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ON AVERAGED, FIRMLY NONEXPANSIVE AND QUASI-NONEXPANSIVE OPERATORS, AND MORE OF THAT

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Abstract. For an operator T on a Hilbert space, inequalities that express the behaviour of the distance between two original points compared to the distance between their images under T, form the base of a hierarchical classification of operators into three different families, each family consisting of classes indexed by a nonnegative real parameter. We investigate proper inclusions between these classes, we range some existing types of operators into the hierarchical classification, and we extend an algorithm to obtain a common fixed point of a family of operators to a more extended family.

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle , \rangle$ and norm $\| \, \|$ derived from it, let $\mathcal{D} \subset \mathcal{H}$ be a nonempty subset of $\mathcal{H}$, and let $T : \mathcal{D} \to \mathcal{H}$ be an (in general nonlinear) operator (we use the words operator and mapping as synonyms). In [10], we considered hierarchical structures, based on a nonnegative real number ν, for each of two families of operators $T : \mathcal{D} \to \mathcal{H}$ but with the added condition that each such T had a nonempty set $\mathrm{Fix}T$ of fixed points. It led to what was called, respectively, the classes of ν-firmly nonexpansive mappings on $\mathcal{D}$ and the classes of ν-quasi nonexpansive mappings on $\mathcal{D}$. As has been remarked in [10] , the demand (and only reason) that in both families each operator had a nonempty set of fixed points was justified by the resulting property that then, for each fixed ν, the corresponding class of the first family was a subset of the corresponding class of the second family.

Continuing the investigation on the hierarchical structure of those operators led to the conclusion that a more complete (and more interesting) description could be obtained when the defining inequality that formed the base of the ν-firmly nonexpansive mappings on $\mathcal{D}$ was also applied for operators $T : \mathcal{D} \to \mathcal{H}$, whether or not they have fixed points. As such, in the present paper a third family could be introduced, in which the classes in each family are still determined by a nonnegative real number ν. At the start of Section 2, these three families have been (re)-introduced, leading for each fixed ν to the class $\mathrm{FNE}(\mathcal{D}, \nu)$ of ν-firmly nonexpansive mappings (whether or not the mapping has fixed points), to the (sub)class $\mathrm{FNEFIX}(\mathcal{D}, \nu)$ of ν-firmly nonexpansive mappings having a nonempty set of fixed points (and as such we have the inclusion that $\mathrm{FNEFIX}(\mathcal{D}, \nu) \subset \mathrm{FNE}(\mathcal{D}, \nu)$, and finally to the class $\mathrm{QNE}(\mathcal{D}, \nu)$ of ν-quasi-nonexpansive mappings, in which by the mere definition each mapping has fixed points. We look in particular at the existence of proper

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inclusions, both within one and the same family but for classes with different values of ν, and between classes corresponding to the same value of ν but belonging to different families.

In Section 3, we investigate some interesting properties of the obtained classes of operators. For instance, we show in Proposition 3.1 that an operator T is averaged (see [5],[6],[8]) if and only if T belongs to some class FNE$(\mathcal{D},\nu)$ for $\nu > 0$. In Proposition 3.4 it is shown that, when T belongs to some class FNE$(\mathcal{D},\nu)$ for $\nu > 0$, then T is strongly nonexpansive (the definition of a strongly nonexpansive operator was introduced in [5]).

As we already remarked in [10], one of the possible applications of the introduced hierarchical structures is that the inclusions that are connected to the hierarchy may make it easier to investigate if results that have been proved formerly for a small class of operators, may be extended to a superset of that class, and if such extension needs to change some side-conditions of the former results. In the last Section 4 of the present paper, we take a closer look at an existing algorithm to obtain a common fixed point of a family of operators. The algorithm was described in [8, Algorithm 1.2] for averaged operators. We look at the possibility to extend it to the classes of operators QNE$(\mathcal{H},\nu)$ for $\nu > 0$, under some stronger condition for the relaxation coefficients.

2. Characterizations and hierarchical structure of the classes of operators

Let $\mathcal{H} \equiv (\mathcal{H}, \langle,\rangle, \|\|)$ be a real Hilbert space, $\mathcal{D}$ a nonempty subset of $\mathcal{H}, T : \mathcal{D} \to \mathcal{H}$ a mapping and ν a nonnegative real number. When T has a nonempty set of fixed points, we denote that set as Fix$T$.
In [10, Theorem 3 and Theorem 4], the following result has been proved for $T : \mathcal{D} \to \mathcal{H}$ and for fixed $\nu \geq 0$:

Theorem 2.1. *The following assertions are equivalent for all $x \in D$ and all $y \in D$:*

(i) $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \nu\|Tx - x - (Ty - y)\|^2$

(ii) $\langle x - Tx - (y - Ty), Ty - Tx\rangle \leq (\dfrac{1-\nu}{2})\|Tx - x - (Ty - y)\|^2$

(iii) $\langle x - Tx - (y - Ty), y - x\rangle \leq -(\dfrac{1+\nu}{2})\|Tx - x - (Ty - y)\|^2$

(iv) $\|x - Tx - (y - Ty) + \dfrac{1}{1+\nu}(y - x)\| \leq \dfrac{1}{1+\nu}\|y - x\|$

(v) $\|Tx - Ty\|^2 \leq (\dfrac{1-\nu}{2})\|Tx - x - (Ty - y)\|^2 + \langle x - y, Tx - Ty\rangle$

(vi) $\|Tx - Ty\|^2 \leq \dfrac{1-\nu}{1+\nu}\|x - y\|^2 + \dfrac{2\nu}{1+\nu}\langle Tx - Ty, x - y\rangle.$

(vii) *When $0 \leq \nu < 1$: $\|Tx - Ty\| \leq \|(1 - \nu)(x - y) + \nu(Tx - Ty)\|$.*

From Theorem 2.1, we derive for each fixed nonnegative ν the following classes of mappings $T : \mathcal{D} \to \mathcal{H}$: in the first place, when for T one of the equivalent assertions in Theorem 2.1 for some given ν is valid, then we say that T is ν-firmly nonexpansive; the class of all such mappings is denoted as $\mathrm{FNE}(\mathcal{D},\nu)$. Secondly, the subclass of mappings T of $\mathrm{FNE}(\mathcal{D},\nu)$ having a nonempty set $\mathrm{Fix}T$ of fixed points, is denoted by $\mathrm{FNEFIX}(\mathcal{D},\nu)$. Hence, from the mere definition, we have that both classes are connected by $\mathrm{FNEFIX}(\mathcal{D},\nu) \subset \mathrm{FNE}(\mathcal{D},\nu)$ (we remark that in [10, Definition 1 and Theorems 3 and 4] only the classes of operators with fixed points have been introduced, but with another notation).

For fixed ν, a third class of operators $T : \mathcal{D} \to \mathcal{H}$ that will be used in this paper has been introduced in [10, Definition 6]. It is, for fixed ν, the class $\mathrm{QNE}(\mathcal{D},\nu)$ of ν-quasi-nonexpansive mappings $T : \mathcal{D} \to \mathcal{H}$, characterized as follows

Theorem 2.2 (see [10, Theorem 7])**.** *Let ν be a fixed nonnegative real number and let $T : \mathcal{D} \to \mathcal{H}$ be an operator with $\mathrm{Fix}T \neq \emptyset$. Then the following assertions are equivalent for all $x \in D$ and all $z \in \mathrm{Fix}T$:*

(i) $\|Tx - z\|^2 \le \|x - z\|^2 - \nu\|x - Tx\|^2$

(ii) $\langle x - Tx, z - Tx\rangle \le (\frac{1-\nu}{2})\|x - Tx\|^2$

(iii) $\langle x - Tx, z - x\rangle \le -(\frac{1+\nu}{2})\|x - Tx\|^2$

(iv) $\|x - Tx + \frac{1}{1+\nu}(z - x)\| \le \frac{1}{1+\nu}\|z - x\|$

(v) $\|Tx - z\|^2 \le (\frac{1-\nu}{2})\|x - Tx\|^2 + \langle x - z, Tx - z\rangle$

(vi) $\|Tx - z\|^2 \le \frac{1-\nu}{1+\nu}\|x - z\|^2 + \frac{2\nu}{1+\nu}\langle Tx - z, x - z\rangle$

(vii) *When $0 \le \nu < 1$: $\|Tx - z\| \le \|(1-\nu)(x - z) + \nu(Tx - z)\|$.*

For a fixed ν, and based on the respective equivalent assertions for operators T in $\mathrm{FNEFIX}(\mathcal{D},\nu)$ (Theorem 2.1) and in $\mathrm{QNE}(\mathcal{D},\nu)$ (Theorem 2.2) we obtain the inclusion $\mathrm{FNEFIX}(\mathcal{D},\nu) \subset \mathrm{QNE}(\mathcal{D},\nu)$ which, contrary to the inclusion $\mathrm{FNEFIX}(\mathcal{D},\nu) \subset \mathrm{FNE}(\mathcal{D},\nu)$, is now an inclusion between classes of operators belonging to truly different families.

For particular values of ν, we recognize some well-known classes of operators. For instance, from Theorem 2.1 (i), we immediately derive that for $\nu = 0$ the class $\mathrm{FNE}(\mathcal{D},0)$ coincides with the class of nonexpansive operators on $\mathcal{D}$ ([2],[3],[11]), while for $\nu = 1$ the class $\mathrm{FNE}(\mathcal{D},1)$ is precisely the class of firmly nonexpansive operators on $\mathcal{D}$ ([2],[13],[14]). From Theorem 2.2 (i), we see that for $\nu = 0$ the class $\mathrm{QNE}(\mathcal{D},0)$ is the class of quasi-nonexpansive operators (or general paracontractions) on $\mathcal{D}$ ([9],[12],[15]). Theorem 2 (ii) shows that for $\nu = 1$ the class $\mathrm{QNE}(\mathcal{D},1)$ coincides with the class $\mathcal{I}$ as defined in [4] and [7].

We now take a closer look at the interesting topic of the existence of proper inclusions, either within one family of operators but corresponding to different values of ν, or between different families of operators but corresponding to the same value of ν.

From the assertions in Theorem 2.1 (i) and Theorem 2.2 (i), we derive that, when μ and ν are real numbers with $\mu > \nu \ge 0$, then $\mathrm{FNE}(\mathcal{D},\mu) \subset \mathrm{FNE}(\mathcal{D},\nu)$, $\mathrm{FNEFIX}(\mathcal{D},\mu) \subset \mathrm{FNEFIX}(\mathcal{D},\nu)$ and $\mathrm{QNE}(\mathcal{D},\mu) \subset \mathrm{QNE}(\mathcal{D},\nu)$. In particular, this leads to $\mathrm{FNE}(\mathcal{D},0) = \bigcup_{\nu=0}^{+\infty}\mathrm{FNE}(\mathcal{D},\nu)$, and analogous expressions are valid for the two other families.

Remark 2.3. In some parts of the proof of the next proposition, use will be made of the following result: when C is a proper nonempty closed convex subset of $\mathcal{H}$, P is the projection operator onto C, $\mathbf{1}$ is the identity operator on $\mathcal{H}$, and $T_\lambda = \mathbf{1} + \lambda(P - \mathbf{1})$ with $0 < \lambda \leq 2$, then $T_\lambda : \mathcal{H} \to \mathcal{H}$ has C as a nonempty set of fixed points, and for arbitrary x and y in $\mathcal{H}$ we have that

$$\|T_\lambda x - T_\lambda y\|^2 \leq \|x - y\|^2 - (\frac{2-\lambda}{\lambda})\|(T_\lambda x - x) - (T_\lambda y - y)\|^2.$$

Hence, $T_\lambda \in \mathrm{FNEFIX}(\mathcal{H},\nu)$ with $\nu = \frac{2-\lambda}{\lambda}$. Evidently, we also have that $T_\lambda \in \mathrm{QNE}(\mathcal{H},\nu)$ with the same value of ν.

Proposition 2.4. *The following proper inclusions are valid for real numbers μ and ν:*

a) *When $\nu > 0$, then*
 (i) $\mathrm{FNEFIX}(\mathcal{D},\nu) \subsetneqq \mathrm{FNEFIX}(\mathcal{D},0)$
 (ii) $\mathrm{QNE}(\mathcal{D},\nu) \subsetneqq \mathrm{QNE}(\mathcal{D},0)$
b) *When $\mu > \nu > 0$, then*
 (i) $\mathrm{FNEFIX}(\mathcal{D},\mu) \subsetneqq \mathrm{FNEFIX}(\mathcal{D},\nu)$
 (ii) $\mathrm{QNE}(\mathcal{D},\mu) \subsetneqq \mathrm{QNE}(\mathcal{D},\nu)$
c) $\mathrm{FNEFIX}(\mathcal{D},0) \subsetneqq \mathrm{QNE}(\mathcal{D},0)$
d) *When $0 < \nu < 1$, then* $\mathrm{FNEFIX}(\mathcal{D},\nu) \subsetneqq \mathrm{QNE}(\mathcal{D},\nu)$

Proof. a)(i) Take for $\mathcal{H}$ the Euclidean space $\mathbb{R}^2$ in which an orthonormal reference system xOy is chosen, and let T_2 be reflection with respect to the x-axis. Then $T_2 = \mathbf{1} + 2(P - \mathbf{1})$ where P is the projection operator onto the x-axis. By Remark 2.3, we know that $T_2 \in \mathrm{FNEFIX}(\mathbb{R}^2,0)$. Taking in particular the points $x = (0,2)$ and $y = (1,3)$, then $\|T_2x - T_2y\|^2 = \|x - y\|^2$, but $\|T_2x - x - (T_2y - y)\|^2 = 4$. Hence, when $\nu > 0$ then $T_2 \notin \mathrm{FNEFIX}(\mathbb{R}^2,\nu)$.

a)(ii) See Example 11 in [10].

b)(i) In order to make the proof somewhat more transparent, let ε be a real number such that $0 < \varepsilon < 1$, take for μ the value 1 and for ν the value $\frac{1-\varepsilon}{1+\varepsilon}$. We'll apply Remark 2.3 for $\mathcal{H} = \mathbb{R}^2$ and C the x-axis of an orthonormal reference system in $\mathbb{R}^2$, in order to obtain that $\mathrm{FNEFIX}(\mathbb{R}^2,1) \subsetneqq \mathrm{FNEFIX}(\mathbb{R}^2,\frac{1-\varepsilon}{1+\varepsilon})$. To this end, we know from Remark 3 that when we take for λ the value $1+\varepsilon$, then $T_{1+\varepsilon}$ belongs to the class $\mathrm{FNEFIX}(\mathbb{R}^2,\frac{1-\varepsilon}{1+\varepsilon})$. We now show that $T_{1+\varepsilon}$ does not belong to $\mathrm{FNEFIX}(\mathbb{R}^2,1)$. Indeed, in order that $T_{1+\varepsilon}$ really should be an element of $\mathrm{FNEFIX}(\mathbb{R}^2,1)$, it would be necessary that the inequality $\|T_{1+\varepsilon}x - T_{1+\varepsilon}y\|^2 \leq \|x-y\|^2 - 1\|T_{1+\varepsilon}x - x - (T_{1+\varepsilon}y - y)\|^2$ should be valid for all $x \in \mathbb{R}^2$ and all $y \in \mathbb{R}^2$. But taking the points $x = (0,2)$ and $y = (1,3)$, it follows from an easy computation that $\|T_{1+\varepsilon}x - T_{1+\varepsilon}y\|^2 = 1 + \varepsilon^2$, that $\|x - y\|^2 = 2$, and that $\|T_{1+\varepsilon}x - x - (T_{1+\varepsilon}y - y)\|^2 = (1+\varepsilon)^2$. Hence, the mentioned inequality is not valid for the chosen pair of points.

b)(ii) Consider the same example as in the proof of b)(i). We know that $T_{1+\varepsilon}$ belongs to $\mathrm{QNE}(\mathbb{R}^2,\frac{1-\varepsilon}{1+\varepsilon})$. In order to show that $T_{1+\varepsilon}$ is not an element of $\mathrm{QNE}(\mathbb{R}^2,1)$, it is sufficient to find a point x in $\mathbb{R}^2$ and a fixed point z of $T_{1+\varepsilon}$ such that the inequality $\|T_{1+\varepsilon}x - z\|^2 \leq \|x - z\|^2 - 1\|x - T_{1+\varepsilon}x\|^2$ is not valid. Take $x = (0,2)$,

$z = (0,0)$. Then $T_{1+\varepsilon}x = (0, -2\varepsilon)$ and $x - T_{1+\varepsilon}x = (0, 2+2\varepsilon)$, and it follows easily that the stated inequality is not valid.

c) (See also Example 2.3 in [15]). Let $\mathcal{H} = \mathbb{R}$, and let $T : \mathbb{R} \to \mathbb{R}$ be the mapping defined by $T0 = 0$ and, for $x \neq 0$, $Tx = \frac{2}{3}x \sin\frac{1}{x}$. Then $\mathrm{Fix}T = \{0\}$, and we see immediately that $|Tx - 0|^2 \leq |x - 0|^2$. Hence, $T \in \mathrm{QNE}(\mathbb{R}, 0)$. But $T \notin \mathrm{FNEFIX}(\mathbb{R}, 0)$. Indeed, taking $x = \frac{2}{\pi}$ and $y = \frac{2}{3\pi}$ we have that $|Tx - Ty| > |x - y|$.

d) We first remark that, when ν is given with $0 < \nu < 1$, then there exists a positive integer N (that we choose to the bigger than 2) such that $\nu \leq \frac{N-1}{N+1}$. Indeed, there exists $\varepsilon > 0$ such that $\nu = 1 - \varepsilon$. Choose $N > 2$ such that $\varepsilon \geq \frac{2}{N+1}$. Then $\nu = 1 - \varepsilon \leq 1 - \frac{2}{N+1}$, and hence $\nu \leq \frac{N-1}{N+1}$.

For this value of N, define $T : \mathbb{R} \to \mathbb{R}$ such that $T0 = 0$ and $Tx = \frac{1}{N}x\cos\frac{1}{x}$, for $x \in \mathbb{R}\backslash\{0\}$. Then $\mathrm{Fix}\ T = \{0\}$. The operator T belongs to $\mathrm{QNE}(\mathcal{D}, \nu)$, since the inequality (i) of Theorem 2.2 is true for $z = 0$ and for all $x \in \mathbb{R}$. Indeed, for $x \neq 0$ we have

$$\left|\frac{1}{N}x\cos\frac{1}{x}\right|^2 \leq |x|^2 - \nu\left|x - \frac{1}{N}x\cos\frac{1}{x}\right|^2 \Longleftrightarrow \frac{1}{N^2}\cos^2\frac{1}{x} \leq 1 - \nu\left|1 - \frac{1}{N}\cos\frac{1}{x}\right|^2.$$

Now, the maximal value of the left-hand side of the obtained inequality is $\frac{1}{N^2}$, and the minimal value of the right-hand side is $1 - \nu(1 + \frac{1}{N})^2$. Hence, in order that the needed inequality is true, it is sufficient that $\frac{1}{N^2} \leq 1 - \nu(1 + \frac{1}{N})^2$, and this is fulfilled since $\nu \leq \frac{N-1}{N+1}$.

We now show that the operator T does not belong to $\mathrm{FNEFIX}(\mathcal{D}, \nu)$. This is done by choosing two points x and y in $\mathbb{R}$ for which inequality (i) in Theorem 2.1 is not true. Choose $x = \frac{1}{(N-1)\pi}, y = \frac{1}{N\pi}$, and suppose for the moment that N is even (for N odd, the following computation is immediately adapted). Then $Tx - Ty = \frac{-2N+1}{N^2(N-1)\pi}$, $x - y = \frac{1}{N(N-1)\pi}$, and $Tx - x - (Ty - y) = \frac{-3N+1}{N^2(N-1)\pi}$. Now, in order that inequality (i) in Theorem (2.1) should be true for those two points, the following inequality has to be true.

$$\left|\frac{-2N+1}{N^2(N-1)\pi}\right|^2 \leq \left|\frac{1}{N(N-1)\pi}\right|^2 - \nu\left|\frac{-3N+1}{N^2(N-1)\pi}\right|^2,$$

and this is the case if and only if $(-2N+1)^2 \leq N^2 - \nu(-3N+1)^2$, and continuing, this is true if and only if $\nu \leq \frac{-3N^2+4N-1}{(-3N+1)^2}$. In the right-hand side of this last inequality, the denominator is positive; however, the numerator is negative when we assume that $N > 2$. Hence, ν should have to be negative, and this is clearly a contradiction. □

3. Some properties of operators in connection to their class

The hierarchical structure of the classes of operators, based on the proper inclusions as described in the foregoing section, makes it acceptable to assume that properties of operators may be connected, in the first place, to the family of operators they belong to (either .-quasi -nonexpansive or .-firmly nonexpansive) and secondly, within each family, to the specific class(es) they belong to defined by one value of ν or by grouping some values of ν (e.g., $\nu \geq 1$, or $\nu \neq 0$, etc.). In this section, we mention a few of those properties.

As for the first property, we recall that an operator $T : \mathcal{D} \to \mathcal{H}$ is called averaged (or α-averaged)([5],[6],[8]) iff there exists a nonexpansive operator N and a real number $\alpha \in]0,1[$ such that $T = (1-\alpha)\mathbf{1}_{\mathcal{D}} + \alpha N$ (where $\mathbf{1}_{\mathcal{D}}$ denotes the identity operator on $\mathcal{D}$). Our first property shows that the family of all averaged operators is precisely the union of all classes FNE$(\mathcal{D},\nu)$ for $\nu > 0$.

Proposition 3.1.

$$T \textit{ is averaged } \Longleftrightarrow T \in \textit{FNE}(\mathcal{D},\nu) \textit{ for } \nu > 0.$$

Proof. Suppose that T is α-averaged. Then there exists $\alpha \in]0,1[$ and a nonexpansive operator N such that $T = (1-\alpha)\mathbf{1}_{\mathcal{D}} + \alpha N$. We know that the family of the nonexpansive operators is precisely the family FNE$(\mathcal{D},0)$. Put $\nu = \frac{1-\alpha}{\alpha}$, or equivalently $\alpha = \frac{1}{1+\nu}$; then $\nu > 0$. Hence, T may also be represented as $T = \frac{1}{1+\nu}N + \frac{\nu}{1+\nu}\mathbf{1}_{\mathcal{D}}$.We show that $T \in$ FNE $(\mathcal{D},\nu)$. Since $N = (1+\nu)T - \nu\mathbf{1}_{\mathcal{D}}$, we have, for $x \in \mathcal{D}$ and $y \in \mathcal{D}$:

$$\begin{aligned}
&\|Nx - Ny\|^2 - \|x-y\|^2 \\
&= \|(1+\nu)(Tx - Ty) - \nu(x-y)\|^2 - \|x-y\|^2 \\
&= (1+\nu)^2\|Tx - Ty\|^2 - (1-\nu^2)\|x-y\|^2 - 2\nu(1+\nu)\langle Tx - Ty, x-y\rangle.
\end{aligned}$$

The left-hand side of the resulting equality is nonpositive, and so the same is true for the right-hand side. Dividing the three terms of the right-hand side by $(1+\nu)^2$ and rearranging, there results

$$\|Tx - Ty\|^2 \leq \frac{1-\nu}{1+\nu}\|x-y\|^2 + \frac{2\nu}{1+\nu}\langle Tx - Ty, x-y\rangle.$$

According to Theorem 2.1, (vi), this means precisely that $T \in$ FNE$(\mathcal{D},\nu)$.

Conversely, suppose that $T \in$ FNE$(\mathcal{D},\nu)$ is given for some $\nu > 0$. Define the operator N by $N = (1+\nu)T - \nu\mathbf{1}_{\mathcal{D}}$. By the same computation as above we have

$$\begin{aligned}
&\frac{1}{(1+\nu)^2}[\|Nx - Ny\|^2 - \|x-y\|^2] \\
&\qquad\qquad = \|Tx - Ty\|^2 - \frac{1-\nu}{1+\nu}\|x-y\|^2 - \frac{2\nu}{1+\nu}\langle Tx - Ty, x-y\rangle.
\end{aligned}$$

The right-hand side of this equality is nonpositive, according to Theorem 2.1,(vi). So the same is true for the left-hand side; since $\frac{1}{(1+\nu)^2} > 0$, we conclude that N is nonexpansive. Writing $\alpha = \frac{1}{1+\nu}$ we obtain that $T = \alpha N + (1-\alpha)\mathbf{1}_{\mathcal{D}}$. Hence, T is α-averaged. □

Let T be an element of FNE$(\mathcal{D},\nu), \nu \geq 0$. Then, the inequality (i) of Theorem 2.1 is true for T. Put $S = \mathbf{1}_{\mathcal{D}} - T$. Replacing in the mentioned inequality T by

$\mathbf{1}_{\mathcal{D}} - S$, and elaborating, we get

$$\begin{aligned} &T \in \mathrm{FNE}(\mathcal{D}, \nu) \\ &\iff \|x - Sx - (y - Sy)\|^2 \leq \|x - y\|^2 - \nu\|x - Sx - x - (y - Sy - y)\|^2 \\ &\iff \|Sx - Sy\|^2 + \|x - y\|^2 - 2\langle Sx - Sy, x - y\rangle \leq \|x - y\|^2 - \nu\|Sx - Sy\|^2. \end{aligned}$$

We formulate the obtained result as follows (cfr. [6, Lemma 2.1])

Proposition 3.2. *For $\nu \geq 0$ and for $S = \mathbf{1}_{\mathcal{D}} - T$ we have $T \in \mathrm{FNE}(\mathcal{D}, \nu)$ if and only if $(\frac{1+\nu}{2})\|Sx - Sy\|^2 \leq \langle Sx - Sy, x - y\rangle$*

In [5], Bruck and Reich introduced the following definition.

Definition 3.3. Let E be a Banach space. An operator $T : \mathcal{D}(T) \subset E \to E$ is called strongly nonexpansive if T is nonexpansive and if the following is true: whenever the sequence $\{x_n - y_n\}$ is bounded and $\|x_n - y_n\| - \|Tx_n - Ty_n\| \to 0$, then $(x_n - y_n) - (Tx_n - Ty_n) \to 0$.

They proved the following result: when E is uniformly convex and T is firmly nonexpansive, then T is strongly nonexpansive.

When working in a Hilbert space, a firmly nonexpansive operator T belongs to $\mathrm{FNE}(\mathcal{D}, 1)$. We are able to generalize the property of Bruck and Reich for all operators $T : \mathcal{D} \to \mathcal{H}$ that belong to $\mathrm{FNE}(\mathcal{D}, \nu)$ for $\nu > 0$, as follows:

Proposition 3.4. *When $T \in \mathrm{FNE}(\mathcal{D}, \nu)$ with $\nu > 0$, then T is strongly nonexpansive.*

Proof. Since $\mathrm{FNE}(\mathcal{D}, \nu) \subset \mathrm{FNE}(\mathcal{D}, 0)$, T is certainly nonexpansive. According to Theorem 2.1, (i), we know that for each positive integer n the following inequality is true

$$\|Tx_n - Ty_n\|^2 - \|x_n - y_n\|^2 \leq -\nu\|Tx_n - x_n - (Ty_n - y_n)\|^2,$$

and so also the next inequality (3.1) is true for each such n

$$(\|Tx_n - Ty_n\| - \|x_n - y_n\|)(\|Tx_n - Ty_n\| + \|x_n - y_n\|) \leq -\nu\|Tx_n - x_n - (Ty_n - y_n)\|^2. \tag{3.1}$$

Now, when the assumptions for controlling strong nonexpansivity are true, then the first factor on the left-hand side tends to zero, in the second factor on the left-hand side the sequence $\{\|x_n - y_n\|\}$ is a bounded sequence of numbers, and it is easy to show that also the sequence of numbers $\|Tx_n - Ty_n\|$ is bounded. Together, this leads to the result that the left-hand side of (3.1) tends to zero. The sequence formed by the numbers of the right-hand side of (3.1) consists of non-positive numbers that are not smaller than those on the left-hand side, and those tend to zero. Hence, also $\nu\|Tx_n - x_n - (Ty_n - y_n)\|^2 \to 0$. So, when $\nu \neq 0$, then $(x_n - y_n) - (Tx_n - Ty_n) \to 0$. $\square$

The former result is of course also valid for the operators belonging to FNEFIX$(\mathcal{D},\nu)$ with $\nu > 0$. Each class FNEFIX$(\mathcal{D},\nu)$, however, is properly contained in the class QNE$(\mathcal{D},\nu)$. We now give a possible adaptation of the notion of strongly nonexpansive operator T for classes of ν-quasi-nonexpansive operators, followed by an extension of Proposition 3.4 for such classes.

Definition 3.5. An operator $T : \mathcal{D} \to \mathcal{H}$ is called strongly quasi-nonexpansive if T is quasi-nonexpansive and if the following is true: whenever for a sequence of fixed points z_n of T the sequence $\{x_n - z_n\}$ is bounded and $\|x_n - z_n\| - \|Tx_n - z_n\| \to 0$, then $x_n - Tx_n \to 0$.

Proposition 3.6. *When $T \in$ QNE$(\mathcal{D},\nu)$ with $\nu > 0$, then T is strongly quasi-nonexpansive.*

Proof. The proof follows completely the same pattern as the proof of Proposition 3.4. So, it will not be elaborated again. □

4. An algorithm to obtain a common fixed point of a family of ν-quasi-nonexpansive operators

The algorithm that we want to investigate, is Algorithm 1.2 in [8]; it goes as follows:

Fix $x_0 \in \mathcal{H}$ and, for every $n \in \mathbb{N}$, set

$$\begin{aligned} x_{n+1} = x_n + \lambda_n(T_{1,n}(T_{2,n}(\cdots T_{m-1,n}(T_{m,n}x_n + e_{m,n}) + e_{m-1,n} \cdots) \\ + e_{2,n}) + e_{1,n} - x_n), \end{aligned}$$

where m is a fixed positive integer, where each $T_{i,n}$ $(1 \le i \le m)$ is an $\alpha_{i,n}$-averaged operator on $\mathcal{H}$, where each $e_{i,n}$ $(1 \le i \le m)$ is a vector in $\mathcal{H}$, and where each $\lambda_n \in]0,1]$.

As we have seen, the averaged operators on $\mathcal{H}$ are those that belong to the classes FNE$(\mathcal{H},\nu)$ for $\nu > 0$. We will investigate that same algorithm for the classes of operators QNE$(\mathcal{H},\nu)$ for $\nu > 0$ which, as we know, form supersets of the former ones. We are interested, in particular, if meaningful results for the algorithm, when applied to the broader class of operators, are still valid, but possibly with suitable adaptations. In order to make the computations somewhat easier to follow, we shall take for m the value 4. So, we consider the following

Algorithm 4.1. Starting from some point $x_0 \in H$ and, for every $n \in \mathbb{N}$, set

$$x_{n+1} = x_n + \lambda_n(T_{1,n}(T_{2,n}(T_{3,n}(T_{4,n}x_n + e_{4,n}) + e_{3,n}) + e_{2,n}) + e_{1,n} - x_n),$$

where, for each i with $1 \le i \le 4$, $T_{i,n}$ belongs to QNE$(\mathcal{H},\nu_{i,n})$ with $\nu_{i,n} > 0$, $e_{i,n}$ belongs to $\mathcal{H}$, and where $\lambda_n \in]0,1]$.

The essential elements needed to assure weak convergence of an orbit of Algorithm 4.1 are given in the following

Theorem 4.2. *Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary orbit of Algorithm 4.1. Suppose that*

(a) $$F = \bigcap_{n=0}^{+\infty} \bigcap_{i=1}^{4} \mathit{Fix} T_{i,n} \neq \emptyset$$

and

(b) $$\forall i, 1 \leq i \leq 4 : \sum_{n\in\mathbb{N}} \lambda_n \|e_{i,n}\| < +\infty.$$

Then:

(i) *The sequence* $(x_n)_{n\in\mathbb{N}}$ *is quasi-Fejér monotone with respect to* F *and, hence, it is bounded.*

(ii) *For fixed* $z \in F$, *put*

$$e_n^* = \lambda_n(T_{1,n}(T_{2,n}(T_{3,n}(T_{4,n}x_n + e_{4,n}) + e_{3,n}) + e_{2,n}) + e_{1,n} - z).$$

Then the sequence $\{\|e_n^*\|\}_{n\in\mathbb{N}}$ *is bounded.*

(iii) *For fixed* $z \in F$, *put* $\zeta = \sup_n \|x_n - z\|$ *and* $\eta = 2\zeta + \sup_n \|e_n^*\|$. *Then*

$$\|x_{n+1} - z\|^2 \leq (1 - \lambda_n)\|x_n - z\|^2 + \eta\|e_n^*\|.$$

(iv) *When* $\sum_{n\in\mathbb{N}} \lambda_n < +\infty$, *then* $\sum_{n\in\mathbb{N}} \|e_n^*\| < +\infty$.

(v) *When* $\sum_{n\in\mathbb{N}} \lambda_n < +\infty$, *we have*

(v)a $\sum_{n\in\mathbb{N}} \lambda_n \nu_{1,n}\|(\mathbf{1} - T_{1,n})T_{2,n}T_{3,n}T_{4,n}x_n\|^2 < +\infty$

(v)b $\sum_{n\in\mathbb{N}} \lambda_n \nu_{2,n}\|(\mathbf{1} - T_{2,n})T_{3,n}T_{4,n}x_n\|^2 < +\infty$

(v)c $\sum_{n\in\mathbb{N}} \lambda_n \nu_{3,n}\|(\mathbf{1} - T_{3,n})T_{4,n}x_n\|^2 < +\infty$

(v)d $\sum_{n\in\mathbb{N}} \lambda_n \nu_{4,n}\|(\mathbf{1} - T_{4,n})x_n\|^2 < +\infty$

(v)e *For fixed* $z \in F$

$$\sum_{n\in\mathbb{N}} \lambda_n \|T_{1,n}T_{2,n}T_{3,n}T_{4,n}x_n - z\|^2 < +\infty.$$

Proof. (i) Take $z \in F$. Then

$$\begin{aligned} x_{n+1} - z &= (1 - \lambda_n)(x_n - z) \\ &\quad + \lambda_n(T_{1,n}(T_{2,n}(T_{3,n}(T_{4,n}x_n + e_{4,n}) + e_{3,n}) + e_{2,n}) + e_{1,n} - z). \end{aligned} \tag{4.1}$$

Since each $T_{i,n}$ is also quasi-nonexpansive, i.e., belongs to $\mathrm{QNE}(\mathcal{H}, 0)$, we obtain after repeated application of the characteristic inequality of quasi-nonexpansivity together with the triangle inequality

$$\begin{aligned} &\|T_{1,n}(T_{2,n}(T_{3,n}(T_{4,n}x_n + e_{4,n}) + e_{3,n}) + e_{2,n}) + e_{1,n} - z\| \\ &\leq \|e_{1,n}\| + \|T_{2,n}(T_{3,n}(T_{4,n}x_n + e_{4,n}) + e_{3,n}) + e_{2,n}) - z\| \\ &\leq \|e_{1,n}\| + \|e_{2,n}\| + \|T_{3,n}(T_{4,n}x_n + e_{4,n}) + e_{3,n} - z\| \\ &\leq \cdots \\ &\leq \|e_{1,n}\| + \|e_{2,n}\| + \|e_{3,n}\| + \|e_{4,n}\| + \|x_n - z\|. \end{aligned} \tag{4.2}$$

Hence, taking norms in (4.1), we obtain

$$\begin{aligned} \|x_{n+1} - z\| &\leq (1 - \lambda_n)\|x_n - z\| + \lambda_n\|x_n - z\| + \lambda_n \sum_{i=1}^{4} \|e_{i,n}\| \\ &= \|x_n - z\| + \lambda_n \sum_{i=1}^{4} \|e_{i,n}\|. \end{aligned} \tag{4.3}$$

According to assumption (b) of Theorem 4.2, we know that $\sum_{n\in\mathbb{N}} \lambda_n(\sum_{i=1}^4 \|e_{i,n}\|) < +\infty$. This means, according to Definition 1.1 in [7], that the sequence $(x_n)_{n\in\mathbb{N}}$ is quasi-Fejér monotone of type I with respect to F. As a consequence, the sequence $(x_n)_{n\in\mathbb{N}}$ is bounded.

We remark that for the proof of this part we only used the fact that all operators are quasi-nonexpansive; hence, this result would still be valid when all $T_{i,n}$ should belong to $\mathrm{QNE}(\mathcal{H}, 0)$.

(ii) Putting $z_n^* = x_n + \lambda_n(z - x_n)$, it follows easily that

$$z_n^* + e_n^* = x_{n+1}. \tag{4.4}$$

From inequality (4.2) in the proof of (i) above we may write that $\frac{\|e_n^*\|}{\lambda_n} \leq \sum_{i=1}^4 \|e_{i,n}\| + \|x_n - z\|$, and hence

$$\|e_n^*\| \leq \lambda_n(\sum_{i=1}^4 \|e_{i,n}\|) + \lambda_n\|x_n - z\|. \tag{4.5}$$

From assumption (b) of this Theorem, the sequence with general term λ_n $(\sum_{i=1}^4 \|e_{i,n}\|)$ is bounded. Moreover, due to the result in (i), we know that also the sequence $\{x_n\}_{n\in\mathbb{N}}$ is bounded, and we know that $\lambda_n \in]0, 1]$ for each n. Hence, $\sup \|e_n^*\| < +\infty$.

(iii) From (4.4) we derive that $\|x_{n+1} - z\| \leq \|z_n^* - z\| + \|e_n^*\|$. Hence, $\|x_{n+1} - z\|^2 \leq \|z_n^* - z\|^2 + (2\|z_n^* - z\| + \|e_n^*\|)\|e_n^*\|$. But, from the definition of z_n^* we derive that $\|z_n^* - z\| = (1 - \lambda_n)\|x_n - z\|$, from which we conclude that $\|z_n^* - z\| \leq \|x_n - z\|$. Using these last two results in the inequality concerning $\|x_{n+1} - z\|^2$ in (iii), we deduce:

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1 - \lambda_n)^2\|x_n - z\|^2 + \eta\|e_n^*\| \\ &\leq (1 - \lambda_n)\|x_n - z\|^2 + \eta\|e_n^*\|. \end{aligned}$$

(iv) The result of (iv) follows immediately from inequality (4.5).

(v) From Theorem 2.2, (i) in the present paper, we know that for each $u \in \mathcal{H}$, for each $z \in F$, for each i with $1 \leq i \leq 4$ and for each $n \in \mathbb{N}$ the following inequality is true:

$$\|T_{i,n}u - z\|^2 \leq \|u - z\|^2 - \nu_{i,n}\|u - T_{i,n}u\|^2. \tag{4.6}$$

This leads, by using (4.6) a number of times, to

$$\begin{aligned} &\|T_{1,n}T_{2,n}T_{3,n}T_{4,n}x_n - z\|^2 \\ &\leq \|T_{2,n}T_{3,n}T_{4,n}x_n - z\|^2 - \nu_{1,n}\|T_{2,n}T_{3,n}T_{4,n}x_n - T_{1,n}T_{2,n}T_{3,n}T_{4,n}x_n\|^2 \\ &= \|T_{2,n}T_{3,n}T_{4,n}x_n - z\|^2 - \nu_{1,n}\|(\mathbf{1} - T_{1,n})(T_{2,n}T_{3,n}T_{4,n}x_n)\|^2 \\ &\leq \cdots \\ &\leq \|x_n - z\|^2 \\ &\quad - \nu_{4,n}\|(\mathbf{1} - T_{4,n})x_n\|^2 \\ &\quad - \nu_{3,n}\|(\mathbf{1} - T_{3,n})T_{4,n}x_n\|^2 \\ &\quad - \nu_{2,n}\|(\mathbf{1} - T_{2,n})T_{3,n}T_{4,n}x_n\|^2 \\ &\quad - \nu_{1,n}\|(\mathbf{1} - T_{1,n})(T_{2,n}T_{3,n}T_{4,n}x_n)\|^2. \end{aligned}$$

Introducing, as in [8], the following notation for operators S_1, S_2, S_3, S_4

$$\sqcap_{k=i}^{4} S_k = \begin{cases} S_i S_{i+1} \cdots S_4, & \text{if } i \leq 4 \\ \mathbf{1}, & \text{otherwise} \end{cases},$$

the finally obtained inequality may be rewritten as

$$\sum_{i=1}^{4} \nu_{i,n} \|(\mathbf{1} - T_{i,n}) \sqcap_{k=i+1}^{4} T_{k,n} x_n\|^2 \leq \|x_n - z\|^2 - \|T_{1,n}T_{2,n}T_{3,n}T_{4,n}x_n - z\|^2. \tag{4.7}$$

Multiplying both members of inequality (4.8) with λ_n and adding to the newly obtained inequality stated in (iii) of the present theorem, there results after some simplification

$$\lambda_n \sum_{i=1}^{4} \nu_{i,n} \|(\mathbf{1} - T_{i,n}) \sqcap_{k=i+1}^{4} T_{k,n} x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 - \lambda_n \|T_{1,n}T_{2,n}T_{3,n}T_{4,n}x_n - z\|^2 + \eta \|e_n^*\|. \tag{4.8}$$

Summing up, for n going from 0 to some positive integer N, we get

$$\sum_{n=0}^{N} \lambda_n \sum_{i=1}^{4} \nu_{i,n} \|(\mathbf{1} - T_{i,n}) \sqcap_{k=i+1}^{4} T_{k,n} x_n\|^2 + \sum_{n=0}^{N} \lambda_n \|T_{1,n}T_{2,n}T_{3,n}T_{4,n}x_n - z\|^2 \leq \|x_0 - z\|^2 + \eta \sum_{n=0}^{N} \|e_n^*\|.$$

Due to the assumption that $\sum_{n\in\mathbb{N}} \lambda_n < +\infty$ and to result (iv) of the present theorem, we know that the right-hand side of the obtained inequality tends to a finite limit when N goes to infinity. This leads to the stated results (v)a,$\cdots$,(v)e. $\square$

We remark that the results in (v) of this theorem may be seen as replacements for the results (ii) and (iii) in Theorem 3.1 in [8], under the additional condition that $\sum_{n\in\mathbb{N}} \lambda_n < +\infty$ but for operators that now are ν-quasi-nonexpansive for $\nu \neq 0$. As a consequence, also the analogous results of Theorem 3.2 and Theorem 3.3 in [8] could be stated.

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