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ABOUT SUBDIFFERENTIAL CALCULUS FOR ABSTRACT CONVEX FUNCTIONS

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ABSTRACT. We introduce a stronger version of the strong globalization property of Rolewicz and examine the corresponding subdifferential calculus for abstract convex functions. In particular, we calculate a formula for the abstract subdifferential of the maximum of a finite set of abstract convex functions. We also present some examples of families of functions, which possess this new strong globalization property.

1. Introduction

In this paper we examine the subdifferential calculus for some classes of abstract convex functions. We are concentrating mainly on the maximum of a finite set of functions. The maximum of abstract convex functions is always abstract convex with respect to the same set of elementary functions. So the question "How is the subdifferential of the maximum of some functions via the subdifferentials of given functions expressed?" is natural. Subdifferential calculus is important for applications of abstract convex analysis, so it is interesting to find conditions that provide the exact formula for the subdifferential of the maximum. We show that such a formula can be given in terms of abstract convex hull with respect to a certain subset of elementary functions (see Corollary 4.1).

Our main goal in this paper is to show that the subdifferential calculus is not a privilege of convex analysis only. We indicate some conditions which guarantee the existence of a calculus in abstract convex case.

In the paper [4] S. Rolewicz introduced the notion of the strong globalization property. He says that a set Φ of functions defined on a topological space X has the strong globalization property if for every Φ-convex function f and for every point $y \in X$ each local Φ-subgradient of f at y can be extended to a global one. Here, into the definition of the strong globalization property, we put a more rigid condition. Namely, we say that Φ has the strong globalization property if for every Φ-convex function f and for every point $y \in X$, each local Φ-subgradient of f at y is also a global one. We show that in such a case subdifferential calculus can be expressed in terms of special functions that approximate in a certain sense the given functions.

In Section 2 we recall some definitions on abstract convexity which will be used in the paper. Section 3 contains some general results related to the subdifferential of a maximum of two abstract convex functions. In Section 4 we examine the subdifferential calculus in the case when the set of elementary functions has the strong globalization property. Section 5 contains some examples.

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2. Preliminaries

Let X be a topological space and H be a set of functions $h : X \to \mathbb{R}$. Recall (see [5]) that a function f defined on X is called H-convex if $f(x) = \sup\{h(x) : h \in \operatorname{supp}(f, H)\}\ \forall x \in X$, where $\operatorname{supp}(f, H) = \{h \in H : \quad h \le f\}$. Here $h \le f$ means that $h(x) \le f(x)$ for all $x \in X$. The mapping $f \mapsto \operatorname{supp}(f, H)$ is called a Minkowski duality. Let $U \subset H$ and $f(x) = \sup\{h(x) : h \in U\}$. Then the set $\operatorname{co}_H U = \{h \in H : h \le f\}$ is called the H-convex hull of U. It is known (see [5], Theorem 7.16) that

$$\operatorname{supp}(\max(f_1, f_2), H) = \operatorname{co}_H(\operatorname{supp}(f_1, H) \cup \operatorname{supp}(f_2, H)), \tag{2.1}$$

where f_1 and f_2 are H-convex functions.

Let L be a set of functions $l : X \to \mathbb{R}$. Here and in the rest of the paper let H_L denote the set of functions $h(x) = l(x) - c$, where $l \in L$ and $c \in \mathbb{R}$. Let $f : X \to \mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\}$ and $y \in \operatorname{dom} f = \{x \in X : \ f(x) < +\infty\}$. Then the set $\partial_L f(y)$ defined by

$$\partial_L f(y) = \{l \in L : \ l(x) - l(y) \le f(x) - f(y)\ \ \forall x \in X\} \tag{2.2}$$

is called the abstract subdifferential of the function f at the point $y \in X$ with respect to L. We will need ε-subdifferentials as well. Let $\varepsilon \ge 0$. The set

$$\partial_{L,\varepsilon} f(y) = \{l \in L : \ l(x) - l(y) \le f(x) - f(y) + \varepsilon\ \ \forall x \in X\}$$

is called the ε-subdifferential of the function f at y with respect to L. Consider also the following set (see [5], p. 364)

$$\mathcal{D}_L f(y) = \{h \in H_L : \ h(x) = l(x) - l(y),\ l \in \partial_L f(y)\}.$$

It is clear that a function $h \in H_L$ belongs to $\mathcal{D}_L f(y)$ if and only if

$$h(y) = 0 \quad \text{and} \quad h(x) \le f(x) - f(y) \quad \forall x \in X.$$

Remark 2.1. For the sake of convenience we assume that for $f(y) = +\infty$ the sets $\partial_L f(y)$, $\partial_{L,\varepsilon} f(y)$ and $\mathcal{D}_L f(y)$ are defined as empty sets. So if we write"the set $\mathcal{D}_L f(y)$ is nonempty" then we mean, in particular, that $y \in \operatorname{dom} f$.

In this paper we provide a basis for further development of applications of abstract convexity to optimization. For this purpose we need to develop subdifferential calculus. Note that if $0 \in L$ then a function f attains its global minimum at a point y if and only if $0 \in \partial_L f(y)$. If $0 \notin L$ then it is more convenient to consider the set $\mathcal{D}_L f(y)$. Then we have the following sufficient condition for a global minimum: if $\mathcal{D}_L f(y)$ contains a nonnegative function then $f(y) \le f(x)$ for all $x \in X$. The equivalent form of this assertion: if $\partial_L f(y)$ contains a function that attains its global minimum at y then $f(y) \le f(x)\ \forall x \in X$. Thus the calculus of subdifferentials is a very important problem.

In this paper we are only presenting some general results related to the calculus of abstract subdifferentials. Detailed examination of some particular nonconvex cases is the theme of further research.

We will use $H_{L,y}$ to denote the set of all $h \in H_L$ such that $h(y) = 0$, that is $H_{L,y} = \{l - l(y) : \quad l \in L\}$. The symbol f_y also denotes the function $f_y(x) = f(x) - f(y)$ (here $y \in \operatorname{dom} f$).

If $T \subset H_{L,y}$ then the $H_{L,y}$-convex hull of T is defined as follows

$$\mathrm{co}_{H_{L,y}} T = \left\{ h \in H_{L,y} : \;\; h(x) \le \sup_{t \in T} t(x) \;\; \forall\, x \in X \right\}. \tag{2.3}$$

3. Subdifferentials of the maximum of two abstract convex functions

It is more convenient to formulate all statements in terms of the set $\mathcal{D}_L f(y)$. First we present some general inclusions for which additional assumptions are not needed.

Proposition 3.1. *Let f_1, f_2 be H_L-convex functions and $f_1(y) = f_2(y)$. Then*

$$\mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)) \subset \mathcal{D}_L(\max\{f_1, f_2\})(y). \tag{3.1}$$

Proof. If $(l - l(y)) \in \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ then

$$\begin{aligned} l(x) - l(y) &\le \sup_{h \in \mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)} h(x) \\ &= \max\left\{ \sup_{t \in \partial_L f_1(y)} (t(x) - t(y)),\; \sup_{t \in \partial_L f_2(y)} (t(x) - t(y)) \right\} \\ &\le \max\{f_1(x) - f_1(y), f_2(x) - f_2(y)\} \\ &= \max\{f_1(x), f_2(x)\} - \max\{f_1(y), f_2(y)\}. \end{aligned}$$

So $(l - l(y)) \in \mathcal{D}_L(\max\{f_1, f_2\})(y)$. □

For some special types of H_L-convex functions f_1, f_2 we can get equality instead of the inclusion in (3.1).

Proposition 3.2. *Let f_1, f_2 be functions defined on X such that the functions f_{1y}, f_{2y} are $H_{L,y}$-convex and $f_1(y) = f_2(y)$. Then*

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)).$$

Proof. If is clear that $\mathcal{D}_L f(y) = \mathrm{supp}(f_y, H_{L,y})$ for any function f. Since $f_1(y) = f_2(y)$ then $(\max\{f_1, f_2\})_y = \max\{f_{1y}, f_{2y}\}$. Formula (2.1) gives us the equality

$$\mathrm{supp}(\max\{f_{1y}, f_{2y}\}, H_{L,y}) = \mathrm{co}_{H_{L,y}}(\mathrm{supp}(f_{1y}, H_{L,y}) \cup \mathrm{supp}(f_{2y}, H_{L,y}))$$

for $H_{L,y}$-convex functions f_{1y}, f_{2y}. Hence

$$\begin{aligned} \mathcal{D}_L(\max\{f_1, f_2\})(y) &= \mathrm{supp}((\max\{f_1, f_2\})_y, H_{L,y}) = \mathrm{supp}(\max\{f_{1y}, f_{2y}\}, H_{L,y}) \\ &= \mathrm{co}_{H_{L,y}}(\mathrm{supp}(f_{1y}, H_{L,y}) \cup \mathrm{supp}(f_{2y}, H_{L,y})) \\ &= \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)). \end{aligned}$$

□

The following example demonstrates that the equality

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$$

does not necessarily hold for arbitrary H_L-convex functions f_1, f_2 with $f_1(y) = f_2(y)$.

Example 3.1. Let $X = \mathbb{R}$ and L consists of all linear functions and the function $l(x) = x^2$. Consider the functions f_1, f_2:

$$f_1(x) = \begin{cases} x^2, & x \le 0, \\ 0, & x \ge 0. \end{cases} \qquad f_2(x) = \begin{cases} 0, & x \le 0, \\ x^2, & x \ge 0. \end{cases}$$

Note that f_1 and f_2 are H_L-convex and $f_1(0) = f_2(0)$. At the same time, both f_{1y} and f_{2y} are not $H_{L,y}$-convex for $y = 0$. It is clear that $\mathcal{D}_L f_1(0) = \mathcal{D}_L f_2(0) = \{0\}$, hence $\mathrm{co}_{H_{L,0}}(\mathcal{D}_L f_1(0) \cup \mathcal{D}_L f_2(0)) = \{0\}$. But the function $f(x) = \max\{f_1(x), f_2(x)\}$ coincides with elementary function $l(x) = x^2$, therefore $l \in \mathcal{D}_L f(0)$. This means that $\mathcal{D}_L(\max\{f_1, f_2\})(0) \neq \mathrm{co}_{H_{L,0}}(\mathcal{D}_L f_1(0) \cup \mathcal{D}_L f_2(0))$.

Further, consider a multifunction $A : X \times 2^{H_L} \times 2^{H_L} \to 2^{H_L}$, where 2^{H_L} is the set of all nonempty subsets of H_L.

Proposition 3.3. *Let $y \in X$. Assume that the inclusion*

$$A(y, \mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)) \subset \mathcal{D}_L(\max\{g_1, g_2\})(y) \tag{3.2}$$

holds for all H_L-convex functions g_1, g_2 such that the sets $\mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)$ are nonempty and $g_1(y) = g_2(y)$. Let f_1, f_2 be H_L-convex functions such that the sets $\mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$. If

$$A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = \mathcal{D}_L(\max\{f_1, f_2\})(y)$$

then

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)). \tag{3.3}$$

Proof. Let f_1, f_2 be H_L-convex functions such that the sets $\mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)$ are nonempty, $f_1(y) = f_2(y)$ and $A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = \mathcal{D}_L(\max\{f_1, f_2\})(y)$. Consider the functions

$$\begin{aligned} g_i(x) &= \sup\{h(x) + f_i(y) : \ h \in \mathcal{D}_L f_i(y)\} \\ &= \sup\{h(x) : \ h(y) = f_i(y), \ h \in \mathrm{supp}(f_i, H_L)\}. \end{aligned}$$

It is clear that $g_1(y) = f_1(y) = f_2(y) = g_2(y)$ and g_{1y}, g_{2y} are $H_{L,y}$-convex. Proposition 3.2 implies the equality $\mathcal{D}_L(\max\{g_1, g_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L g_1(y) \cup \mathcal{D}_L g_2(y))$. Since

$$\begin{aligned} \mathcal{D}_L g_i(y) &= \{h \in H_{L,y} : \ h \le g_i - g_i(y)\} \\ &= \left\{ h \in H_{L,y} : \ h(x) \le \sup_{h' \in \mathcal{D}_L f_i(y)} h'(x) \ \ \forall x \in X \right\} = \mathcal{D}_L f_i(y) \end{aligned}$$

then $A(y, \mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)) = A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y))$. Hence

$$\begin{aligned} \mathcal{D}_L(\max\{f_1, f_2\})(y) &= A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = A(y, \mathcal{D}_L g_1(y), \mathcal{D}_L g_2(y)) \\ &\subset \mathcal{D}_L(\max\{g_1, g_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L g_1(y) \cup \mathcal{D}_L g_2(y)) \\ &= \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)). \end{aligned} \tag{3.4}$$

Combining the above inclusion with Proposition 3.1 yields the equality $\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$. $\square$

Proposition 3.4. *Let $y \in X$. Assume that*

$$\mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)) \subset A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) \subset \mathcal{D}_L(\max\{f_1, f_2\})(y)$$

for all H_L-convex functions f_1, f_2 such that the sets $\mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$. Then for all such functions f_1, f_2

$$A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)).$$

Proof. Using the same functions g_i as in the proof of Proposition 3.3 we conclude that $A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y)) \subset \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ (see (3.4)). However, $\mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y)) \subset A(y, \mathcal{D}_L f_1(y), \mathcal{D}_L f_2(y))$ by our assumptions. So we obtain the desired equality. □

Example 3.1 and Proposition 3.3 show that, in general, the set $\mathcal{D}_L(\max\{f_1, f_2\})(y)$ cannot be described in terms of the sets $\mathcal{D}_L f_1(y)$ and $\mathcal{D}_L f_2(y)$.

At the same time the equality $\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ is valid for broad classes of H_L-convex functions. However the mapping $\mathrm{co}_{H_{L,y}}$ can be very complicated.

Proposition 3.5. *Let $\mathcal{L}$ be a set of functions defined on a set X. Let L consist of all functions $l(x) = \max\{l_1(x), l_2(x) + c\}$, where $l_1, l_2 \in \mathcal{L}$ and $c \in \mathbb{R}$. Then*

$$\mathcal{D}_L(\max\{f_1, f_2\})(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$$

for all H_L-convex functions f_1, f_2 and all points $y \in X$ such that the sets $\mathcal{D}_L f_1(y)$, $\mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$.

Proof. It is clear that $H_{\mathcal{L}} \subset H_L$ and a function is H_L-convex if and only if it is $H_{\mathcal{L}}$-convex. Let f_1 and f_2 be H_L-convex functions (then they are also $H_{\mathcal{L}}$-convex). Let $y \in X$ be a point such that the sets $\mathcal{D}_L f_1(y)$ and $\mathcal{D}_L f_2(y)$ are nonempty and $f_1(y) = f_2(y)$. First we will prove that

$$\sup\{h_i(x) : \ h_i \in \mathcal{D}_L f_i(y)\} = f_i(x) - f_i(y) \ \ \forall\, x \in X \ \forall\, i = 1, 2. \tag{3.5}$$

For this purpose we only need to check that $\sup\{h_i(x) : \ h_i \in \mathcal{D}_L f_i(y)\} \geq f_i(x) - f_i(y)$. For each $i = 1, 2$ choose an arbitrary function $h_i' \in \mathcal{D}_L f_i(y)$. Since $h_i' \in H_L$ then $h_i'(x) = \max\{l_i^1(x), l_i^2(x) + c_i\} + c_i'$, where $l_i^1, l_i^2 \in \mathcal{L}$ and $c_i, c_i' \in \mathbb{R}$. For the sake of definiteness assume that $h_i'(y) = l_i^1(y) + c_i'$. Then $l_i^1(y) + c_i' = 0$ and $l_i^1(x) + c_i' \leq f_i(x) - f_i(y)$ for all $x \in X$. For every $t_i \in \mathrm{supp}(f_i, H_{\mathcal{L}})$ consider the function h_{t_i} defined by

$$h_{t_i}(x) = \max\{l_i^1(x) + c_i', t_i(x) - f_i(y)\}.$$

We see that $h_{t_i} \in H_L$, $h_{t_i}(y) = 0$ and $h_{t_i}(x) \leq f_i(x) - f_i(y) \ \ \forall\, x \in X$, that is $h_{t_i} \in \mathcal{D}_L f_i(y)$. Since f_i is $H_{\mathcal{L}}$-convex then

$$\sup\{t_i(x) - f_i(y) : \ t_i \in \mathrm{supp}(f_i, H_{\mathcal{L}})\} = f_i(x) - f_i(y) \ \ \forall\, x \in X.$$

Hence

$$\begin{aligned} \sup\{h_i(x) : \ h_i \in \mathcal{D}_L f_i(y)\} &\geq \sup\{h_{t_i}(x) : \ t_i \in \mathrm{supp}(f_i, H_{\mathcal{L}})\} \\ &\geq \sup\{t_i(x) - f_i(y) : \ t_i \in \mathrm{supp}(f_i, H_{\mathcal{L}})\} \\ &= f_i(x) - f_i(y) \ \ \forall\, x \in X. \end{aligned}$$

So the equalities (3.5) hold true. This means that

$$\begin{aligned}&\mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))\\ &= \{h \in H_{L,y} :\ h(x) \le \max\{f_1(x) - f_1(y), f_2(x) - f_2(y)\}\ \forall\, x \in X\}\\ &= \mathcal{D}_L(\max\{f_1, f_2\})(y).\end{aligned} \qquad \square$$

Under the assumptions of Proposition 3.5, in order to describe the sets $\mathcal{D}_L f_1(y)$ and $\mathcal{D}_L f_2(y)$ we need to know all support functions of f_1 and f_2 with respect to $H_{\mathcal{L}}$. In other words, we need to know the values of the functions f_1 and f_2 at each point $x \in X$. These sets can be very complicated, and therefore the set $\mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) \cup \mathcal{D}_L f_2(y))$ is also complicated.

In the next sections we will consider one special case when the subdifferential calculus is possible. We will assume that the subdifferential has a local nature. This means that for the description of a set $\mathcal{D}_L f(y)$ we need to know the behaviour of the function f only in a neighbourhood of the point y. This allows us to give a sufficiently simple description of $\mathcal{D}_L f(y)$.

4. Subdifferential calculus in case that H_L has the strong globalization property

Let H be a set of functions defined on a topological space X. We say that H has the *strong globalization property* if for any H-convex function f, for any point $y \in X$ and for any $h \in H$ the following implication holds

$$(4.1)\quad (h(y) = f(y),\ h(x) \le f(x) \text{ in a neighbourhood of } y) \implies (h(x) \le f(x) \text{ for all } x \in X).$$

For instance, it was shown in [4] (see Example 5.4) that the set H of all continuous affine functions defined on a topological linear space X has the strong globalization property.

Remark 4.1. Assume that H has the strong globalization property. Then every subset $H' \subset H$ also has the strong globalization property since any H'-convex function is H-convex.

Now let L be a set of functions defined on X. As above, H_L denotes the set of all vertical shifts of functions $l \in L$. Assume that H_L has the strong globalization property. Take an H_L-convex function f and a point $y \in X$. Let U be a neighbourhood of y. Then the following equality holds

$$(4.2)\qquad \partial_L f(y) = \{l \in L :\ l(x) - l(y) \le f(x) - f(y)\ \forall\, x \in U\}.$$

Indeed, let $l \in L$ and $l(x) - l(y) \le f(x) - f(y)\ \forall\, x \in U$. Then the function $h(x) = l(x) - l(y) + f(y)$ belongs to H_L. Moreover, $h(y) = f(y)$ and $h(x) \le f(x)\ \forall\, x \in U$. Hence $h(x) \le f(x)$ for all $x \in X$. This implies $l \in \partial_L f(y)$.

Similarly, we have the equality for the set $\mathcal{D}_L f(y)$

$$(4.3)\qquad \mathcal{D}_L f(y) = \{h \in H_L :\ h(y) = 0,\ h(x) \le f(x) - f(y)\ \forall\, x \in U\}.$$

The following proposition demonstrates a technique that can be applied for subdifferential calculus. We assume that the space $\mathbb{R}^n$ is equipped with the usual coordinate-wise order relation: $a \le b$ if and only if $a_i \le b_i$ for all $i = 1, \dots, n$

($a, b \in \mathbb{R}^n$). We will consider increasing continuous mappings $M : \mathbb{R}^n \to \mathbb{R}$. For example, the mappings $M(a) = \sum_i a_i$ and $M(a) = \max_i a_i$ ($a = (a_1, \dots, a_n) \in \mathbb{R}^n$) are increasing and continuous on $\mathbb{R}^n$. Moreover, the maximum of abstract convex functions is always abstract convex. Hence the mapping $M(a) = \max_i a_i$ verifies the assumptions of Propositions 4.1 and 4.4 irrespective of the set H_L.

For $y \in X$ let $\mathcal{U}(y)$ denote the set of all neighbourhoods of y.

Proposition 4.1. *Let $M : \mathbb{R}^n \to \mathbb{R}$ be an increasing continuous mapping such that for all $h_1, \dots, h_n \in H_L$ the function $M(h_1(x), \dots, h_n(x))$ is H_L-convex. Let $y \in X$ and $f_1, \dots, f_n$ be H_L-convex functions such that $f_i(x) < +\infty$ for all x from a neighbourhood of the point y. Assume that H_L has the strong globalization property. Then for every $h \in \mathcal{D}_L M(f_1, \dots, f_n)(y)$ the following inequalities hold*

$$h(x) \le M(F_1(x), \dots, F_n(x)) - M(F_1(y), \dots, F_n(y)) \quad \textit{for all } x \in X,$$

where

$$F_i(x) = \inf_{\varepsilon > 0} \inf_{U \in \mathcal{U}(y)} \sup_{z \in U} \sup \{h_i(x) : \ h_i \in \operatorname{supp}(f_i, H_L),\ h_i(z) \ge f_i(z) - \varepsilon\}.$$

Proof. Since the mapping $M : \mathbb{R}^n \to \mathbb{R}$ is continuous and increasing then for any sets $A_i \subset \mathbb{R}$ ($i = 1, \dots, n$)

$$(4.4) \qquad \begin{aligned} M\left(\sup_{a_1 \in A_1} a_1, \dots, \sup_{a_n \in A_n} a_n\right) &= \sup_{a_i \in A_i} M(a_1, \dots, a_n), \\ M\left(\inf_{a_1 \in A_1} a_1, \dots, \inf_{a_n \in A_n} a_n\right) &= \inf_{a_i \in A_i} M(a_1, \dots, a_n). \end{aligned}$$

(Here we assume that $M(b_1, \dots, b_j, \dots, b_n) = \lim_{b \to b_j} M(b_1, \dots, b, \dots, b_n)$ for $b_j = \pm\infty$).

Consider the following sets

$$T_i(U, \varepsilon) = \bigcup_{z \in U} \{h_i \in \operatorname{supp}(f_i, H_L) : \ h_i(z) \ge f_i(z) - \varepsilon\},$$

where $U \in \mathcal{U}(y)$ and $\varepsilon > 0$.

Let U' be a neighbourhood of y such that $f_i(x) < +\infty$ for all i and $x \in U'$. Since all functions f_i are H_L-convex then for any $U_i \in \mathcal{U}(y)$ and $\varepsilon_i > 0$

$$(4.5) \qquad f_i(x) = \sup_{h_i \in T_i(U_i, \varepsilon_i)} h_i(x) \qquad \forall\, x \in U_i \cap U'.$$

In particular, the equality $f_i(y) = \sup_{h_i \in T_i(U_i, \varepsilon_i)} h_i(y)$ holds for any $U_i \in \mathcal{U}(y)$ and $\varepsilon_i > 0$. Hence

$$(4.6) \qquad F_i(y) = \inf_{\varepsilon_i > 0} \inf_{U_i \in \mathcal{U}(y)} \sup_{h_i \in T_i(U_i, \varepsilon_i)} h_i(y) = f_i(y).$$

So let $h \in \mathcal{D}_L M(f_1, \dots, f_n)(y)$. Then

$$h(x) \le M(f_1(x), \dots, f_n(x)) - M(f_1(y), \dots, f_n(y)) \quad \text{for all } x \in X.$$

Due to (4.5), (4.6) and (4.4) we have for any neighbourhoods $U_i \in \mathcal{U}(y)$ and $\varepsilon_i > 0$

(4.7)

$$h(x) \le M\left(\sup_{h_1\in T_1(U_1,\varepsilon_1)} h_1(x),\dots,\sup_{h_n\in T_n(U_n,\varepsilon_n)} h_n(x)\right) - M(F_1(y),\dots,F_n(y))$$
$$= \sup_{h_i\in T_i(U_i,\varepsilon_i)} M(h_1(x),\dots,h_n(x)) - M(F_1(y),\dots,F_n(y)) \quad \forall\, x\in\bigcap_{i=1}^n (U_i\cap U').$$

Note that, by our assumptions, each function $M(h_1(x),\dots,h_n(x))$ is H_L-convex, and therefore for any $U_i\in\mathcal{U}(y)$ and $\varepsilon_i>0$ the function

$$\sup_{h_i\in T_i(U_i,\varepsilon_i)} M(h_1(x),\dots,h_n(x)) - M(F_1(y),\dots,F_n(y)) \qquad (x\in X) \tag{4.8}$$

is H_L-convex as well. Moreover, by (4.7), each function (4.8) is not less than $h(x)$ in a neighbourhood of the point y and interpolates h at y (it is equal to zero at y). Since H_L has the strong globalization property it follows that

$$h(x) \le \sup_{h_i\in T_i(U_i,\varepsilon_i)} M(h_1(x),\dots,h_n(x)) - M(F_1(y),\dots,F_n(y)) \qquad \forall\, x\in X.$$

This implies (see (4.4))

$$h(x) \le \inf_{\varepsilon_i>0}\inf_{U_i\in\mathcal{U}(y)}\sup_{h_i\in T_i(U_i,\varepsilon_i)} M(h_1(x),\dots,h_n(x)) - M(F_1(y),\dots,F_n(y))$$
$$= M(F_1(x),\dots,F_n(x)) - M(F_1(y),\dots,F_n(y)) \qquad \forall\, x\in X. \qquad \square$$

Assume that the set $\mathcal{D}_L f(y)$ is nonempty. Then we can introduce the following function defined on X

$$\mathrm{app}_{f,y}(x) = \inf_{U\in\mathcal{U}(y)}\ \sup_{z\in U,\ \mathcal{D}_L f(z)\neq\emptyset}\ \sup_{h\in\mathcal{D}_L f(z)} (h(x)+f(z)), \qquad (x\in X). \tag{4.9}$$

We will show that the function $\mathrm{app}_{f,y}$ can be considered as an approximation of the function f near the point y. In the classical convex case we can estimate this function using ε-subdifferentials (see Proposition 4.5 and Example 4.1).

First we prove some properties of the function $\mathrm{app}_{f,y}$.

Proposition 4.2. *Let $y\in X$ and $f: X\to\mathbb{R}_{+\infty}$ be a function such that $\mathcal{D}_L f(y)\neq\emptyset$. Then*

$$\mathrm{app}_{f,y}(y) = f(y), \qquad \sup_{h\in\mathcal{D}_L f(y)} (h(x)+f(y)) \le \mathrm{app}_{f,y}(x) \le f(x) \quad \forall\, x\in X. \tag{4.10}$$

Moreover, $\mathcal{D}_L(\mathrm{app}_{f,y})(y) = \mathcal{D}_L f(y)$.

Proof. Since $y\in U$ for each $U\in\mathcal{U}(y)$ we have that $\mathrm{app}_{f,y}(x) \ge \sup_{h\in\mathcal{D}_L f(y)}(h(x)+f(y))$ for all $x\in X$. In particular, $\mathrm{app}_{f,y}(y) \ge \sup_{h\in\mathcal{D}_L f(y)}(h(y)+f(y)) = f(y)$. Inequality $\mathrm{app}_{f,y}\le f$ is trivial since $h(x)+f(z)\le f(x)$ for any $h\in\mathcal{D}_L f(z)$.

Since $\mathrm{app}_{f,y}(y) = f(y)$ and $\mathrm{app}_{f,y}\le f$ then $\mathcal{D}_L(\mathrm{app}_{f,y})(y)\subset\mathcal{D}_L f(y)$. Take a function $h\in\mathcal{D}_L f(y)$. Then

$$h(x) = (h(x)+f(y))-f(y) \le \sup_{h'\in\mathcal{D}_L f(y)} (h'(x)+f(y)) - f(y) \le \mathrm{app}_{f,y}(x) - \mathrm{app}_{f,y}(y),$$

hence $h \in \mathcal{D}_L(\mathrm{app}_{f,y})(y)$. □

Proposition 4.3. *Let $y \in X$ and f, g be H_L-convex functions such that $\mathcal{D}_L f(y) \neq \emptyset$ and $\mathcal{D}_L g(y) \neq \emptyset$. Assume that H_L has the strong globalization property. If $f(x) = g(x)$ in a neighbourhood U' of y then $\mathrm{app}_{f,y} = \mathrm{app}_{g,y}$.*

Proof. Let $z \in U' \cap \operatorname{dom} f \cap \operatorname{dom} g$. Then U' is also a neighbourhood of z and, by (4.3), we have

$$\begin{aligned}\mathcal{D}_L f(z) &= \{h \in H_L : \ h(z) = 0, \ h(x) \leq f(x) - f(z) \ \ \forall\, x \in U'\} \\ &= \{h \in H_L : \ h(z) = 0, \ h(x) \leq g(x) - g(z) \ \ \forall\, x \in U'\} = \mathcal{D}_L g(z).\end{aligned}$$

Since we can take in (4.9) $\inf_{(U \in \mathcal{U}(y), U \subset U')}$ instead of $\inf_{U \in \mathcal{U}(y)}$ then $\mathrm{app}_{f,y} = \mathrm{app}_{g,y}$. □

So if H_L has the strong globalization property and f is an H_L-convex function such that $\mathcal{D}_L f(y) \neq \emptyset$ then, in view of Propositions 4.2 and 4.3, we can say that the function $\mathrm{app}_{f,y}$ approximates the function f near the point y in the following sense: the function $\mathrm{app}_{f,y}$ depends only on the local behaviour of f at y, interpolates f at y and does not exceed f on the whole space X. The equality $\mathcal{D}_L(\mathrm{app}_{f,y})(y) = \mathcal{D}_L f(y)$ shows that such an approximation is closely connected with the notion of the subdifferential. Note that the function $t(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ enjoys all these properties as well. However, due to the inequalities $t(x) \leq \mathrm{app}_{f,y}(x) \leq f(x)$ (see (4.10)), the approximation $\mathrm{app}_{f,y}(x)$ is better than $t(x)$.

A main question now is to establish conditions which guarantee that the approximations $\mathrm{app}_{f,y}(x)$ and $t(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ coincide on X.

If $\mathcal{D}_L f(z)$ is nonempty in a neighbourhood of y then

$$\mathrm{app}_{f,y}(x) = \limsup_{z \to y} \sup_{h \in \mathcal{D}_L f(z)} (h(x) + f(z)) \quad \text{for all } x \in X. \tag{4.11}$$

So for each fixed $x \in X$ we have: $\mathrm{app}_{f,y}(x) = \sup_{h \in \mathcal{D}_L f(y)}(h(x) + f(y))$ if and only if the function $u(z) = \sup_{h \in \mathcal{D}_L f(z)}(h(x) + f(z))$ is upper semicontinuous at the point y.

Proposition 4.4. *Let $M : \mathbb{R}^n \to \mathbb{R}$ be an increasing continuous mapping such that for all $h_1, \ldots, h_n \in H_L$ the function $M(h_1(x), \ldots, h_n(x))$ is H_L-convex. Let $y \in X$ and $f_1, \ldots, f_n$ be H_L-convex functions such that the sets $\mathcal{D}_L f_1(z), \ldots, \mathcal{D}_L f_n(z)$ are nonempty in a neighbourhood of y. If H_L has the strong globalization property then*

$$\mathcal{D}_L M(f_1, \ldots, f_n)(y) = \mathcal{D}_L M\left(\mathrm{app}_{f_1,y}, \ldots, \mathrm{app}_{f_n,y}\right)(y). \tag{4.12}$$

Proof. If $h \in \mathcal{D}_L M\left(\mathrm{app}_{f_1,y}, \ldots, \mathrm{app}_{f_n,y}\right)(y)$ then

$$h(x) \leq M\left(\mathrm{app}_{f_1,y}(x), \ldots, \mathrm{app}_{f_n,y}(x)\right) - M\left(\mathrm{app}_{f_1,y}(y), \ldots, \mathrm{app}_{f_n,y}(y)\right) \qquad \forall\, x \in X.$$

Since the mapping M is increasing then, due to Proposition 4.2,

$$h(x) \leq M(f_1(x), \ldots, f_n(x)) - M(f_1(y), \ldots, f_n(y)) \qquad \forall\, x \in X,$$

hence $h \in \mathcal{D}_L M(f_1, \ldots, f_n)(y)$.

Conversely, let $h \in \mathcal{D}_L M(f_1, \ldots, f_n)(y)$. In this part of the proof we will use the same arguments as those in the proof of Proposition 4.1. Let U' be a neighbourhood of y such that the sets $\mathcal{D}_L f_i(z)$ are nonempty for all i and $z \in U'$. By definition of $\mathcal{D}_L f_i(z)$ we have for all $z \in U'$

$$f_i(z) = \sup_{h_i \in \mathcal{D}_L f_i(z)} (h_i(z) + f_i(z)), \qquad f_i(x) \geq \sup_{h_i \in \mathcal{D}_L f_i(z)} (h_i(x) + f_i(z)) \qquad \forall\, x \in X.$$

Hence for any neighbourhood $U_i \in \mathcal{U}(y)$

$$f_i(x) = \sup_{z \in U_i \cap U'} \sup_{h_i \in \mathcal{D}_L f_i(z)} (h_i(x) + f_i(z)) \qquad \forall\, x \in U_i \cap U',$$

and therefore (see (4.4))

$$\begin{aligned}
&h(x) \\
&\leq M(f_1(x), \dots, f_n(x)) - M(f_1(y), \dots, f_n(y)) \\
&= M\left(\sup_{z \in U_1 \cap U'} \sup_{h_1 \in \mathcal{D}_L f_1(z)} (h_1(x) + f_1(z)), \dots, \sup_{z \in U_n \cap U'} \sup_{h_n \in \mathcal{D}_L f_n(z)} (h_n(x) + f_n(z)) \right) \\
&\quad - M\left(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)\right) \\
&= \sup_{z_i \in U_i \cap U'} \sup_{h_i \in \mathcal{D}_L f_i(z_i)} M\left((h_1(x) + f_1(z_1)), \dots, (h_n(x) + f_n(z_n))\right) \\
&\quad - M\left(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)\right) \quad \forall\, x \in \bigcap_{i=1}^{n} (U_i \cap U').
\end{aligned}$$

Since H_L has the strong globalization property then this inequality holds for all $x \in X$. So, due to (4.4), we conclude that for all $x \in X$

$$\begin{aligned}
&h(x) \\
&\leq \inf_{U_i \in \mathcal{U}(y)} \sup_{z_i \in U_i \cap U'} \sup_{h_i \in \mathcal{D}_L f_i(z_i)} M\left((h_1(x) + f_1(z_1)), \dots, (h_n(x) + f_n(z_n))\right) \\
&\quad - M\left(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)\right) \\
&= \inf_{U_i \in \mathcal{U}(y)} \sup_{(z_i \in U_i,\ \mathcal{D}_L f_i(z_i) \neq \emptyset)} \sup_{h_i \in \mathcal{D}_L f_i(z_i)} M\left((h_1(x) + f_1(z_1)), \dots, (h_n(x) + f_n(z_n))\right) \\
&\quad - M\left(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)\right) \\
&= M\left(\mathrm{app}_{f_1,y}(x), \dots, \mathrm{app}_{f_n,y}(x)\right) - M\left(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)\right). \qquad \square
\end{aligned}$$

Corollary 4.1. *Assume that H_L has the strong globalization property. Assume that $f_1, \dots, f_n$ are H_L-convex functions and take a point $y \in X$ such that the sets $\mathcal{D}_L f_1(z), \dots, \mathcal{D}_L f_n(z)$ are nonempty in a neighbourhood of y and*

$$\mathrm{app}_{f_i,y}(x) = \sup_{h \in \mathcal{D}_L f_i(y)} (h(x) + f_i(y)) \qquad \textit{for all } x \in X, \ \ i = 1, \dots, n.$$

If $f_1(y) = \dots = f_n(y)$ then

$$\mathcal{D}_L(\max\{f_1, \dots, f_n\})(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i=1}^{n} \mathcal{D}_L f_i(y). \tag{4.13}$$

If all functions f_i are continuous at y then

$$\mathcal{D}_L(\max\{f_1, \dots, f_n\})(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i \in I} \mathcal{D}_L f_i(y), \tag{4.14}$$

where $I = \{i : \ f_i(y) = \max\{f_1(y), \dots, f_n(y)\}\}$.

Proof. Let $M(a_1, \dots, a_n) = \max\{a_1, \dots, a_n\}$. Then M satisfies the conditions of Proposition 4.4. Hence, by (4.12)

$$\mathcal{D}_L(\max\{f_1, \dots, f_n\})(y) = \mathcal{D}_L(\max\{\mathrm{app}_{f_1,y}, \dots, \mathrm{app}_{f_n,y}\})(y). \tag{4.15}$$

Let $f_1(y) = \dots = f_n(y)$. Since $\mathrm{app}_{f_i,y}(y) = f_i(y)$ (see (4.10)) then we have

$$\begin{aligned}
&\max\{\mathrm{app}_{f_1,y}(x), \dots, \mathrm{app}_{f_n,y}(x)\} - \max\{\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)\} \\
&= \max\left\{\sup_{h\in\mathcal{D}_L f_1(y)} h(x) + f_1(y), \dots, \sup_{h\in\mathcal{D}_L f_n(y)} h(x) + f_n(y)\right\} - \max\{f_1(y), \dots, f_n(y)\} \\
&= \max_i \sup_{h\in\mathcal{D}_L f_i(y)} h(x).
\end{aligned}$$

So a function $h' \in H_{L,y}$ belongs to $\mathcal{D}_L(\max\{\mathrm{app}_{f_1,y}, \dots, \mathrm{app}_{f_n,y}\})(y)$ if and only if $h'(x) \le \max_i \sup_{h\in\mathcal{D}_L f_i(y)} h(x)$ for all $x \in X$. In other words (see (2.3))

$$\mathcal{D}_L(\max\{\mathrm{app}_{f_1,y}, \dots, \mathrm{app}_{f_n,y}\})(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i=1}^{n} \mathcal{D}_L f_i(y).$$

This and (4.15) give us the required equality (4.13).

If all functions f_i are continuous at the point y then there exists a neighbourhood U of y such that $\max\{f_1(x), \dots, f_n(x)\} = \max_{i\in I} f_i(x)$ for all $x \in U$. Since H_L has the strong globalization property then

$$\mathcal{D}_L(\max\{f_1, \dots, f_n\})(y) = \mathcal{D}_L\left(\max_{i\in I} f_i\right)(y).$$

At the same time, $f_i(y) = f_j(y)$ for any $i, j \in I$. Then it follows from the first part of the proof that

$$\mathcal{D}_L\left(\max_{i\in I} f_i\right)(y) = \mathrm{co}_{H_{L,y}} \bigcup_{i\in I} \mathcal{D}_L f_i(y).$$

Thus the equality (4.14) holds true. $\square$

Corollary 4.2. *Let conditions of Proposition 4.4 hold,* $M(h_1, \dots, h_n) \in H_L$ *for all* $h_i \in H_L$ *and*

$$\mathrm{app}_{f_i,y}(x) = \sup_{h\in\mathcal{D}_L f_i(y)} (h(x) + f_i(y)) \qquad \text{for all } x \in X \text{ and } i = 1, \dots, n.$$

Then

$$\begin{aligned}
&\mathcal{D}_L M(f_1, \dots, f_n)(y) \\
&\qquad = \mathrm{co}_{H_{L,y}} \left[M(\mathcal{D}_L f_1(y) + f_1(y), .., \mathcal{D}_L f_n(y) + f_n(y)) - M(f_1(y), .., f_n(y))\right],
\end{aligned}$$

where $[M(\mathcal{D}_L f_1(y) + f_1(y), \dots, \mathcal{D}_L f_n(y) + f_n(y)) - M(f_1(y), \dots, f_n(y))]$ *is the set of all functions of the form*

$$h(x) = M(h_1(x) + f_1(y), \dots, h_n(x) + f_n(y)) - M(f_1(y), \dots, f_n(y))$$

with $h_i \in \mathcal{D}_L f_i(y)$ *for all* $i = 1, \dots, n$.

Proof. It is sufficient to note that, by our conditions, every function

$h(x) = M(h_1(x)+f_1(y), \dots, h_n(x)+f_n(y)) - M(f_1(y), \dots, f_n(y))$ with $h_i \in \mathcal{D}_L f_i(y)$ belongs to $H_{L,y}$.

Due to (4.12) a function $h' \in H_{L,y}$ belongs to $\mathcal{D}_L M(f_1, \dots, f_n)(y)$ if and only if

$$\begin{aligned}
&h'(x) \\
&\le M(\mathrm{app}_{f_1,y}(x), \dots, \mathrm{app}_{f_n,y}(x)) - M(\mathrm{app}_{f_1,y}(y), \dots, \mathrm{app}_{f_n,y}(y)) \\
&= M\left(\sup_{h\in\mathcal{D}_L f_1(y)} (h(x) + f_1(y)), \dots, \sup_{h\in\mathcal{D}_L f_n(y)} (h(x) + f_n(y))\right) \\
&\quad - M(f_1(y), \dots, f_n(y)) \\
&= \sup_{h_i\in\mathcal{D}_L f_i(y)} [M(h_1(x) + f_1(y), \dots, h_n(x) + f_n(y)) - M(f_1(y), \dots, f_n(y)] \quad \forall\, x \in X.
\end{aligned}$$

The proof is completed. $\square$

For example, if $M(a_1, \dots, a_n) = a_1+\dots+a_n$ then, under assumptions of Corollary 4.2, the sum $(f_1 + \dots + f_n)$ of H_L-convex functions f_i is H_L-convex as well and

$$\mathcal{D}_L(f_1 + \dots + f_n)(y) = \mathrm{co}_{H_{L,y}}(\mathcal{D}_L f_1(y) + \dots + \mathcal{D}_L f_n(y)).$$

So the main problem now is to find conditions which guarantee the equality $\mathrm{app}_{f,y}(x) = \sup_{h\in\mathcal{D}_L f(y)}(h(x) + f(y))$. Since $\mathrm{app}_{f,y}(x) \ge \sup_{h\in\mathcal{D}_L f(y)}(h(x) + f(y))$ then we are interested in the inverse inequality. In the following proposition we estimate function $\mathrm{app}_{f,y}$ using ε-subdifferentials.

Proposition 4.5. *Let $y \in X$. Assume that for any H_L-convex function g the following implication holds:*

$$\limsup_{x\to y} g(x) < +\infty \implies g \text{ is continuous at } y. \tag{4.16}$$

Let a function f be H_L-convex and continuous at y. If the set $\mathcal{D}_L f(y)$ is nonempty then

$$\mathrm{app}_{f,y}(x) \le \lim_{\varepsilon\to+0} \sup_{l\in\partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)) \qquad \text{for all } x \in X. \tag{4.17}$$

Proof. First we will prove that for each $\varepsilon > 0$ a neighbourhood U_ε of the point y exists such that

$$l(y) - l(z) + f(z) \ge f(y) - \varepsilon \quad \text{for all } z \in U_\varepsilon,\ l \in \partial_L f(z). \tag{4.18}$$

Assume it is not true. Then a number $\varepsilon > 0$ exists such that for any neighbourhood U of the point y we can find $z \in U$ and $l \in \partial_L f(z)$, for which the inequality $(l(y) - l(z) + f(z)) < (f(y) - \varepsilon)$ holds.

Then consider the function

$$g(x) = \sup\{l(x) - l(z) + f(z) :\ z \in X,\ l \in \partial_L f(z),\ l(y) - l(z) + f(z) < f(y) - \varepsilon\}.$$

This function is H_L-convex, $g(x) \le f(x)$ for all $x \in X$ and $g(y) \le f(y) - \varepsilon$. Moreover, due to our assumption, for any neighbourhood U of the point y a point $z \in U$ exists such that $g(z) \ge f(z)$, hence $\limsup_{z\to y} g(z) \ge \liminf_{z\to y} f(z)$. Since f is continuous at the point y and $g(y) \le f(y) - \varepsilon$ then $\limsup_{z\to y} g(z) \ge f(y) >$

$f(y) - \varepsilon \geq g(y)$. Hence g is discontinuous at y and, by (4.16), we conclude that $\limsup_{z\to y} g(z) = +\infty$. On the other hand, since $g \leq f$ and f is continuous at y then $\limsup_{z\to y} g(z) \leq \limsup_{z\to y} f(z) = f(y) < +\infty$, which contradicts the equality $\limsup_{z\to y} g(z) = +\infty$.

So for each $\varepsilon > 0$ a neighbourhood U_ε of y exists such that (4.18) holds. Then for any $z \in U_\varepsilon$ and $l \in \partial_L f(z)$ we have

$$l(x) - l(y) = (l(x) - l(z) + f(z)) - (l(y) - l(z) + f(z)) \leq f(x) - f(y) + \varepsilon \quad \forall\, x \in X.$$

This implies that $l \in \partial_{L,\varepsilon} f(y)$ for all $l \in \partial_L f(z)$ with $z \in U_\varepsilon$. Therefore

$$\sup_{l\in\partial_L f(z),\, z\in U_\varepsilon} (l(x)-l(y)+f(y)) \leq \sup_{l\in\partial_{L,\varepsilon} f(y)} (l(x)-l(y)+f(y)) \quad \text{for all } x \in X. \tag{4.19}$$

At the same time, since $(-l(z) + f(z)) \leq (-l(y) + f(y))$ whenever $l \in \partial_L f(z)$ then

$$\sup_{l\in\partial_L f(z),\, z\in U_\varepsilon} (l(x) - l(z) + f(z)) \leq \sup_{l\in\partial_L f(z),\, z\in U_\varepsilon} (l(x) - l(y) + f(y)) \quad \text{for all } x \in X. \tag{4.20}$$

It follows from the inequalities (4.19) and (4.20) that

$$\sup_{l\in\partial_L f(z),\, z\in U_\varepsilon} (l(x) - l(z) + f(z)) \leq \sup_{l\in\partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)) \quad \text{for all } x \in X.$$

Hence

$$\begin{aligned}
&\mathrm{app}_{f,y}(x)\\
&= \inf_{U\in\mathcal{U}(y)} \sup_{z\in U,\, l\in\partial_L f(z)} (l(x) - l(z) + f(z))\\
&\leq \inf_{\varepsilon>0} \sup_{z\in U_\varepsilon,\, l\in\partial_L f(z)} (l(x) - l(z) + f(z))\\
&\leq \inf_{\varepsilon>0} \sup_{l\in\partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)) = \lim_{\varepsilon\to+0} \sup_{l\in\partial_{L,\varepsilon} f(y)} (l(x) - l(y) + f(y)). \qquad \square
\end{aligned}$$

Remark 4.2. Implication (4.16) means that every H_L-convex function g is continuous at y whenever a neighbourhood U of y and a number $c \in \mathbb{R}$ exist such that $g(u) \leq c$ for all $u \in U$. Note that this implication can be false even in the case when all elements of H_L are continuous. For example, let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by: $g(x) = 0$ if $x \leq 0$ and $g(x) = 1$ if $x > 0$. Then g can be represented as the supremum of a family of continuous functions. We see that g is uniformly bounded on $\mathbb{R}$. However g is discontinuous at zero.

Example 4.1. Let L be the set of all linear continuous functions defined on a normed space X. Then every H_L-convex function is convex in usual sense. It was proved in ([2], Proposition 2.2.6) that a convex function g defined on X is Lipschitz continuous at $y \in X$ provided that g is bounded above in a neighbourhood of y. Thus we conclude that the condition (4.16) is valid in the classical convex case.

The other approach to examining the equality $\mathrm{app}_{f,y}(x) = \sup_{h\in\mathcal{D}_L f(y)}(h(x) + f(y))$ is based on upper semicontinuity of the mapping $\mathcal{D}_L f(\cdot)$. We will use the following definition of upper semicontinuity of set valued mappings due to Berge [1]. Let X and T be topological spaces. We say that a mapping $D : X \to 2^T$ is upper semicontinuous at $y \in X$ if, for any open set $G \subset T$ such that $D(y) \subset G$, a neighbourhood U of y exists such that $D(u) \subset G$ for all $u \in U$.

Proposition 4.6. *Let L consist of continuous functions. Assume that the set H_L is equipped with the topology of pointwise convergence. Let f be an H_L-convex function, $y \in X$ and let the sets $\mathcal{D}_L f(z)$ be nonempty in a neighbourhood of y. If f is continuous at the point y and $\mathcal{D}_L f(\cdot)$ is upper semicontinuous at y then* $\operatorname{app}_{f,y}(x) = \sup_{h \in \mathcal{D}_L f(y)} (h(x) + f(y))$ *for all $x \in X$.*

Proof. Firstly, since f is continuous at y then (see (4.11))

$$\operatorname{app}_{f,y}(x) = \limsup_{z \to y} \sup_{h \in \mathcal{D}_L f(z)} (h(x) + f(z)) = f(y) + \limsup_{z \to y} \sup_{h \in \mathcal{D}_L f(z)} h(x).$$

Take $\varepsilon > 0$ and $x \in X$. Let t denote the function $t(z) = \sup_{h \in \mathcal{D}_L f(z)} h(x)$. Let $G_\varepsilon = \{h \in H_L : \ \exists g \in \mathcal{D}_L f(y) \ \ |h(x) - g(x)| < \varepsilon\}$. Then G_ε is an open set and $\mathcal{D}_L f(y) \subset G_\varepsilon$. Since the mapping $\mathcal{D}_L f(\cdot)$ is upper semicontinuous at y then there is a neighbourhood U of the point y such that $T := \cup_{z \in U} \mathcal{D}_L f(z) \subset G_\varepsilon$, hence

$$\begin{aligned}
\sup_{z \in U} t(z) &= \sup_{z \in U} \sup_{h \in \mathcal{D}_L f(z)} h(x) = \sup_{h \in T} h(x) \leq \sup_{h \in G_\varepsilon} h(x) \\
&= \sup\{h(x) : \ h \in H_L, \ \exists g \in \mathcal{D}_L f(y) \ \ |h(x) - g(x)| < \varepsilon\} \\
&\leq \sup\{h(x) : \ h \in H_L, \ \exists g \in \mathcal{D}_L f(y) \ \ (h(x) - g(x)) < \varepsilon\} \\
&\leq \sup_{g \in \mathcal{D}_L f(y)} g(x) + \varepsilon = t(y) + \varepsilon.
\end{aligned}$$

This means that t is upper semicontinuous at y, therefore

$$\operatorname{app}_{f,y}(x) = f(y) + \limsup_{z \to y} t(z) \leq f(y) + t(y) = \sup_{h \in \mathcal{D}_L f(y)} (h(x) + f(y)).$$

The reverse inequality $\sup_{h \in \mathcal{D}_L f(y)} (h(x) + f(y)) \leq \operatorname{app}_{f,y}(x)$ follows from Proposition 4.2. $\square$

5. Examples

Let X and Y be topological spaces and $\omega : X \to Y$ an open continuous mapping. Let $\mathcal{L}$ be a set of functions defined on $\omega(X) = \{\omega(x) : \quad x \in X\}$. Let L be the set of all functions $l(x) = \ell(\omega(x))$ defined on X, where $\ell \in \mathcal{L}$. Then the set of all H_L-convex functions coincides with the set of functions $f(x) = g(\omega(x))$, where g is $H_{\mathcal{L}}$-convex.

Proposition 5.1. *If $H_{\mathcal{L}}$ has the strong globalization property then also H_L has the strong globalization property.*
If g is an $H_{\mathcal{L}}$-convex function, $y = \omega(x)$ and

$$\operatorname{app}_{g,y}(z) = \sup_{h \in \mathcal{D}_{\mathcal{L}} g(y)} (h(z) + g(y)) \qquad \forall\, z \in \omega(X),$$

then the following equalities hold for the function $f = g \circ \omega$

$$\operatorname{app}_{f,x}(z) = \sup_{h \in \mathcal{D}_L f(x)} (h(z) + f(x)) \qquad \forall\, z \in X.$$

Proof. Assume that $H_{\mathcal{L}}$ has the strong globalization property, and let $h \in H_L$. Let $f(x) = g(\omega(x))$ be an H_L-convex function such that

$$h(y) = f(y), \quad h(x) \leq f(x) \quad \forall\, x \in U,$$

where U is a neighbourhood of y. Since $h(x) = \ell(\omega(x)) - c$ then $\ell(\omega(y)) - c = g(\omega(y))$, $\ell(\omega(x)) - c \le g(\omega(x))\ \ \forall x \in U$. Since ω is an open mapping then $U' = \omega(U)$ is a neighbourhood of the point $\omega(y)$. Because $H_{\mathcal{L}}$ has the strong globalization property, we have $\ell(z) - c \le g(z)$ for all $z \in \omega(X)$ and $h(x) \le f(x)$ for all $x \in X$. So we proved that H_L has the strong globalization property.

Let us prove the second part of proposition. Let g be $H_{\mathcal{L}}$-convex, $y = \omega(x)$ and $\mathrm{app}_{g,y}(z) = \sup_{h\in\mathcal{D}_{\mathcal{L}}g(y)}(h(z)+g(y))$ for all $z \in \omega(X)$. It is clear that $(\ell - \ell(\omega(t))) \in \mathcal{D}_{\mathcal{L}}g(\omega(t))$ if and only if $(\ell \circ \omega - \ell(\omega(t))) \in \mathcal{D}_L f(t)$. Hence

$$\mathrm{app}_{f,x}(z) = \inf_{U\in\mathcal{U}(x)} \sup_{t\in U} \sup_{h\in\mathcal{D}_{\mathcal{L}}g(\omega(t))} (h(\omega(z)) + g(\omega(t))).$$

Since ω is a continuous and open mapping then

$$\mathrm{app}_{f,x}(z) = \inf_{U'\in\mathcal{U}(\omega(x))} \sup_{t\in U'} \sup_{h\in\mathcal{D}_{\mathcal{L}}g(t)} (h(\omega(z)) + g(t)).$$

Thus, using also our assumption, we obtain

$$\begin{aligned}\mathrm{app}_{f,x}(z) = \mathrm{app}_{g,y}(\omega(z)) &= \sup_{h\in\mathcal{D}_{\mathcal{L}}g(y)} (h(\omega(z)) + g(y)) \\ &= \sup_{h\in\mathcal{D}_{\mathcal{L}}g(\omega(x))} (h(\omega(z)) + g(\omega(x))) = \sup_{h\in\mathcal{D}_L f(x)} (h(z) + f(x)). \qquad \square\end{aligned}$$

Note that, under the conditions of Proposition 5.1, we have a simple isomorphism between $H_{\mathcal{L}}$-convex and H_L-convex functions. If $f = g \circ \omega$ then $\inf_{x\in X} f(x) = \inf_{y\in\omega(X)} g(y)$. So if $H_{\mathcal{L}}$ has the strong globalization property but the elementary functions $h \in H_{\mathcal{L}}$ seem difficult then we can use such isomorphism in order to get a more convenient equivalent form of abstract convex functions.

Proposition 5.2. *Let X and V be topological spaces. Let H be a set of functions $h : X \to \mathbb{R}$. Assume that for each two points $x, y \in X$ there exists a continuous mapping $\omega : V \to X$ such that $x, y \in \omega(V)$ and H^ω has the strong globalization property, where H^ω is the set of all functions $h' : V \to \mathbb{R}$ defined by $h'(v) = h(\omega(v))$, $(h \in H)$. Then H has the strong globalization property.*

Proof. Let $f : X \to \mathbb{R}_{+\infty}$ be H-convex function. Let $y \in X$ and $h \in H$ be a function such that $h(y) = f(y)$ and $h(x) \le f(x)$ for all x from a neighbourhood U of the point y. Take a point $x \in X$ and consider a mapping $\omega : V \to X$, which satisfies the conditions of our proposition for the points x, y. Let $\omega(v_1) = y$ and $\omega(v_2) = x$. Consider the functions h', f' defined on V by the formulas: $h'(v) = h(\omega(v))$, $f'(v) = f(\omega(v))$. Then h' belongs to H^ω, and f' is H^ω-convex. Since ω is continuous then a neighbourhood U' of the point v_1 exists such that $\omega(v) \in U$ for all $v \in U'$. Hence $h'(v_1) = h(y) = f(y) = f'(v_1)$ and $h'(v) = h(\omega(v)) \le f(\omega(v)) = f'(v)$ for all $v \in U'$. Since H^ω has the strong globalization property then $h'(v) \le f'(v)$ for all $v \in V$. In particular, $h(x) = h'(v_2) \le f'(v_2) = f(x)$. $\square$

Now consider the simplest case $X = \mathbb{R}$.

Proposition 5.3. *Let L be a set of continuous functions defined on $\mathbb{R}$. Assume that for any functions $h_1, h_2 \in H_L$ and for any points $x_1, x_2 \in X$ the following implication holds*

$$(5.1) \qquad (h_1(x_1) = h_2(x_1),\ \ h_1(x_2) = h_2(x_2),\ \ x_1 \ne x_2) \implies (h_1 = h_2).$$

Let $y \in \mathbb{R}$ and f be an H_L-convex function such that the sets $\mathcal{D}_L f(z)$ are nonempty in a neighbourhood U of y. Then for any $h \in H_L$ implication (4.1) holds.

Proof. Let U be a neighbourhood of y such that $\mathcal{D}_L f(z) \neq \emptyset$ for all $z \in U$. Let $h \in H_L$ be an elementary function such that $h(y) = f(y)$ and $h(x) \leq f(x)$ for all $x \in U'$, where U' is a neighbourhood of y. We need to check that $h(x) \leq f(x)$ for all $x \in \mathbb{R}$.

First we will show that $h(x) \leq f(x)$ for any $x > y$. Let $x > y$. Then a point $z \in U \cap U'$ exists such that $x > z > y$. Since $z \in U$ then $\mathcal{D}_L f(z) \neq \emptyset$. Take an arbitrary function $h_z \in \mathcal{D}_L f(z)$. Then $h_z(y) + f(z) \leq f(y) = h(y)$. Moreover, since $z \in U'$ then $h(z) \leq f(z) = h_z(z) + f(z)$. Consider the function $h'(t) = h_z(t) + f(z)$. Since H_L is closed under vertical shifts and $h_z \in H_L$ then $h' \in H_L$. So for these z, y and $h, h' \in H_L$ we have

$$z > y, \quad h'(y) \leq h(y), \quad h(z) \leq h'(z). \tag{5.2}$$

Note that, under our assumptions, H_L consists of continuous functions. Then, due to (5.2), a point $t_1 \in [y, z]$ exists such that $h'(t_1) = h(t_1)$.

Now suppose that $h(x) > h'(x)$. This means, in particular, that $h \neq h'$. It follows from (5.1) that $h'(t) \neq h(t)$ for any $t \neq t_1$. Then, by (5.2), either $h'(y) < h(y)$ or $h(z) < h'(z)$. If $h(z) < h'(z)$ then a point $t_2 \in (z, x)$ exists such that $h'(t_2) = h(t_2)$, which contradicts our assumption. Hence $h'(y) < h(y)$ and $y < t_1$. Take a positive number ε such that $\varepsilon < \min\{h(y) - h'(y), h(x) - h'(x)\}$ and consider the function $h_\varepsilon(t) = h'(t) + \varepsilon$. Then $h_\varepsilon \in H_L$. Moreover, the following inequalities hold

$$h_\varepsilon(t_1) > h(t_1), \quad h_\varepsilon(y) < h(y), \quad h_\varepsilon(x) < h(x). \tag{5.3}$$

Since $y < t_1 < x$ and the functions h_ε and h are continuous then, by (5.3), we can find two different points $a \in (y, t_1)$ and $b \in (t_1, x)$ such that $h_\varepsilon(a) = h(a)$ and $h_\varepsilon(b) = h(b)$. Then, by (5.1), $h_\varepsilon = h$, which contradicts (5.3).

So we conclude that $h(x) \leq h'(x)$. Since $h'(x) = h_z(x) + f(z)$ and $h_z \in \mathcal{D}_L f(z)$ then $h'(x) \leq f(x)$. Thus we have proved that $h(x) \leq f(x)$ for any $x > y$.

The same arguments show that $h(x) \leq f(x)$ for all $x < y$. □

Proposition 5.4. *Let L be a set of continuous functions defined on $\mathbb{R}$ such that (5.1) is valid for H_L. Assume also that for any sequence $\{h_i\} \subset H_L$ the following holds: if a function $h \in H_L$ and an interval $(a, b) \subset \mathbb{R}$ exist such that $\lim_{i \to +\infty} h_i(x) = h(x)$ for all $x \in (a, b)$ then $\lim_{i \to +\infty} h_i(x) = h(x)$ for all $x \in \mathbb{R}$. Then H_L has the strong globalization property.*

Proof. Let f be an H_L-convex function and $y \in \mathbb{R}$. Let $h \in H_L$ be an elementary function such that $h(y) = f(y)$ and $h(x) \leq f(x)$ in a neighbourhood U of the point y. We need to check that $h(x) \leq f(x)$ for all $x \in \mathbb{R}$. Here we will show only that $h(x) \leq f(x)$ for all $x < y$. The proof of the inequality $h(x) \leq f(x)$ for $x > y$ is analogous.

First suppose that a sequence $\{y_i\} \subset \mathbb{R}$ exists such that $y_i < y \; \forall \, i$, $\lim_{i \to +\infty} y_i = y$ and $h(y_i) < f(y_i)$ for all i. Since f is H_L-convex then for each i a function $h_i \in \mathrm{supp}(f, H_L)$ exists such that $f(y_i) \geq h_i(y_i) > h(y_i)$. We have for each i

$$y_i < y, \quad h_i(y_i) > h(y_i), \quad h_i(y) \leq f(y) = h(y). \tag{5.4}$$

Since the functions h_i and h are continuous then we can find a point $t \in (y_i, y]$ such that $h_i(t) = h(t)$. Assume that $h_i(x) < h(x)$ for certain $x < y_i$. Then a point $t' \in (x, y_i)$ exists such that $h_i(t') = h(t')$, and therefore, by (5.1), $h_i = h$, which contradicts (5.4). Hence $h(x) \leq h_i(x) \leq f(x)$ for all $x < y_i$. Since $y_i \to y$ then $h(x) \leq f(x)$ for all $x < y$.

Now suppose that such a sequence $\{y_i\}$ does not exist. Since $h(x) \leq f(x)$ for all $x \in U$ then $h(x) = f(x)$ for all $x \in [a, y]$, where a is a point from the neighbourhood U and $a < y$. Assume that a point $y_0 < a$ exists such that $h(y_0) > f(y_0)$. We will get some contradictions for such a situation. So take a small enough $\varepsilon > 0$ such that $h(y_0) - f(y_0) > 2\varepsilon$. Let $\{\varepsilon_i\}$ be a decreasing sequence of positive numbers and $\lim_{i\to+\infty} \varepsilon_i = 0$, $\varepsilon_1 = \varepsilon$. Since f is H_L-convex and H_L is closed under shifts then a sequence $\{h_i\} \subset \operatorname{supp}(f, H_L)$ exists such that $h_i(a) = f(a) - \varepsilon_i$ for each i. Consider two cases:

1.) Let a point $y' \in (a, y)$ and an index i exist such that $f(y') - h_i(y') > f(a) - h_i(a) = \varepsilon_i$. Choose a positive number δ such that $\min\{f(y') - h_i(y'), 2\varepsilon_i\} > \delta > f(a) - h_i(a) = \varepsilon_i$. Then consider the function $h'(x) = h_i(x) + \delta$. We have

$$h'(y') = h_i(y') + \delta < f(y') = h(y'), \qquad h'(a) = h_i(a) + \delta > f(a) = h(a),$$
$$h'(y_0) = h_i(y_0) + \delta < f(y_0) + 2\varepsilon_i \leq f(y_0) + 2\varepsilon < h(y_0).$$

Since $y_0 < a < y'$, these inequalities contradict (5.1) and the continuity of the elementary functions.

2.) Let $f(y') - h_i(y') \leq f(a) - h_i(a) = \varepsilon_i$ for all i and $y' \in (a, y)$. Since $f(y') - h_i(y') \geq 0$ then

$$\lim_{i\to+\infty} h_i(x) = f(x) = h(x) \qquad \text{for all } x \in (a, y).$$

Due to the assumptions of this proposition $\lim_{i\to+\infty} h_i(x) = h(x)$ for all $x \in X$. Hence $h(y_0) = \lim_{i\to+\infty} h_i(y_0) \leq f(y_0)$ because $h_i \in \operatorname{supp}(f, H_L)$. But this contradicts the assumption $h(y_0) > f(y_0)$. □

Example 5.1. Let $a_0 > 0$ and $X = \mathbb{R}$. Let L be the set of all functions $l(x) = -a_0(x - a)^2$, where $a \in \mathbb{R}$. Then conditions of Proposition 5.4 hold for H_L, and therefore H_L has the strong globalization property. But we do not have the tools here for necessary or for sufficient conditions for global minimum of H_L-convex functions since H_L does not contain any constant and each function $h(x) = -a_0(x - a)^2 - c$ has no global minimum over X.

So we should consider only examples where some elementary functions attain their global minimum. In the following example zero belongs to L. Hence we will have necessary and sufficient condition for the global minimum.

Example 5.2. Let $l_1(x)$ and $l_2(x)$ be continuous strictly decreasing and strictly increasing functions respectively ($x \in \mathbb{R}$). Assume that L consists of all the functions $al_1(x)$, $al_2(x)$ with $a \geq 0$. It is easy to check that the set H_L verifies the assumptions of Proposition 5.4. For example, we can take

$$l_1(x) = -e^x, \qquad l_2(x) = -e^{-x}.$$

We see that the set H_L here is closed under horizontal and vertical shifts. Moreover, the set of all H_L-convex functions is bigger than the set of all lower semicontinuous convex functions defined on $\mathbb{R}$. Indeed, let $t(x) = ax - c$ be an affine function. If $a = 0$ then $t \in H_L$. If $a > 0$ then for each $y \in \mathbb{R}$ we have that $(-ae^y e^{-x} + a + t(y)) \leq t(x)$ for any $x \in \mathbb{R}$, the function $h(x) = -ae^y e^{-x} + a + t(y)$ interpolates t in y and belongs to H_L. The same can be done for $a < 0$. Hence every affine function is H_L-convex.

Example 5.3. Let $l_1, \dots, l_m, a_1, \dots, a_m$ be strictly increasing continuous functions defined on $\mathbb{R}$. Let L denote the set of all functions $l^t(x) = a_1(t)l_1(x) + \dots + a_m(t)l_m(x)$ with $t \in \mathbb{R}$. We will check that (5.1) is valid for H_L. So let

$$h_1(x) = a_1(t_1)l_1(x) + \dots + a_m(t_1)l_m(x) - c_1,$$
$$h_2(x) = a_1(t_2)l_1(x) + \dots + a_m(t_2)l_m(x) - c_2.$$

Let $x \neq y$ and $h_1(x) = h_2(x)$, $h_1(y) = h_2(y)$. Then $(h_1(x) - h_1(y)) - (h_2(x) - h_2(y)) = 0$, that is

$$(a_1(t_1) - a_1(t_2))(l_1(x) - l_1(y)) + \dots + (a_m(t_1) - a_m(t_2))(l_m(x) - l_m(y)) = 0. \tag{5.5}$$

Since $x \neq y$ and the functions l_i are strictly increasing then all the quantities $(l_i(x) - l_i(y))$ have the same sign and are not equal to zero. Since all a_i are strictly increasing then the equality (5.5) is possible only for $t_1 = t_2$. It follows from the equality $h_1(y) = h_2(y)$ that $c_1 = c_2$, hence $h_1 = h_2$.
Now let the sequences $\{t_k\}, \{c_k\}$ and an interval (a, b) be such that

$$\lim_{k\to+\infty} \left(\sum_{i=1}^m a_i(t_k)l_i(x) - c_k \right) = \sum_{i=1}^m a_i(t_0)l_i(x) - c_0 \qquad \text{for all } x \in (a, b).$$

Let $x, y \in (a, b)$ and $x > y$. Then

$$\begin{aligned} &\lim_{k\to+\infty} \left(\sum_{i=1}^m a_i(t_k)l_i(x) - c_k \right) - \lim_{k\to+\infty} \left(\sum_{i=1}^m a_i(t_k)l_i(y) - c_k \right) \\ &\qquad = \sum_{i=1}^m a_i(t_0)l_i(x) - \sum_{i=1}^m a_i(t_0)l_i(y) \\ &\qquad \implies \lim_{k\to+\infty} ((a_1(t_k) - a_1(t_0))(l_1(x) - l_1(y)) \\ &\qquad\qquad + \dots + (a_m(t_k) - a_m(t_0))(l_m(x) - l_m(y))) = 0. \end{aligned}$$

Since all the quantities $(l_i(x) - l_i(y))$ are positive and all the functions a_i are continuous and strictly increasing then $\lim_{k\to+\infty} t_k = t_0$. The equality $\lim_{k\to+\infty} c_k = c_0$ is valid as well. Hence, due to Proposition 5.4, H_L has the strong globalization property.

Now consider the usual convex functions defined on a topological linear space.

Example 5.4. Let L be the set of all linear continuous functions defined on a topological linear space X. Let $\mathcal{L}$ be the set of all linear functions defined on $\mathbb{R}$. It follows from Example 5.3 (with $m = 1$, $a_1(t) = t$, $l_1(x) = x$) that the set $H_{\mathcal{L}}$ of all affine functions defined on $\mathbb{R}$ has the strong globalization property. Take two arbitrary points $x, y \in X$ and consider the function $\omega : \mathbb{R} \to X$ defined by $\omega(v) = vx + (1 - v)y$. Then $\omega(0) = y$ and $\omega(1) = x$. Moreover, ω is continuous and for any $h \in H_L$ the function $h'(v) = h(\omega(v))$ belongs to $H_{\mathcal{L}}$. Indeed, if $h(z) = l(z) + c \ \forall z \in X$, where $l \in L$ and $c \in \mathbb{R}$, then $h'(v) = l(vx + (1-v)y) + c = v(l(x) - l(y)) + (l(y) + c)$. Thus, by Proposition 5.2 (see also Remark 4.1), H_L has the strong globalization property.

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