\mathcal{V}_{22}$ of $\mathcal{A}_2$. By (A5) and (3.12) there exist a positive number

$$\delta_2 \in (0, 8^{-1} \min\{\gamma_1, \Delta_0\delta_1\}) \tag{3.15}$$

and an open neighborhood $\mathcal{V}_{21}$ of a_2 in $\mathcal{A}_2$ with the weak topology such that the following property holds:

(Piii) For each $x \in \cup\{S_b : b \in \mathcal{V}_{21}\}$ satisfying $\phi_{\bar{a}_1}(x) \le D$ and each $y \in X$ satisfying $\rho_s(x, y) \le \delta_2$ the inequality $|\phi_{\bar{a}_1}(x) - \phi_{\bar{a}_1}(y)| \le \Delta_0\delta_1/16$ holds.

It follows from (A6) and (3.12) that there exists an open neighborhood $\mathcal{V}_{22}$ of a_2 in $\mathcal{A}_2$ with the weak topology such that the following property holds:

(Piv) For each $h_1, h_2 \in \mathcal{V}_{22}$ and each $x \in S_{h_1}$ satisfying $\phi_{\bar{a}_1}(x) \le D$ there exists $y \in S_{h_2}$ such that $\rho_s(x, y) \le \delta_2$.

Since $\mathcal{D}$ is an open subset of $\mathcal{A}_1 \times \mathcal{A}_2$ with the weak topology we may assume without loss of generality that

$$\mathcal{V}_1 \times [\mathcal{V}_{21} \cap \mathcal{V}_{22}] \subset \mathcal{D}. \tag{3.16}$$

Set

$$\mathcal{W} = \mathcal{V}_1 \times [\mathcal{V}_{21} \cap \mathcal{V}_{22}] \times \mathcal{U}. \tag{3.17}$$

Clearly $\mathcal{W}$ is an open set in $\mathcal{D} \times \mathcal{A}_3$ with the weak topology,

$$\bar{a}_1 \in \mathcal{V}_1,\ a_2 \in \mathcal{V}_{21} \cap \mathcal{V}_{22}. \tag{3.18}$$

We will show that (H1) holds with $x = \bar{x}$, $\eta = \Delta_0\delta_1/4$, $\alpha = \inf(f_{(\bar{a}_1, a_2, \bar{a}_3)})$ and $\mathcal{W}$ and γ.

In view of (3.8) there is $b \in \mathcal{U}$ such that $d_3(b, a_3) < \delta_1$. Together with (3.18), (3.17), (3.12), (3.4) and (3.3) this implies that $(\bar{a}_1, a_2, b) \in \mathcal{W}$ and

$$d((a_1, a_2, a_3), (\bar{a}_1, a_2, b)) \le r + \delta_1 < \epsilon.$$

Thus

$$\mathcal{W} \cap \{b \in \mathcal{A} : d(a, b) < \epsilon\} \neq \emptyset. \tag{3.19}$$

We will show that $\inf(f_b) < D - 2$ for all $b \in \mathcal{W}$.

Let $\xi \in \mathcal{U}$. By (3.7), (3.10), (A1), (3.1), (3.9) and (3.13) $\bar{x} \in S_{a_2} \cap Q_\xi$ and

$$-\infty < \inf(f_{(\bar{a}_1, a_2, \xi)}) \le \phi_{\bar{a}_1}(\bar{x}) < \infty \text{ for all } \xi \in \mathcal{U}. \tag{3.20}$$

Assume that

$$b = (b_1, b_2, b_3) \in \mathcal{W}. \tag{3.21}$$

In view of (3.21), (3.17), (3.18), (3.7), (3.14) and (Piv) there is

$$y \in S_{b_2} \tag{3.22}$$

such that

$$\rho_s(y, \bar{x}) \le \delta_2. \tag{3.23}$$

Together with (3.18), (3.7), (3.14), (3.4) and (Piii) this implies that

$$\phi_{\bar{a}_1}(y) \le \phi_{\bar{a}_1}(\bar{x}) + \Delta_0\delta_1/16 < D - 3.$$

It follows from this inequality, (3.21), (3.17), (Pii), (3.14), (3.4) and (3.7) that

$$\phi_{b_1}(y) \le \phi_{\bar{a}_1}(y) + \Delta_0\delta_1/4 < \phi_{\bar{a}_1}(\bar{x}) + \Delta_0\delta_1(5/16) < D - 2. \tag{3.24}$$

In view of (3.23), (3.15), (3.4), (3.10), (3.21) and (3.17) $y \in Q_{b_3}$. Together with (3.24), (3.22), (3.1) and (3.2) this implies that

$$\inf(f_b) \le \phi_{b_1}(y) < \phi_{\bar{a}_1}(\bar{x}) + \Delta_0\delta_1(5/16) < D - 2.$$

Thus we have shown that

$$\inf(f_b) < \phi_{\bar{a}_1}(\bar{x}) + \Delta_0\delta_1(5/16) < D - 2 \text{ for all } b \in \mathcal{W}. \tag{3.25}$$

Assume that

$$b = (b_1, b_2, b_3) \in \mathcal{W}. \tag{3.26}$$

By (3.26) and (3.17)

$$(\bar{a}_1, b_2, b_3) \in \mathcal{W}. \tag{3.27}$$

It follows from (3.25), (3.26), (3.17), (3.27), (3.1) (3.2) and property (Pii) that

$$\begin{aligned}\inf(f_b) &\le \inf\{f_b(z) :\ z \in X \text{ and } f_{(\bar{a}_1,b_2,b_3)}(z) \le D\}\\ &\le \Delta_0\delta_1/4 + \inf\{f_{(\bar{a}_1,b_2,b_3)}(z) :\ z \in X \text{ and } f_{(\bar{a}_1,b_2,b_3)}(z) \le D\}\\ &= \Delta_0\delta_1/4 + \inf(f_{(\bar{a}_1,b_2,b_3)})\end{aligned}$$

and

$$\begin{aligned}\inf(f_{(\bar{a}_1,b_2,b_3)}) &\le \inf\{f_{(\bar{a}_1,b_2,b_3)}(z) :\ z \in X \text{ and } f_b(z) \le D\}\\ &\le \inf\{f_b(z) :\ z \in X \text{ and } f_b(z) \le D\} + \Delta_0\delta_1/4\\ &= \Delta_0\delta_1/4 + \inf(f_b).\end{aligned}$$

Thus

$$|\inf(f_b) - \inf(f_{(\bar{a}_1,b_2,b_3)})| \le \Delta_0\delta_1/4 \text{ for all } b = (b_1, b_2, b_3) \in \mathcal{W}. \tag{3.28}$$

Relations (3.1), (3.2), (3.7), (3.9) (3.13) and (A1) imply that

$$\begin{aligned}-\infty < \inf(f_{(\bar{a}_1,a_2,\bar{a}_3)}) &\le \phi_{\bar{a}_1}(\bar{x}) \le \phi_{a_1}(\bar{x}) + r\delta(\gamma_0)\\ &\le r\delta(\gamma_0) + \delta_1 + \inf\{\phi_{a_1}(z) :\ z \in S_{a_2} \cap Q_{\bar{a}_3}\}\\ &\le r\delta(\gamma_0) + \delta_1 + \inf(f_{(\bar{a}_1,a_2,\bar{a}_3)}).\end{aligned} \tag{3.29}$$

Assume that

$$b = (b_1, b_2, b_3) \in \mathcal{W},\ z \in X,\ f_b(z) \le \inf(f_b) + \Delta_0\delta_1/4. \tag{3.30}$$

By (3.30), (3.1), (3.2) and (3.25)

$$z \in S_{b_2} \cap Q_{b_3},\ f_b(z) = \phi_{b_1}(z). \tag{3.31}$$

It follows from (3.31), (3.30), (3.25), (3.7), (3.4) that

$$\phi_{b_1}(z) \le \inf(f_b) + \Delta_0\delta_1/4 < D - 1. \tag{3.32}$$

Together with (Pii), (3.30) and (3.17) this inequality implies that

$$|\phi_{\bar{a}_1}(z) - \phi_{b_1}(z)| \le \Delta_0\delta_1/4. \tag{3.33}$$

Relations (3.33), (3.31), (3.32) and (3.28) imply that

$$
\begin{aligned}
(3.34)\qquad & |\phi_{\bar{a}_1}(z) - \inf(f_{(\bar{a}_1,b_2,b_3)})| \\
& \leq |\phi_{\bar{a}_1}(z) - \phi_{b_1}(z)| + |\phi_{b_1}(z) - \inf(f_b)| + |\inf(f_b) - \inf(f_{(\bar{a}_1,b_2,b_3)})| \\
& \leq 3\Delta_0\delta_1/4.
\end{aligned}
$$

Relations (3.34), (3.25), (3.27), (3.7) and (3.4) imply that

$$
(3.35)\qquad \phi_{\bar{a}_1}(z) < D.
$$

By (Piv) (with $h_1 = b_2$, $h_2 = a_2$, $x = z$), (3.30), (3.31), (3.17) and (3.35) there exists $y \in X$ such that

$$
(3.36)\qquad y \in S_{a_2},\ \rho_s(z, y) \leq \delta_2.
$$

In view of (3.30), (3.17), (Piii), (3.31), (3.35) and (3.36)

$$
(3.37)\qquad |\phi_{\bar{a}_1}(y) - \phi_{\bar{a}_1}(z)| \leq \Delta_0\delta_1/16.
$$

It follows from (3.36), (3.15), (3.4), (3.31), (3.30), (3.17) and (3.11) that

$$
(3.38)\qquad y \in \{u \in X :\ \rho_s(z, u) \leq \Delta_0\} \subset Q_{\bar{a}_3}.
$$

By (3.37), (3.34), (3.30), (3.25), (3.29), (3.7) and (3.4)

$$
\begin{aligned}
(3.39)\qquad \phi_{\bar{a}_1}(y) & \leq \phi_{\bar{a}_1}(z) + 16^{-1}\Delta_0\delta_1 \leq \inf(f_{(\bar{a}_1,b_2,b_3)}) + 3\Delta_0\delta_1/4 + \Delta_0\delta_1/16 \\
& < \phi_{\bar{a}_1}(\bar{x}) + \Delta_0\delta_1[(5/16) + 3/4 + 1/16] \\
& \leq \inf(f_{(\bar{a}_1,a_2,\bar{a}_3)}) + r\delta(\gamma_0) + \delta_1 + \Delta_0\delta_1(5/4) \\
& \leq \inf(f_{(\bar{a}_1,a_2,\bar{a}_3)}) + 2r\delta(\gamma_0).
\end{aligned}
$$

In view of (3.39), (3.36), (3.38), (3.1), (3.2) and (Pi) $\rho(\bar{x}, y) \leq \gamma_0$. Together with (3.36), (3.15), (3.6) and (3.3) this inequality implies that

$$
\rho(z, \bar{x}) \leq \rho(z, y) + \rho(y, \bar{x}) \leq \rho(z, y) + \gamma_0 \leq \gamma_0/16 + \gamma_0 < \gamma.
$$

It follows from (3.33), (3.37), (3.36), (3.38), (3.39) and (3.4) that

$$
\begin{aligned}
& |\phi_{b_1}(z) - \inf(f_{(\bar{a}_1,a_2,\bar{a}_3)})| \\
& \leq |\phi_{b_1}(z) - \phi_{\bar{a}_1}(z)| + |\phi_{\bar{a}_1}(z) - \phi_{\bar{a}_1}(y)| + |\phi_{\bar{a}_1}(y) - \inf(f_{(\bar{a}_1,a_2,\bar{a}_3)})| \\
& \leq \Delta_1\delta_1/4 + \Delta_0\delta_1/16 + 2r\delta(\gamma_0) \leq 3r\delta(\gamma_0) < \gamma_0 < \gamma.
\end{aligned}
$$

This completes the proof of Proposition 3.1. □

4. The main result

Let $(Y, ||\cdot||)$ be a Banach space and $-\infty < a < b < \infty$. A function $x : [a, b] \to Y$ is strongly measurable on $[a, b]$ if there exists a sequence of functions $x_n : [a, b] \to Y$, $n = 1, 2, \dots$ such that for any integer $n \geq 1$ the set $x_n([a, b])$ is countable and the set $\{t \in [a, b] :\ x_n(t) = y\}$ is Lebesgue measurable for any $y \in Y$, and $x_n(t) \to x(t)$ as $n \to \infty$ in $(Y, ||\cdot||)$ for almost every $t \in [a, b]$.

The function $x : [a, b] \to Y$ is Bochner integrable if it is strongly measurable and there exists a finite $\int_a^b ||x(t)||dt$. Denote by $S(Y)$ the set of all nonempty closed convex subsets of Y. For each $x \in Y$ and each $C \subset Y$ set $d_Y(x, C) = \inf\{||x - y|| : y \in C\}$. For each $C_1, C_2 \subset Y$ define

$$d_Y(C_1, C_2) = \max\{\sup_{y \in C_1} d_Y(y, C_2), \sup_{x \in C_2} d_Y(x, C_1)\}.$$

For the space $S(Y)$ we consider the uniformity determined by the following base:

$$\mathcal{E}_Y(\epsilon) = \{(C_1, C_2) \in S(Y) \times S(Y) : d_Y(C_1, C_2) \leq \epsilon\}, \tag{4.1}$$

where $\epsilon > 0$. It is well known that the space $S(Y)$ with this uniformity is metrizable (by a Hausdorff type metric) and complete. We endow the set $S(Y)$ with the topology induced by this uniformity.

Denote by $\mathcal{P}_Y([a, b])$ the set of all set-valued mappings $A : [a, b] \to S(Y)$. For the space $\mathcal{P}_Y([a, b])$ we consider the uniformity determined by the following base:

$$\mathcal{E}^{(Y)}_{[a,b]}(\epsilon) = \{(A_1, A_2) \in (\mathcal{P}_Y([a, b]))^2 : d_Y(A_1(t), A_2(t)) \leq \epsilon \text{ for all } t \in [a, b]\}, \tag{4.2}$$

where $\epsilon > 0$. It is easy to see that the space $\mathcal{P}_Y([a, b])$ with this uniformity is metrizable and complete. We equip the space $\mathcal{P}_Y([a, b])$ with the topology induced by this uniformity.

Let $(E, || \cdot ||)$, $(F, || \cdot ||)$ be Banach spaces. We equip the space E with the metric $d_E(x, y) = ||x - y||$, $x, y \in E$ and equip the space F with the metric $d_F(u, v) = ||u - v||$, $u, v \in F$.

Let $-\infty < \tau_1 < \tau_2 < \infty$. Denote by $W^{1,1}(\tau_1, \tau_2; E)$ the set of all functions $x : [\tau_1, \tau_2] \to E$ for which there exists a Bochner integrable function $u : [\tau_1, \tau_2] \to E$ such that

$$x(t) = x(\tau_1) + \int_{\tau_1}^{t} u(s)ds, \ t \in (\tau_1, \tau_2].$$

(see, e.g. [2]). It is known that if $x \in W^{1,1}(\tau_1, \tau_2; E)$ then this equation defines a unique Bochner integrable function u which is called the derivative of x and is denoted by x'.

Let $0 \leq T_1 < T_2 < \infty$. Denote by X the set of all pairs of functions (x, u) where $x \in W^{1,1}(T_1, T_2; E)$ and $u : [T_1, T_2] \to F$ is a strongly measurable function. To be more precise, we have to define elements of X as classes of pairs equivalent in the sense that (x_1, u_1) and (x_2, u_2) are equivalent if and only if $x_2(t) = x_1(t)$ for all $t \in [T_1, T_2]$ and $u_2(t) = u_1(t)$, $t \in [T_1, T_2]$ almost everywhere (a.e.). For the set X we consider the metric ρ defined by

$$\begin{aligned} &\rho((x_1, u_1), (x_2, u_2)) \\ &= \inf\{\epsilon > 0 : \text{mes}\{t \in [T_1, T_2] : ||x_1(t) - x_2(t)|| + ||u_1(t) - u_2(t)|| \geq \epsilon\} \leq \epsilon\}, \\ &\qquad (x_1, u_1), \ (x_2, u_2) \in X. \end{aligned} \tag{4.3}$$

In the sequel we consider the space X endowed with the metric ρ and with the topology induced by the metric ρ. For each (x_1, u_1), $(x_2, u_2) \in X$ set

$$\begin{aligned} &\tilde{\rho}_s((x_1, u_1), (x_2, u_2)) = \infty \text{ if } \text{mes}(\{t \in [T_1, T_2] : u_1(t) \neq u_2(t)\}) > 0, \\ &\text{otherwise } \tilde{\rho}_s((x_1, u_1), (x_2, u_2)) = \sup\{||x_1(t) - x_2(t)|| : t \in [T_1, T_2]\} \end{aligned} \tag{4.4}$$

and

$$\rho_s((x_1,u_1),(x_2,u_2)) = \tilde{\rho}_s((x_1,u_1),(x_2,u_2))(1+\tilde{\rho}_s((x_1,u_1),(x_2,u_2)))^{-1}.$$

Clearly ρ_s is a metric.

Set

$$\mathcal{P}_E = \mathcal{P}_E([T_1,T_2]),\ \mathcal{P}_F = \mathcal{P}_F([T_1,T_2]),\ \tilde{A}(t) = E \text{ for all } t \in [T_1,T_2]. \tag{4.5}$$

For each continuous mapping $G : [T_1,T_2] \times E \times F \to E$, each nonempty closed subset $B \subset E$, each $A \in \mathcal{P}_E$ and each $U \in \mathcal{P}_F$ we denote by $X(B,G,A,U)$ the set of all pairs $(x,u) \in X$ such that

$$x(T_1) \in B, \tag{4.6}$$

$$x(t) \in A(t),\ t \in [T_1,T_2], \tag{4.7}$$

$$u(t) \in U(t),\ t \in [T_1,T_2] \text{ a.e.}, \tag{4.8}$$

$$x'(t) = G(t,x(t),u(t)),\ t \in [T_1,T_2] \text{ a.e.}. \tag{4.9}$$

Denote by $\mathcal{M}$ the set of all functions $f : [T_1,T_2] \times E \times F \to R^1$ with the following properties:

(i) f is measurable with respect to the σ-algebra generated by products of Lebesgue measurable subsets of $[T_1,T_2]$ and Borel subsets of $E \times F$;

(ii) for each $\epsilon, M > 0$ there exists $\delta > 0$ such that for almost every $t \in [T_1,T_2]$ the inequality $|f(t,x_1,u_1) - f(t,x_2,u_2)| \le \epsilon$ holds for each $x_1, x_2 \in E$ and each $u_1, u_2 \in F$ satisfying

$$||x_i||, ||u_i|| \le M,\ i = 1,2 \text{ and } ||x_1 - x_2||, ||u_1 - u_2|| \le \delta;$$

(iii) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that for almost every $t \in [T_1,T_2]$ the inequality

$$|f(t,x_1,u) - f(t,x_2,u)| \le \epsilon \max\{|f(t,x_1,u)|, |f(t,x_2,u)|\} + \epsilon$$

is valid for each $x_1, x_2 \in E$ and each $u \in F$ satisfying

$$||x_1||, ||x_2|| \le M,\ ||u|| \ge \Gamma,\ ||x_1 - x_2|| \le \delta;$$

(iv) there exists an integrable scalar function $\Lambda(t) \le 0$, $t \in [T_1,T_2]$ such that $f(t,x,u) \ge \Lambda(t)$ for all $(t,x,u) \in [T_1,T_2] \times E \times F$;

(v) there is a constant $c_f > 0$ such that $|f(t,0,0)| \le c_f$ for almost every $t \in [T_1,T_2]$.

It follows from property (i) that for any $f \in \mathcal{M}$, each continuous function $x : [T_1,T_2] \to E$ and each strongly measurable function $u : [T_1,T_2] \to F$ the function $f(t,x(t),u(t))$, $t \in [T_1,T_2]$ is measurable.

Now we equip the set $\mathcal{M}$ with the strong and weak topologies.

For each $f, g \in \mathcal{M}$ set

$$\tilde{d}_{\mathcal{M}}(f,g) = \sup\{|f(t,x,u) - g(t,x,u)| :\ (t,x,u) \in [T_1,T_2] \times E \times F\} + \sup\{\mathrm{Lip}(f(t,\cdot,\cdot) - g(t,\cdot,\cdot)) :\ t \in [T_1,T_2]\},$$

$$d_{\mathcal{M}}(f,g) = \tilde{d}_{\mathcal{M}}(f,g)(1 + \tilde{d}_{\mathcal{M}}(f,g))^{-1}. \tag{4.10}$$

Clearly $(\mathcal{M}, d_{\mathcal{M}})$ is a complete metric space. The metric $d_{\mathcal{M}}$ induces in $\mathcal{M}$ a topology which is called the strong topology.

For each $\epsilon > 0$ we set

$$\mathcal{E}_{\mathcal{M}w}(\epsilon) = \{(f, g) \in \mathcal{M} \times \mathcal{M} : \text{ there exists a nonnegative } \phi \in L^1(T_1, T_2)$$
$$\text{such that } \int_{T_1}^{T_2} \phi(t)dt \leq 1 \text{ and for every } t \in [T_1, T_2] \text{ and every } (x, u) \in E \times F$$

$$|f(t, x, u) - g(t, x, u)| < \epsilon + \epsilon \max\{|f(t, x, u)|, |g(t, x, u)|\} + \epsilon\phi(t)\}. \tag{4.11}$$

It was shown in [27] that for the set $\mathcal{M}$ there exists the uniformity which is determined by the base $\mathcal{E}_{\mathcal{M}w}(\epsilon)$, $\epsilon > 0$. This uniformity induces in $\mathcal{M}$ the weak topology.

Denote by $\mathcal{M}^l$ (respectively, $\mathcal{M}^c$) the set of all lower semicontinuous (respectively, continuous) functions $f : [T_1, T_2] \times E \times F \to R^1$ in $\mathcal{M}$. Denote by $\mathcal{M}_L$ the set of all functions $f \in \mathcal{M}$ such that for almost every $t \in [T_1, T_2]$ the function $f(t, \cdot, \cdot)$ is Lipschitzian on bounded subsets of $E \times F$. Denote by $\mathcal{M}_{Ll}$ the set of all functions $f \in \mathcal{M}$ such that for almost every $t \in [T_1, T_2]$ the function $f(t, \cdot, \cdot) : E \times F \to R^1$ is locally Lipschitzian. Clearly $\mathcal{M}^l$, $\mathcal{M}^c$, $\mathcal{M}_L$, $\mathcal{M}_{Ll}$ are closed subsets of $\mathcal{M}$ with the strong topology. We consider the topological subspaces $\mathcal{M}^l$, $\mathcal{M}^c$, $\mathcal{M}_L$, $\mathcal{M}_{Ll}$, $\mathcal{M}^l \cap \mathcal{M}_{Ll}$, $\mathcal{M}^c \cap \mathcal{M}_{Ll}$, $\mathcal{M}^l \cap \mathcal{M}_L$, $\mathcal{M}^c \cap \mathcal{M}_L \subset \mathcal{M}$ with the relative weak and strong topologies.

For each $f \in \mathcal{M}$ we define $I^f : X \to R^1 \cup \{\infty\}$ by

$$I^f(x, u) = \int_{T_1}^{T_2} f(t, x(t), u(t))dt, \ (x, u) \in X. \tag{4.12}$$

We study the optimal control problem

$$I^f(x, u) \to \min, \ (x, u) \in X(B, G, A, U) \tag{4.13}$$

where $f \in \mathcal{M}$, B is a nonempty closed subset of E, $A \in \mathcal{P}_E$, $U \in \mathcal{P}_F$ and $G : [T_1, T_2] \times E \times F \to E$ belongs to a space of mappings described below.

Denote by $\mathcal{L}$ the set of all continuous mappings $G : [T_1, T_2] \times E \times F \to E$. It is not difficult to see that for each $G \in \mathcal{L}$, each continuous function $x : [T_1, T_2] \to E$ and each strongly measurable function $u : [T_1, T_2] \to F$ the function $G(t, x(t), u(t))$, $t \in [T_1, T_2]$ is strongly measurable.

For each $G_1, G_2 \in \mathcal{L}$ we set

$$\tilde{d}_{\mathcal{L}s}(G_1, G_2) = \sup\{||G_1(t, x, u) - G_2(t, x, u)|| : \ t \in [T_1, T_2], \ x \in E, \ u \in F\}$$
$$+ \sup\{\text{Lip}(G_1(t, \cdot, \cdot) - G_2(t, \cdot, \cdot)) : \ t \in [T_1, T_2]\}, \tag{4.14}$$

$$d_{\mathcal{L}s}(G_1, G_2) = \tilde{d}_{\mathcal{L}s}(G_1, G_2)(1 + \tilde{d}_{\mathcal{L}s}(G_1, G_2))^{-1}.$$

It is not difficult to see that $(\mathcal{L}, d_{\mathcal{L}s})$ is a complete metric space. The metric $d_{\mathcal{L}_s}$ induces in $\mathcal{L}$ the topology which is called the strong topology.

For each $G_1, G_2 \in \mathcal{L}$ we set

$$\tilde{d}_{\mathcal{L}w}(G_1, G_2) = \sup\{||G_1(t, x, u) - G_2(t, x, u)|| : \ t \in [T_1, T_2], \ x \in E, \ u \in F\}$$
$$+ \sup\{\text{Lip}(G_1(t, \cdot, u) - G_2(t, \cdot, u)) : \ t \in [T_1, T_2], \ u \in F\},$$

$$d_{\mathcal{L}w}(G_1,G_2)=\tilde{d}_{\mathcal{L}w}(G_1,G_1)(1+\tilde{d}_{\mathcal{L}w}(G_1,G_2))^{-1}. \tag{4.15}$$

It is not difficult to see that $(\mathcal{L}, d_{\mathcal{L}w})$ is a complete metric space. The metric $d_{\mathcal{L}w}$ induces in $\mathcal{L}$ the topology which is called the weak topology.

Let B be a nonempty closed subset of E and let $\tilde{\mathcal{M}}$ be one of the following spaces: $\mathcal{M}$; $\mathcal{M}^l$; $\mathcal{M}^c$; $\mathcal{M}_{Ll}$; $\mathcal{M}_L$; $\mathcal{M}^l\cap\mathcal{M}_{Ll}$; $\mathcal{M}^c\cap\mathcal{M}_{Ll}$; $\mathcal{M}^l\cap\mathcal{M}_L$; $\mathcal{M}^c\cap\mathcal{M}_L$. Denote by $\mathcal{N}(B)$ the set of all pairs $(f,G)\in\tilde{\mathcal{M}}\times\mathcal{L}$ with the following properties:

(vi) there exist numbers $c_0,c_1>0$, an integrable scalar function $\lambda(t)\geq 0$, $t\in[T_1,T_2]$ and an increasing function $\psi:[0,\infty)\to[0,\infty)$ such that

$$||G(t,y_1,v)-G(t,y_2,v)||\leq c_0||y_1-y_2||\psi(||v||) \quad \text{for each } t\in[T_1,T_2], \text{ each } y_1,y_2\in E \text{ and each } v\in F, \tag{4.16}$$

$$f(t,y,v)\geq c_1\psi(||v||)-\lambda(t) \text{ for all } (t,y,v)\in[T_1,T_2]\times E\times F; \tag{4.17}$$

(vii) for each $\epsilon>0$ there is an integrable scalar function $\phi_\epsilon(t)\geq 0$, $t\in[T_1,T_2]$ such that $||G(t,y,v)||\leq\phi_\epsilon(t)+\epsilon f(t,y,v)$ for all $(t,y,v)\in[T_1,T_2]\times E\times F$;

(viii) there is $(x,u)\in X$ such that $I^f(x,u)<\infty$, $x(T_1)\in B$ and

$$x'(t)=G(t,x(t),u(t)),\ t\in[T_1,T_2] \text{ a.e..}$$

The space $\mathcal{N}(B)$ was introduced in [27] where it was noted that $\mathcal{N}(B)=\mathcal{N}(E)$ (see Corollary 11.1 of [27]). For simplicity we set

$$\mathcal{N}=\mathcal{N}(E)=\mathcal{N}(B). \tag{4.18}$$

We assume that $\mathcal{N}\neq\emptyset$. The following result was proved in [27, Proposition 4.1].

Proposition 4.1. *The set $\mathcal{N}$ is an open closed subset of $\tilde{\mathcal{M}}\times\mathcal{L}$ with the weak topology.*

By Proposition 9.1 of [27] for each $f\in\tilde{\mathcal{M}}$, $I^f: X\to R^1\cup\{\infty\}$ is a lower semicontinuous functional on (X,ρ).

We study the optimal control problem

$$I^f(x,u)\to\min,\ (x,u)\in X(B,G,A,U)$$

where $(f,G)\in\mathcal{N},\ A\in\mathcal{P}_E,\ U\in\mathcal{P}_F$.

For each $a=(f,G,A,U)\in\mathcal{N}\times\mathcal{P}_E\times\mathcal{P}_F$ we define $J_a: X\to R^1\cup\{\infty\}$ by

$$J_a(x,u)=I^f(x,u) \text{ if } (x,u)\in X(B,G,A,U), \text{ otherwise } J_a(x,u)=\infty. \tag{4.19}$$

We will show that for each $a=(f,G,A,U)\in\mathcal{N}\times\mathcal{P}_E\times\mathcal{P}_F$, $J^a:\ X\to R^1\cup\{\infty\}$ is a lower semicontinuous functional on (X,ρ) (see Propositions 5.1 and 5.2).

The following theorem is our first main result.

Theorem 4.1. *Assume that the set B is bounded and let $\mathcal{A}$ be the closure of the set $\{a=(f,G,A,U)\in\mathcal{N}\times\mathcal{P}_E\times\mathcal{P}_F:\ \inf(J_a)<\infty\}$ in the space $\mathcal{N}\times\mathcal{P}_E\times\mathcal{P}_F$ with the strong topology. Assume that $\mathcal{A}$ is nonempty. Then there exists an everywhere dense (in the strong topology) subset $\mathcal{B}\subset\mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for each $a\in\mathcal{B}$ the minimization problem for J_a on (X,ρ) is strongly well-posed with respect to $\mathcal{A}$.*

We also study the optimal control problem

$$I^f(x,u) \to \min,\ (x,u) \in X(B,G,\tilde{A},U)$$

where $(f,G) \in \mathcal{N},\ U \in \mathcal{P}_F$.

For each $a = (f,G,U) \in \mathcal{N} \times \mathcal{P}_F$ we define $\widehat{J}_a : X \to R^1 \cup \{\infty\}$ by

$$\widehat{J}_a(x,u) = I^f(x,u) \text{ if } (x,u) \in X(B,G,\tilde{A},U), \text{ otherwise } \widehat{J}_a(x,u) = \infty. \tag{4.20}$$

We will show that for each $a \in (f,G,U) \in \mathcal{N} \times \mathcal{P}_F$, $\widehat{J}^a :\ X \to R^1 \cup \{\infty\}$ is a lower semicontinuous functional on (X,ρ) (see Propositions 5.1 and 5.2).

The following theorem is our second main result.

Theorem 4.2. *Assume that the set B is bounded and let $\mathcal{A}$ be the closure of the set $\{(f,G,U) \in \mathcal{N} \times \mathcal{P}_F :\ \inf(\widehat{J}_a) < \infty\}$ in the space $\mathcal{N} \times \mathcal{P}_F$ with the strong topology. Assume that $\mathcal{A}$ is nonempty. Then there exists an everywhere dense (in the strong topology) subset $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{B}$ the minimization problem for $\widehat{J}_a$ on (X,ρ) is strongly well-posed with respect to $\mathcal{A}$.*

5. Auxiliary results

In this section we collect auxiliary results which will be used in the proofs of Theorems 4.1 and 4.2.

Proposition 5.1 ([27, Proposition 9.1])**.** *Let $f \in \mathcal{M}$, $(x,u) \in X$, $\{(x_i,u_i)\}_{i=1}^{\infty} \subset X$ and let*

$$\rho((x_i,u_i),(x,u)) \to 0 \text{ as } i \to \infty.$$

Then

$$\int_{T_1}^{T_2} f(t,x(t),u(t))dt \le \liminf_{i\to\infty} \int_{T_1}^{T_2} f(t,x_i(t),u_i(t))dt.$$

Proposition 5.2 ([27, Proposition 9.2])**.** *Assume that $(f,G) \in \mathcal{N}$, $\{(x_i,u_i)\}_{i=1}^{\infty} \subset X$ is a Cauchy sequence in the space (X,ρ), the sequence $\{I^f(x_i,u_i)\}_{i=1}^{\infty}$ is bounded and that for all natural numbers i*

$$x_i'(t) = G(t,x_i(t),u_i(t)),\ t \in [T_1,T_2] \text{ a.e.}.$$

Then there is $(x_*,u_*) \in X$ such that $\rho((x_i,u_i),(x_*,u_*)) \to 0$ as $i \to \infty$, $x_i(t) \to x_*(t)$ as $i \to \infty$ uniformly on $[T_1,T_2]$ and that

$$x_*'(t) = G(t,x_*(t),u_*(t)),\ t \in [T_1,T_2] \text{ a.e.}.$$

Proposition 5.3 ([27, Proposition 9.3])**.** *Let $f \in \mathcal{M}$, $\epsilon \in (0,1)$ and $D > 0$. Then there is a neighborhood $\mathcal{V}$ of f in $\mathcal{M}$ with the weak topology such that for each $g \in \mathcal{V}$ and each $(x,u) \in X$ satisfying $\min\{I^f(x,u), I^g(x,u)\} \le D$ the inequality $|I^f(x,u) - I^g(x,u)| \le \epsilon$ holds.*

For each $f \in \mathcal{M}$ and each $A \subset X$ set

$$\inf(I^f;A) = \inf\{I^f(x,u) :\ (x,u) \in A\}.$$

Analogously to Lemma 5.1 [24] we can prove the following auxiliary result.

Proposition 5.4. *For each $\gamma \in (0,1)$ there exists $\delta(\gamma) \in (0,1)$ such that for each $f \in \mathcal{M}$, each nonempty set $A \subset X$ for which $\inf(I^f; A) < \infty$, each $r \in (0,1]$ and each $(\bar{x}, \bar{u}) \in A$ satisfying $I^f(\bar{x}, \bar{u}) \le \inf(I^f; A) + r\delta(\gamma)/2$ there exists a continuous function $h : [T_1, T_2] \times E \times F \to R^1$ which satisfies*

$$0 \le h(t,x,u) \le r/2 \text{ for all } (t,x,u) \in [T_1,T_2] \times E \times F,$$

$$|h(t,x_1,u_1) - h(t,x_2,u_2)| \le 2^{-1} r(||x_1 - x_2|| + ||u_1 - u_2||)$$
$$\text{for each } x_1, x_2 \in E, \quad \text{each } u_1, u_2 \in F \quad \text{and each } t \in [T_1, T_2]$$

such that the function $\bar{f}$ defined by

$$\bar{f}(t,x,u) = f(t,x,u) + h(t,x,u), \ (t,x,u) \in [T_1,T_2] \times E \times F$$

belongs to $\mathcal{M}$, satisfies

$$I^{\bar{f}}(\bar{x}, \bar{u}) \le I^f(x,u) + r\delta(\gamma)$$

and has the following property:

For each $(y,v) \in A$ satisfying

$$I^{\bar{f}}(y,v) \le \inf(I^{\bar{f}}; A) + 2\delta(\gamma) r$$

the inequality $\rho((\bar{x},\bar{u}),(y,v)) \le \gamma$ is valid.

Moreover h is the sum of two functions, one of them depending only on (t,x) while the other depending only on (t,u).

Proposition 5.5 ([27, Lemma 9.2])**.** *Let the set B be bounded, $(f,G) \in \mathcal{N}$, $M > 0$ and $\epsilon > 0$. Then there exists $\delta > 0$ such that for each $H \in \mathcal{L}$ satisfying $\tilde{d}_{\mathcal{L}w}(H,G) \le 1$, each $(x,u) \in X$ which satisfies*

$$x(T_1) \in B, \ I^f(x,u) \le M, \ x'(t) = H(t, x(t), u(t)), \ t \in [T_1, T_2] \text{ a.e.}$$

and each continuous function $z : [T_1, T_2] \to E$ satisfying $||z(t) - x(t)|| \le \delta, \ t \in [T_1, T_2]$ the following inequality holds:

$$\left| \int_{T_1}^{T_2} f(t, x(t), u(t))dt - \int_{T_1}^{T_2} f(t, z(t), u(t)) \right| \le \epsilon.$$

Proposition 5.6 ([27, Lemma 6.2])**.** *Let $(f,G) \in \mathcal{N}$, $\epsilon > 0$ and $M > 0$. Then there exist $D > 0$ and a neighborhood $\mathcal{V}$ of G in $\mathcal{L}$ with the weak topology such that for each $H, \Lambda \in \mathcal{V}$, each $(x,u) \in X$ satisfying*

$$x(T_1) \in B, \ x'(t) = H(t, x(t), u(t)), \ t \in [T_1, T_2] \text{ a.e. and } I^f(x,u) \le M,$$

and each $\xi \in B$ there exists $z \in W^{1,1}(T_1, T_2; E)$ such that $(z,u) \in X$, $z(T_1) = \xi$,

$$x'(t) = \Lambda(t, x(t), u(t)), \ t \in [T_1, T_2] \text{ a.e.}$$

and

$$||z(t) - x(t)|| \le \epsilon + D||\xi - x(T_1)|| \text{ for all } t \in [T_1, T_2].$$

We need the following result (see Lemma 2.1 of [18]).

Proposition 5.7. *Let $(Z, ||\cdot||)$ be a norm linear space and let $B(0,r) = \{y \in Z : ||y|| \le r\}$. Assume that r is a positive number and C is a closed convex subset of Z such that for all $y \in B(0,r)$, $\inf_{x \in C} ||y - x|| \le r$. Then $0 \in C$.*

The following proposition is an auxiliary result for (A7).

Proposition 5.8. *Let B be a nonempty closed subset of E, $a = (f, G, A, U) \in \mathcal{N} \times \mathcal{P}_E \times \mathcal{P}_F$, $\inf(J_a) < \infty$ and let $\epsilon, \delta > 0$. Then there are*

$$\gamma \in (0, 1/8),\ \bar{A} \in \mathcal{P}_E,\ \bar{U} \in \mathcal{P}_F,\ (\bar{x}, \bar{u}) \in X(B, G, \bar{A}, \bar{U}) \tag{5.1}$$

and an open set $\mathcal{W}$ in $\mathcal{P}_E \times \mathcal{P}_F$ such that

$$A(t) \subset \bar{A}(t),\ U(t) \subset \bar{U}(t) \text{ for all } t \in [T_1, T_2], \tag{5.2}$$

$$(A, \bar{A}) \in \mathcal{E}^{(E)}_{[T_1,T_2]}(\epsilon),\ (U, \bar{U}) \in \mathcal{E}^{(F)}_{[T_1,T_2]}(\epsilon),$$

$$\mathcal{W} \cap \{(B, V) \in \mathcal{P}_E \times \mathcal{P}_F :\ (B, A) \in \mathcal{E}^{(E)}_{[T_1,T_2]}(\epsilon),\ (V, U) \in \mathcal{E}^{(F)}_{[T_1,T_2]}(\epsilon)\} \neq \emptyset, \tag{5.3}$$

$$I^f(\bar{x}, \bar{u}) \le \inf\{I^f(x, u) :\ (x, u) \in X(B, G, \bar{A}, \bar{U})\} + \delta < \infty \tag{5.4}$$

and that for all $(B, V) \in \mathcal{W}$ the following properties hold:

(Ci) *if $(x, u) \in X$ satisfies*

$$||x(t) - \bar{x}(t)|| + ||u(t) - \bar{u}(t)|| \le 2\gamma \text{ for a.e. } t \in [T_1, T_2], \tag{5.5}$$

then $x(t) \in B(t)$ for all $t \in [T_1, T_2]$, $u(t) \in V(t)$ for almost every $t \in [T_1, T_2]$;

(Cii) *if $(x, u) \in X$ satisfies $x(t) \in B(t)$ for all $t \in [T_1, T_2]$ and $u(t) \in V(t)$ for a.e. $t \in [T_1, T_2]$ and if $(z, v) \in X$ satisfies*

$$||x(t) - z(t)|| + ||u(t) - v(t)|| \le 2\gamma \text{ for a. e. } t \in (T_1, T_2), \tag{5.6}$$

then

$$z(t) \in \bar{A}(t) \text{ for all } t \in [T_1, T_2],\ v(t) \in \bar{U}(t) \text{ for a.e. } t \in [T_1, T_2]. \tag{5.7}$$

Proof. For each $r \in (0, 1]$ define $A^{(r)} \in \mathcal{P}_E$, $B^{(r)} \in \mathcal{P}_F$ by

$$A^{(r)}(t) = \{x \in E :\ d_E(x, A(t)) \le r\},\ t \in [T_1, T_2], \tag{5.8}$$

define

$$U^{(r)}(t) = \{u \in F :\ d_F(u, U(t)) \le r\},\ t \in [T_1, T_2] \tag{5.9}$$

and set

$$\mu(r) = \inf(J_{(f,G,A^{(r)},U^{(r)})}). \tag{5.10}$$

Clearly $\mu(r)$ is finite for all $r \in (0, 1]$ and the function μ is monotone decreasing. There is $r_0 \in (0, 8^{-1}\epsilon)$ such that μ is continuous ar r_0. Choose $r_1 \in (0, r_0)$ such that

$$|\mu(r_1) - \mu(r_0)| < 16^{-1}\delta. \tag{5.11}$$

There is

$$(\bar{x}, \bar{u}) \in X(B, G, A^{(r_1)}, B^{(r_1)}) \tag{5.12}$$

such that

$$I^f(\bar{x}, \bar{u}) \le \mu(r_1) + 16^{-1}\delta. \tag{5.13}$$

Relations (5.13) and (5.11) imply that

$$I^f(\bar{x}, \bar{u}) \le \mu(r_0) + 8^{-1}\delta. \tag{5.14}$$

Set

$$r_2 = (r_0 + r_1)/2. \tag{5.15}$$

Clearly

$$(A^{(r_i)}, A) \in \mathcal{E}^E_{[T_1,T_2]}(\epsilon),\ i = 0, 1, 2, \tag{5.16}$$
$$(U^{(r_i)}, U) \in \mathcal{E}^F_{[T_1,T_2]}(\epsilon),\ i = 0, 1, 2.$$

Choose a positive number γ such that

$$0 < \gamma < 1/8,\ \gamma < (r_0 - r_2)/8 = (r_0 - r_1)/16 = (r_2 - r_1)/8 \tag{5.17}$$

and define

$$\bar{A} = A^{(r_0)},\ \bar{U} = U^{(r_0)}, \tag{5.18}$$

$$\mathcal{W} = \text{the interior of the set}$$

$$\{(B, V) \in \mathcal{P}_E \times \mathcal{P}_F :\ (B, A^{(r_2)}) \in \mathcal{E}^{(E)}_{[T_1,T_2]}(\gamma),\ (V, U^{(r_2)}) \in \mathcal{E}^{(F)}_{[T_1,T_2]}(\gamma)\}. \tag{5.19}$$

It follows from (5.17), (5.18), (5.12), (5.19), (5.8), (5.16), (5.14) and (5.10) that (5.1)-(5.4) hold. We will show that (Ci) and C(ii) hold.

Assume that $(B, V) \in \mathcal{W}$ and let $(x, u) \in X$ satisfy (5.5). By (5.5)

$$||u(t) - \bar{u}(t)|| \le 2\gamma \text{ for a.e. } t \in [T_1, T_2],\ ||x(t) - \bar{x}(t)|| \le 2\gamma \text{ for all } t \in [T_1, T_2]. \tag{5.20}$$

In view of (5.19) and the inclusion $(B, V) \in \mathcal{W}$ for all $t \in [T_1, T_2]$

$$d_E(B(t), A^{(r_2)}(t)) \le \gamma,\ d_F(V(t), U^{(r_2)}(t)) \le \gamma.$$

This implies that for all $t \in [T_1, T_2]$

$$d_E(B(t) - x(t), A^{(r_2)}(t) - x(t)) \le \gamma,\ d_F(V(t) - u(t), V^{(r_2)}(t) - u(t)) \le \gamma. \tag{5.21}$$

It follows from (5.12), (5.8), (5.9) and (5.15) that for all $t \in [T_1, T_2]$

$$\{z \in E :\ ||z - \bar{x}(t)|| \le r_2 - r_1\} \subset \{z \in E :\ d_E(z, A(t)) \le r_2\} = A^{(r_2)}(t) \tag{5.22}$$

and that for almost every $t \in [T_1, T_2]$

$$\{v \in F :\ ||v - \bar{u}(t)|| \le r_2 - r_1\} \subset \{v \in F :\ d_F(v, U(t)) \le r_2\} = U^{(r_2)}(t). \tag{5.23}$$

In view of (5.20), (5.17) and (5.22) for all $t \in [T_1, T_2]$

$$\{z \in E :\ ||z - x(t)|| \le \gamma\} \subset \{z \in E :\ ||z - \bar{x}(t)|| \le 3\gamma\} \subset A^{(r_2)}(t),$$
$$\{z \in E :\ ||z|| \le \gamma\} \subset A^{(r_2)}(t) - x(t). \tag{5.24}$$

By (5.20), (5.17) and (5.23) for almost every $t \in [T_1, T_2]$

$$\{v \in F :\ ||v - u(t)|| \le \gamma\} \subset \{v \in F :\ ||v - \bar{u}(t)|| \le 3\gamma\} \subset U^{(r_2)}(t),$$
$$\{v \in F :\ ||v|| \le \gamma\} \subset U^{(r_2)}(t) - u(t). \tag{5.25}$$

It follows from (5.24), (5.21) and Proposition 5.7 that for all $t \in [T_1, T_2]$ we have $0 \in B(t) - x(t)$. In view of (5.21), (5.25) and Proposition 5.7 for almost every $t \in [T_1, T_2]$ we have $0 \in V(t) - u(t)$. Therefore C(i) holds.

Let us show that (Cii) hold. Assume that $(x, u) \in X$ satisfies

$$x(t) \in B(t) \text{ for all } t \in [T_1, T_2], \tag{5.26}$$

$$u(t) \in V(t) \text{ for almost every } t \in [T_1, T_2] \tag{5.27}$$

and $(z, v) \in X$ satisfies (5.6). By (5.26), (5.8), (5.17)-(5.19) and the inclusion $(B, V) \in \mathcal{W}$, for all $t \in [T_1, T_2]$

$$d_E(z(t), A(t)) \le ||z(t) - x(t)|| + d_E(x(t), Ax(t))$$
$$\le ||z(t) - x(t)|| + d_E(B(t), A^{(r_2)}(t)) + r_2 \le 3\gamma + r_2 \le r_0,$$
$$z(t) \in A^{(r_0)}(t) = \bar{A}(t).$$

In view of (5.27), (5.9), (5.6), (5.17)-(5.19) and the inclusion $(B, V) \in \mathcal{W}$, for almost every $t \in [T_1, T_2]$

$$d_F(v(t), U(t)) \le ||v(t) - u(t)|| + d_F(u(t), U(t))$$
$$\le ||v(t) - u(t)|| + d_F(v(t), U^{(r_2)}(t)) + r_2 \le 3\gamma + r_2 \le r_0,$$
$$v(t) \in U^{(r_0)}(t) = \bar{U}(t).$$

Therefore C(ii) holds. This completes the proof of Proposition 5.8. □

6. Proofs of Theorems 4.1 and 4.2

Set $\mathcal{A}_1 = \tilde{\mathcal{M}}$ and $\mathcal{A}_2 = \mathcal{L}$. In the case of Theorem 4.1 we put $\mathcal{A}_3 = \mathcal{P}_E \times \mathcal{P}_F$ and in the case of Theorem 4.2 put $\mathcal{A}_3 = \mathcal{P}_F$. Set $\mathcal{D} = \mathcal{N}$. For each $a \in \tilde{\mathcal{M}}$ put $\phi_a = I^a$ and for each $H \in \mathcal{L}$ set

$$S_H = \{(x, u) \in X : \ x(T_1) \in B \text{ and } x'(t) = H(t, x(t), u(t)), \ t \in [T_1, T_2] \text{ a.e.}\}.$$

In the case of Theorem 4.1 for any $a = (A, U) \in \mathcal{P}_E \times \mathcal{P}_F$ set

$$Q_a = \{(x, u) \in X : \ x(t) \in A(t), \ t \in [T_1, T_2] \text{ and } u(t) \in U(t), \ t \in [T_1, T_2] \text{ a.e.}\}.$$

In the case of Theorem 4.2 for any $a \in \mathcal{P}_F$ set

$$Q_a = \{(x, u) \in X : \ u(t) \in a(t), \ t \in [T_1, T_2] \text{ a.e.}\}.$$

By (4.17) and Propositions 5.1 and 5.2, (A1) holds. In view of (4.17) and property (viii), (A2) holds. By Theorem 2.1 we need to show that (H1) and (H2) hold. (H2) follows from Proposition 5.2. Therefore we need only to verify (H1). In view of Proposition 3.1 it is sufficient to show that (A3)-(A7) hold. Note that (A3) follows from Proposition 5.3. Proposition 5.4 implies (A4). By Proposition 5.5, (A5) is true. It is clear that (A6) follows from Proposition 5.6 and (A7) follows from Proposition 5.8. This completes the proof of Theorems 4.1 and 4.2.

7. Extensions of Theorems 4.1 and 4.2

We study the optimal control problem

$$I^f(x, u) \to \min, \ (x, u) \in X(\{z\}, G, A, U)$$

where $(f, G) \in \mathcal{N}$, $z \in E$, $A \in \mathcal{P}_E$, $U \in \mathcal{P}_F$.

For each $a = (f, G, z, A, U) \in \mathcal{N} \times E \times \mathcal{P}_E \times \mathcal{P}_F$ we define $J_a : X \to R^1 \cup \{\infty\}$ by

(7.1) $\quad J_a(x, u) = I^f(x, u)$ if $(x, u) \in X(\{z\}, G, A, U)$, otherwise $J_a(x, u) = \infty$.

By Propositions 5.1 and 5.2 for each $a \in \mathcal{N} \times E \times \mathcal{P}_E \times \mathcal{P}_F$, $J^a : \ X \to R^1 \cup \{\infty\}$ is a lower semicontinuous functional on (X, ρ).

We will prove the following result.

Theorem 7.1. *Let $\mathcal{A}$ be the closure of the set $\{a \in \mathcal{N} \times E \times \mathcal{P}_E \times \mathcal{P}_F : \inf(J_a) < \infty\}$ in the space $\mathcal{N} \times E \times \mathcal{P}_E \times \mathcal{P}_F$ with the strong topology. Then there exists an everywhere dense (in the strong topology) subset $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{B}$ the minimization problem for J_a on (X, ρ) is strongly well-posed with respect to $\mathcal{A}$.*

We also study the optimal control problem

$$I^f(x,u) \to \min, \ (x,u) \in X(\{z\}, G, \tilde{A}, U)$$

where $(f, G) \in \mathcal{N}, \ z \in E, \ U \in \mathcal{P}_F$.

For each $a = (f, G, z, U) \in \mathcal{N} \times E \times \mathcal{P}_F$ we define $\widehat{J}_a : X \to R^1 \cup \{\infty\}$ by

$$\widehat{J}_a(x,u) = I^f(x,u) \text{ if } (x,u) \in X(\{z\}, G, \tilde{A}, U), \text{ otherwise } \widehat{J}_a(x,u) = \infty. \tag{7.2}$$

By Propositions 5.1 and 5.2 for each $a \in \mathcal{N} \times E \times \mathcal{P}_F$, $\widehat{J}^a : X \to R^1 \cup \{\infty\}$ is a lower semicontinuous functional on (X, ρ).

We will prove the following result.

Theorem 7.2. *Let $\mathcal{A}$ be the closure of the set $\{a \in \mathcal{N} \times E \times \mathcal{P}_F : \inf(\widehat{J}_a) < \infty\}$ in the space $\mathcal{N} \times E \times \mathcal{P}_F$ with the strong topology. Then there exists an everywhere dense (in the strong topology) subset $\mathcal{B} \subset \mathcal{A}$ which is a countable intersection of open (in the weak topology) subsets of $\mathcal{A}$ such that for each $a \in \mathcal{B}$ the minimization problem for $\widehat{J}_a$ on (X, ρ) is strongly well-posed with respect to $\mathcal{A}$.*

Proofs of Theorems 7.1 and 7.2. Set $\mathcal{A}_1 = \tilde{\mathcal{M}}$ and $\mathcal{A}_2 = \mathcal{L} \times E$. In the case of Theorem 7.1 we put $\mathcal{A}_3 = \mathcal{P}_E \times \mathcal{P}_F$ and in the case of Theorem 7.2 put $\mathcal{A}_3 = \mathcal{P}_F$. Set $\mathcal{D} = \mathcal{N} \times E$. For each $a \in \tilde{\mathcal{M}}$ put $\phi_a = I^a$ and for each $a = (H, z) \in \mathcal{L} \times E$ set

$$S_a = \{(x,u) \in X : \ x(T_1) = z \text{ and } x'(t) = H(t, x(t), u(t)), \ t \in [T_1, T_2] \text{ a.e.}\}.$$

In the case of Theorem 7.1 for any $a = (A, U) \in \mathcal{P}_E \times \mathcal{P}_F$ set

$$Q_a = \{(x,u) \in X : \ x(t) \in A(t), \ t \in [T_1, T_2] \text{ and } u(t) \in U(t), \ t \in [T_1, T_2] \text{ a.e.}\}.$$

In the case of Theorem 7.2 for any $a \in \mathcal{P}_F$ set

$$Q_a = \{(x,u) \in X : \ u(t) \in a(t), \ t \in [T_1, T_2] \text{ a.e.}\}.$$

By (4.17) and Propositions 5.1 and 5.2, (A1) holds. In view of (4.17), property (viii) and the definition of $\mathcal{N}$, (A2) holds. By Theorem 2.1 we need to show that (H1) and (H2) hold. (H2) follows from Proposition 5.2. Therefore we need only to verify (H1). In view of Proposition 3.1 it is sufficient to show that (A3)-(A7) hold. Note that (A3) follows from Proposition 5.3. Proposition 5.4 implies (A4). By Proposition 5.5, (A5) is true. It is clear that (A6) follows from Proposition 5.6 and (A7) follows from Proposition 5.8. This completes the proof of Theorems 7.1 and 7.2. □

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