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EVOLUTION INCLUSIONS WITH PLN FUNCTIONS AND APPLICATION TO VISCOSITY AND CONTROL

C. CASTAING AND S. MARCELLIN

Abstract. We present some existence and uniqueness of absolutely continuous solutions for the evolution inclusion

$$\begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + F(t, u(t)) & \text{a.e. } t \in [T_0, T] \\ u(T_0) = x_0 \end{cases}$$

in a separable Hilbert space H, here ∂f is the proximal subdifferential of a lower semicontinuous primal lower nice function f defined on H, $F : [T_0, T] \times H \rightrightarrows H$ is a convex weakly compact valued upper semicontinuous multifunction. Applications to Control and Viscosity problems involving Young measures are investigated.

1. Introduction and Preliminaries

The present work deals with an evolution inclusion governed by the subdifferential of a nonconvex function and its applications to control and viscosity problems. Throughout all the paper, H stands for a real separable Hilbert space. A proper function $f : H \to \mathbf{R} \cup \{+\infty\}$ is *primal lower nice* (pln for short) at $x_0 \in \text{dom}\, f$, if there exist positive constant real numbers, s_0, c_0, Q_0 such that for all x in the closed ball $\overline{B}_H(x_0, s_0)$, for all $q \geq Q_0$ and for $v \in \partial_P f(x)$ with $||v|| \leq c_0 q$, one has

$$f(y) \geq f(x) + \langle v, y - x \rangle - \frac{q}{2} ||y - x||^2$$

for each $y \in \overline{B}_H(x_0, s_0)$, here $\partial_P f(x)$ denotes the proximal subdifferential of f at x ([18], [19]). It is straightforward to observe that each extended real valued convex function is primal lower nice at any point of its domain as well as functions that are convex up to a square. Another example of pln functions is given by qualified convexely composite functions. To learn more on the study of pln functions, we refer to ([10], [14], [17], [18], [21]). Recall that if f is pln at u_0 with constants s_0, c_0, Q_0, one has

$$\text{(local hypomonotonicity)} \qquad \langle v_1 - v_2, x_1 - x_2 \rangle \geq -q ||x_1 - x_2||^2$$

for any $v_i \in \partial_P f(x_i)$ with $||v_i|| \leq c_0 q$ whenever $q \geq Q_0$ and $x_i \in \overline{B}_H(u_0, s_0)$, $i = 1, 2$. A more general class of pln functions involving the one of Φ-convex functions in considered in [11] in which evolution problems without lack of convexity where studied.

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In section 2, we present some existence and uniqueness of absolutely continuous solutions for the evolution inclusion

$$\begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + F(t, u(t)) & \text{a.e. } t \in [T_0, T] \\ u(T_0) = x_0 \end{cases}$$

where f is primal lower nice at $x_0 \in \operatorname{dom} f$ with constants, $s_0, c_0, Q_0 > 0$ and $F : [T_0, T] \times H \rightrightarrows H$ is a convex weakly compact valued upper semicontinuous multifunction.

In section 3, we give some applications to Control theory, namely we study some viscosity properties of a value function V_J defined on $[0, T] \times H$ by

$$V_J(\tau, x) := \sup_{\nu \in \mathcal{Z}} \inf_{\mu \in \mathcal{Y}} \{ \int_\tau^T [\int_Z [\int_Y J(t, u_{x,\mu,\nu}(t), y, z)\, \mu_t(dy)]\, \nu_t(dz)]\, dt \},$$

where the cost function $J : [0, T] \times \mathbf{R}^d \times Y \times Z \to \mathbf{R}$ is bounded and continuous, the control spaces Y and Z are compact metric spaces, and the control measure μ (resp. ν) belongs to the space of Young measures $\mathcal{Y} := \mathcal{Y}([0, T], \mathcal{M}^1_+(Y))$ (resp. $\mathcal{Z} := \mathcal{Y}([0, T], \mathcal{M}^1_+(Z)))$ that is the set of all Lebesgue-measurable mappings from $[0, T]$ into the space $\mathcal{M}^1_+(Y)$ (resp. $\mathcal{M}^1_+(Z)$) of all probability Radon measures on Y (resp. Z) endowed with the vague topology $\sigma(\mathcal{C}(Y)', \mathcal{C}(Y))$ (resp. $\sigma(\mathcal{C}(Z)', \mathcal{C}(Z))$), $u_{x,\mu,\nu}$ is the trajectory solution on $[0, T]$ of the evolution inclusion

$$\begin{cases} \dot{u}_{x_0,\mu,\nu}(t) \in -\partial f(u_{x_0,\mu,\nu}(t)) + \int_Z [\int_Y g(t, u_{x_0,\mu,\nu}(t), y, z)\, \mu_t(dy)]\, \nu_t(dz), \\ u_{x_0,\mu,\nu}(t_0) = x_0 \in \operatorname{dom} f, \end{cases}$$

here $f : H \to \mathbf{R}$ is a Lipschitz continuous function that is pln on each closed ball centered at the origin with the same constants, $g : [0, T] \times H \times Y \times Z \to H$, is a bounded continuous mapping and uniformly lipschitzean on H. A bang-bang type theorem in Control theory and the study of the solutions set of a class of functional evolution inclusions are also investigated.

Unless specified, in all the sequel, ∂ stands for the proximal subdifferential operator.

We refer to [16] for pioneer results on evolution problems associated with the subdifferential of lower semicontinuous (lsc) primal lower nice functions.

2. Evolution inclusions associated with the subdifferential of a lsc pln function

Throughout H is a separable Hilbert space. For the convenience of the reader, let us recall and summarize the following theorem and its remarks ([15], Theorem 4.1.2 and Remark 4.1.4) since the proof of Theorem 2.2 below involves results from them.

Theorem 2.1 (alias Theorem 4.1.2 in [15]). *Let $f : H \to \mathbf{R} \cup \{+\infty\}$ be a proper lsc function. Consider some point $u_0 \in$ dom f such that f is pln at u_0 with constants s_0, c_0, Q_0 and let some real number $\eta_0 \in]0, s_0[$ be such that*

$$\inf\{f(x) : x \in \overline{B}_H(u_o, \eta_0)\} \text{ is finite.}$$

(Such η_0 always exists by lower semicontinuity of f at u_0). Consider also a real number $T_0 \geq 0$ and some $h \in L^2_{loc}([T_0, +\infty[; H)$.

Then, there exist some real number $\tau > T_0$ and a unique mapping $u : [T_0, \tau] \to \overline{B}_H(u_o, \eta_0)$ that is absolutely continuous on $[T_0, \tau]$ and such that

$$\text{(I)} \qquad \begin{cases} \dot{u}(t) + \partial f(u(t)) \ni h(t) & \text{for a.e } t \in [T_0, \tau], \\ u(T_0) = u_0. \end{cases}$$

In addition, the following properties hold:

(a) $\{u(t) : t \in [T_0, \tau]\} \subset dom\, f$;
(b) $\dot{u} \in L^2([T_0, \tau]; H)$;
(c) *for all* $s, t \in [T_0, \tau]$ *with* $s \leq t$,

$$(2.1)\quad (\int_s^t \|\dot{u}(r)\|^2 dr)^{\frac{1}{2}} \leq [f(u_0) - f(u(t)) + \frac{1}{4}\int_{T_0}^t \|h(r)\|^2 dr]^{\frac{1}{2}} + \frac{1}{2}(\int_{T_0}^t \|h(r)\|^2 dr)^{\frac{1}{2}},$$

which implies that

$$(2.2)\qquad \int_s^t \|\dot{u}(r)\|^2 dr \leq 2(f(u_0) - f(u(t))) + \int_{T_0}^t \|h(r)\|^2 dr.$$

Moreover, the solution $u(\cdot)$ is "slow", that is :

$$\dot{u}(t) = -(\partial f(u(t)) - h(t))^0 \text{ for almost every } t \in]T_0, \tau[,$$

where $(\partial f(u(t)) - h(t))^0$ is the element of minimum norm of the closed convex set $\partial f(u(t)) - h(t)$.

Remark 2.1 (alias Remark 4.1.4 in [15]). With the notations of Theorem 2.1, note that, as $\eta_0 < s_0$, f is pln at any point of $\overline{B}_H(u_0, \frac{\eta_0}{2}) \cap dom\, f$ with the same constants $\frac{\eta_0}{2}$, c_0, Q_0. So, given $M > 0$, for all $x_0 \in \overline{B}_H(u_0, \frac{\eta_0}{2})$ and all $h \in L^2([T_0, T]; H)$ such that $f(x_0) \leq M$ and $\|h\|_{L^2([T_0,T];H)} \leq M$, for any real number $\tau \in]T_0, T]$ satisfying

$$(\tau - T_0)^{\frac{1}{2}} [2(M - \inf_{\overline{B}_H(u_0,\eta_0)} f + M^2)]^{\frac{1}{2}} < \frac{\eta_0}{2},$$

there is an absolutely continuous mapping $u : [T_0, \tau] \to \overline{B}_H(u_0, \eta_0)$ such that

- $\dot{u}(t) + \partial f(u(t)) \ni h(t)$ a.e in $[T_0, \tau]$, $u(T_0) = x_0$,
- $u([T_0, \tau]) \subset dom\, f$,
- $\dot{u} \in L^2([T_0, \tau]; H)$, and for all $s, t \in [T_0, \tau]$ with $s \leq t$,

$$\begin{aligned} \int_s^t \|\dot{u}(r)\|^2 dr &\leq 2(f(x_0) - f(u(t))) + \int_{T_0}^t \|h(r)\|^2 dr \\ &\leq 2(M - \inf_{\overline{B}_H(u_0,\eta_0)} f) + M^2. \end{aligned}$$

We begin with a local existence of solutions for the evolution inclusion under consideration.

Theorem 2.2. *Assume that H is a separable Hilbert space and $f : H \to \mathbf{R} \cup \{\infty\}$ is proper lsc primal lower nice at $x_0 \in dom\ f$ with constants $s_0, c_0, Q_0 > 0$ satisfying:*

(i) $\inf\{f(x) : x \in \overline{B}_H(x_0, s_0)\} \in\ R$,

(ii) *for each positive real number λ, the truncated sublevel set*

$$L_f(\lambda) := \{x \in H : ||x - x_0|| \leq s_0;\ f(x) \leq \lambda\}$$

is compact in $(H, ||.||)$.

Let $F : [T_0, +\infty[\times H \Rightarrow H$ *be a nonempty convex weakly compact valued multifunction satisfying:*

(j) $F(.,.)$ *is separately scalarly Lebesgue-measurable on* $[T_0, +\infty[$ *and separately scalarly upper semicontinuous on* H,

(jj) *there exists a nonegative function* $k \in L^2_{\mathbf{R}}([T_0, +\infty[)$ *such that,* $\forall t \in [T_0, +\infty[$, $\forall x \in H$

$$F(t, x) \subset k(t)(1 + ||x||)\overline{B}_H(0, 1).$$

Let us fix an arbitrary number $T > T_0$. *Then there exist* $\tau \in]T_0, T]$ *and at least one absolutely continuous mapping* $u : [T_0, \tau] \to \overline{B}_H(x_0, s_0)$ *satisfying*

$$(\mathcal{I}_{\partial f, F}) \qquad \begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + F(t, u(t)) & a.e.\ t \in [T_0, \tau] \\ u(T_0) = x_0. \end{cases}$$

More precisely, there exists $\beta \in L^2([T_0, \tau]; H)$ *such that*

$$\beta(t) \in F(t, u(t)) \quad a.e. \quad t \in [T_0, \tau]$$

and

$$0 \in \dot{u}(t) + \partial f(u(t)) + \beta(t)\ \ a.e.\ t \in [T_0, \tau];\ u(T_0) = x_0,$$

with

$$\int_{T_0}^{t} ||\dot{u}(s)||^2 ds \leq 2(f(x_0) - f(u(t))) + \int_{T_0}^{t} ||\beta(s)||^2 ds, \quad \forall t \in [T_0, \tau].$$

Proof. Let us set $M := (1 + ||x_0|| + s_0)||k||_{L^2([T_0,T];H)}$. According to Theorem 2.1 and Remark 2.1, there exists $\tau \in]T_0, T]$ satisfying

$$(\tau - T_0)^{\frac{1}{2}}[2(f(x_0) - \inf\{f(x) : x \in \overline{B}_H(x_0, s_0)\} + M^2)]^{\frac{1}{2}} < s_0,$$

such that for any $h \in L^2([T_0, T]; H)$ with $||h||_{L^2([T_0,T];H)} \leq M$, there exists a unique absolutely continuous mapping $u_h : [T_0, \tau] \to \overline{B}_H(x_0, s_0)$ such that

$$\begin{cases} 0 \in \dot{u}_h(t) + \partial f(u_h(t)) + h(t) & \text{a.e. } t \in [T_0, \tau], \\ u_h(T_0) = x_0 \end{cases}$$

with $u_h([T_0, \tau]) \subset dom f$ and f is pln at $u(t), t \in [T_0, \tau]$, and

$$(2.3) \qquad \forall t \in [T_0, \tau], \int_{T_0}^{t} ||\dot{u}_h(s)||^2 ds \leq 2(f(x_0) - f(u_h(t))) + \int_{T_0}^{t} ||h(s)||^2 ds.$$

In particular

$$(2.4) \qquad ||\dot{u}_h||^2_{L^2_H([T_0,\tau])} \leq 2(f(x_0) - \inf\{f(x) : x \in \overline{B}_H(x_0, s_0)\}) + M^2.$$

Let us consider the convex weakly compact set in $L^2([T_0, \tau]; H)$

$$\overline{B}_{L^2([T_0,\tau];H)}(0, M) = \{h \in L^2([T_0, \tau]; H) : ||h||_{L^2([T_0,\tau];H)} \leq M\},$$

and define the solution map

$$\overline{B}_{L^2([T_0,\tau];H)}(0,M) \to \mathcal{C}([T_0,\tau];\overline{B}_H(x_0,s_0))$$
$$h \mapsto u_h$$

here $\mathcal{C}([T_0,\tau];\overline{B}_H(x_0,s_0)$ denotes the space of all continuous mappings defined on $[T_0,\tau]$ with values in $\overline{B}_H(x_0,s_0)$, endowed with the norm of uniform convergence. Using (j)-(jj), it is not difficult to see that, for any $h \in L^2([T_0,\tau];H)$, the set-valued map $F(.,h(.))$ admits Lebesgue-measurable selections ([7], Theorem VI-6). Next, for each $h \in L^2([T_0,\tau];H)$, put

$$\Gamma(h) := \{\gamma \in L^2([T_0,\tau];H) : \gamma(t) \in F(t,u_h(t)) \ \text{ a.e. } \ t \in [T_0,\tau]\}.$$

Now, we prove the main fact of the proof which provides the existence of solutions of our evolution inclusion on $[T_0,\tau]$.

Main fact. Γ *is an nonempty convex weakly compact-valued upper semicontinuous multifunction from* $\overline{B}_{L^2([T_0,\tau];H)}(0,M)$ *to* $\overline{B}_{L^2([T_0,\tau];H)}(0,M)$*, here* $\overline{B}_{L^2([T_0,\tau];H)}(0,M)$ *is endowed with the weak topology of* $L^2([T_0,\tau];H)$ *.*

Let $h \in \overline{B}_{L^2([T_0,\tau];H)}(0,M)$ and let γ be a Lebesgue-measurable selection of $F(.,u_h(.))$. By (ii) we have that $||\gamma(t)|| \leq k(t)(1+||u_h(t)||)$ for a.e $t \in [T_0,\tau]$. As the choice of τ ensures that $u_h(t) \in \overline{B}_H(x_0,s_0)$ for all $t \in [T_0,\tau]$, making use of (ii) we see that for a.e. $t \in [T_0,\tau]$, $||\gamma(t)|| \leq k(t)(1+||x_0||+s_0)$ which implies that $\gamma \in L^2([T_0,\tau];H)$ and

$$||\gamma||_{L^2([T_0,\tau];H)} \leq (1+||x_0||+s_0)||k||_{L^2([T_0,\tau])} = M.$$

Hence $\Gamma(h) \subset \overline{B}_{L^2([T_0,\tau];H)}(0,M)$ for any $h \in \overline{B}_{L^2([T_0,\tau];H)}(0,M)$. Since F has closed convex values in H, it is obvious that $\Gamma(h)$ is closed and convex in $L^2([T_0,\tau];H)$ and $\overline{B}_{L^2([T_0,\tau];H)}(0,M)$ is $\sigma(L^2([T_0,\tau];H),L^2([T_0,\tau];H))$ compact, by what has been proved, we conclude that $\Gamma(h)$ is a nonempty convex weakly compact subset of $\overline{B}_{L^2([T_0,\tau];H)}(0,M)$. It remains to check that

$$\Gamma : \overline{B}_{L^2([T_0,\tau];H)}(0,M) \rightrightarrows \overline{B}_{L^2([T_0,\tau];H)}(0,M)$$

is upper semicontinuous. As H is separable, $\overline{B}_{L^2([T_0,\tau];H)}(0,M)$ is compact metrizable for the weak topology on $L^2([T_0,\tau];H)$, it is enough to prove that the graph of Γ is sequentially compact for this topology. Let h_n, h and γ_n, γ in $\overline{B}_{L^2([T_0,\tau];H)}(0,M)$ be such that $h_n \to h$ and $\gamma_n \to \gamma$ weakly with

$$\gamma_n(t) \in F(t,u_{h_n}(t)) \text{ a.e. } t \in [T_0,\tau].$$

Indeed, according to the estimate (2.3), for every $t \in [T_0,\tau]$, and for every $n \in \mathbf{N}$, $u_{h_n}(t)$ lies in the truncated sublevel set $L_f(f(x_0)+\frac{M^2}{2})$ that is compact in $(H,||.||)$ and by the estimate (2.4), (u_{h_n}) is equi-Holder continuous. Hence, by Ascoli's theorem, we may assume that up to an extracted subsequence (u_{h_n}) converges uniformly on $[T_0,\tau]$, and actually, by virtue of Proposition 4.1.8 in [15], (u_{h_n}) converges uniformly to u_h. Consequently, we may apply now the closure theorem in ([7], Theorem VI-4) to get $\gamma(t) \in F(t,u_h(t))$ a.e. $t \in [T_0,\tau]$. In view of the Kakutani-Ky-Fan fixed point theorem, there is $\overline{h} \in \overline{B}_{L^2([T_0,\tau];H)}(0, M)$ such that $\overline{h} \in \Gamma(\overline{h})$. In other words, the absolutely continuous mapping $u_{\overline{h}} : [T_0, \tau] \to \overline{B}_H(x_0, s_0)$ satisfies

$$\begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + \overline{h}(t) \\ \overline{h}(t) \in F(t, u(t)) \text{ a.e. } t \in [T_0, \tau] \\ u(T_0) = x_0 \end{cases}$$

and is a solution of the evolution inclusion $(\mathcal{I}_{\partial f,F})$ on $[T_0, \tau]$. □

Now we proceed to the global existence result.

Theorem 2.3. *Let $T_0 \in \mathbf{R}^+$. Let H be a separable Hilbert space, and let $f : H \to \mathbf{R} \cup \{\infty\}$ be a proper lsc function that is pln on its domain dom f. Suppose that for some real number $\alpha > 0$,*

- (H_1) $f(x) \geq -\alpha(1 + ||x||), \forall x \in H$.
- (H_2) *f is inf-ball compact around each point of dom f, i.e, $\forall x \in$ dom f, there exists $r > 0$ such that, $\forall \lambda > 0$, the set $\{f \leq \lambda\} \cap \overline{B}_H(x, r)$ is compact in $(H, ||.||)$.*

Let $F : [T_0, +\infty[\times H \Rightarrow H$ be a nonempty convex weakly compact valued multifunction satisfying the conditions (j) and (jj) of Theorem 2.2. Then, for each $x_0 \in$ dom f, there exists a locally absolutely continuous mapping $u : [T_0, +\infty[\to H$ that satisfies

$$(\mathcal{I}_{\partial f,F}) \qquad \begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + F(t, u(t)) \quad a.e. \ t \in [T_0, +\infty[\\ u(T_0) = x_0 \\ u([T_0, +\infty[) \subset \text{dom } f. \end{cases}$$

The following inequality holds for any $r, t \in [T_0, +\infty[, r \leq t$

$$\int_r^t ||\dot{u}(s)||^2 ds \leq 2(f(x_0) - f(u(t))) + \int_{T_0}^t ||\beta(s)||^2 ds$$

here β is a $L^2_{loc}([T_0, +\infty[; H)$-selection of $F(., u(.))$ such that

$$0 \in \dot{u}(t) + \partial f(u(t)) + \beta(t) \quad a.e. \quad t \in [T_0, +\infty[.$$

Proof. Denote by $u : [T_0, \theta[\to H$ with $\theta \leq +\infty$, the *maximal* locally absolutely continuous solution [1] of the inclusion

$$(\mathcal{I}_{\partial f,F}) \qquad \begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + F(t, u(t)) \quad \text{a.e.} \ \ t \in [T_0, \theta[\\ u(T_0) = x_0 \\ u([T_0, \theta[) \subset \text{dom } f \end{cases}$$

for which there exists $\beta \in L^2_{loc}([T_0, \theta[; H)$ satisfying $\beta(t) \in F(t, u(t))$ for a.e. $t \in [T_0, \theta[$ with

$$0 \in \dot{u}(t) + \partial f(u(t)) + \beta(t) \text{ a.e. } \ t \in [T_0, \theta[$$

and

$$(2.5) \qquad \forall t \in [T_0, \theta[, \int_{T_0}^t ||\dot{u}(s)||^2 ds \leq 2(f(x_0) - f(u(t))) + \int_{T_0}^t ||\beta(s)||^2 ds.$$

[1]The choice of such a maximal solution is classically made possible by Zorn's lemma.

Our aim is to show that $\theta = +\infty$. First, let us observe a few facts. Fix any $t \in [T_0, \theta[$. By virtue of (2.5) and (jj), one has

$$f(u(t)) \le f(x_0) + \frac{1}{2} \int_{T_0}^{t} k^2(s)(1 + ||u(s)||)^2 ds \tag{2.6}$$

while (H_1) and (jj) lead to

$$\int_{T_0}^{t} ||\dot{u}(s)||^2 ds \le 2(f(x_0) + \alpha(1 + ||u(t)||)) + \int_{T_0}^{t} k^2(s)(1 + ||u(s)||)^2 ds \tag{2.7}$$

for all $t \in [T_0, \theta[$, and hence

$$\begin{aligned} ||u(t) - x_0||^2 \le 2(t - T_0)[f(x_0) + \alpha + \int_{T_0}^{t} k^2(s) ds] \\ + 2(t - T_0)[\alpha ||u(t)|| + \int_{T_0}^{t} k^2(s) ||u(s)||^2 ds]. \end{aligned}$$

This implies that

$$\begin{aligned} &||u(t)||^2 - 4\alpha(t - T_0)||u(t)|| \\ &\qquad \le 2||x_0||^2 + 4(t - T_0)[f(x_0) + \alpha + \int_{T_0}^{t} k^2(s) ds + \int_{T_0}^{t} k^2(s) ||u(s)||^2 ds]. \end{aligned}$$

Then it is not difficult to deduce that

$$\begin{aligned} &||u(t)|| \le 4\alpha(t - T_0) \\ &\qquad + 2[2||x_0||^2 + 4(t - T_0)(f(x_0) + \alpha + \int_{T_0}^{t} k^2(s) ds + \int_{T_0}^{t} k^2(s) ||u(s)||^2 ds)]^{\frac{1}{2}} \end{aligned}$$

and hence

$$\begin{aligned} ||u(t)||^2 \le 8(4\alpha^2(t - T_0)^2 + 2||x_0||^2 + 4(t - T_0)(f(x_0) + \alpha + \int_{T_0}^{t} k^2(s) ds)) \\ + 32(t - T_0) \int_{T_0}^{t} k^2(s) ||u(s)||^2 ds. \end{aligned}$$

Thus, applying Gronwall's inequality yields

$$||u(t)||^2 \le a(t) + 32(t - T_0) \int_{T_0}^{t} a(s) k^2(s) \exp(32 \int_{s}^{t} k^2(\tau)(\tau - T_0) d\tau) ds \tag{2.8}$$

where

$$a(t) := 8[4\alpha^2(t - T_0)^2 + 2||x_0||^2 + 4(t - T_0)(f(x_0) + \alpha + \int_{T_0}^{t} k^2(s) ds)]$$

for each $t \in [T_0, \theta[$.

Now, to show that $\theta = +\infty$, we proceed by contradiction. Assume that $\theta < +\infty$. Then we easily deduce from (2.8) that

$$M_\theta := \sup_{t \in [T_0, \theta[} ||u(t)|| < +\infty. \tag{2.9}$$

Then, by (2.7) for any $s, t \in [T_0, \theta[$ with $s \le t$,

$$||u(t) - u(s)|| \le (t-s)^{\frac{1}{2}}[2(f(x_0) + \alpha(1 + M_\theta)) + (1 + M_\theta)^2 ||k||^2_{L^2([T_0,\theta])}]^{\frac{1}{2}}$$

which implies, by Cauchy's criterion that $\overline{u} := \lim_{t \uparrow \theta} u(t)$ exists in $(H, ||.||)$. As

$$\forall t \in [T_0, \theta[,\ f(u(t)) \le f(x_0) + \frac{1}{2}(1 + M_\theta)^2 ||k||^2_{L^2([T_0,\theta])}$$

in view of (2.6), the lower semicontinuity of f ensures that $\overline{u} \in \text{dom } f$ and hence f is pln at $\overline{u}$. Considering θ as initial time and $\overline{u}$ as initial value, under our assumptions, the local existence Theorem 2.2 guarantees that there exist $\delta > 0$ and an absolutely continuous mapping $v : [\theta, \theta + \delta] \to H$ satisfying

$$\begin{cases} 0 \in \dot{v}(t) + \partial f(v(t)) + \gamma(t) \ \text{ a.e. } \ t \in [\theta, \theta + \delta] \\ \gamma(t) \in F(t, v(t)) \ \text{ a.e. } \ t \in [\theta, \theta + \delta] \\ v(\theta) = \overline{u} \\ v([\theta, \theta + \delta]) \subset \text{dom } f, \end{cases}$$

where $\gamma \in L^2([\theta, \theta + \delta]; H)$ and for each $t \in [\theta, \theta + \delta]$,

$$\int_\theta^t ||\dot{v}(s)||^2 ds \le 2(f(\overline{u}) - f(v(t))) + \int_\theta^t ||\gamma(s)||^2 ds.$$

As a result, defining $w : [T_0, \theta + \delta] \to H$ by

$$w(t) = \begin{cases} u(t) & \text{if } t \in [T_0, \theta[\\ v(t) & \text{if } t \in [\theta, \theta + \delta] \end{cases}$$

and $\psi = \mathbb{1}_{[T_0,\theta[}\beta + \mathbb{1}_{[\theta,\theta+\delta]}\gamma$, we see that w is absolutely continuous on $[T_0, \theta+\delta]$ and $\psi \in L^2([T_0, \theta + \delta]; H)$ and one has

$$\begin{cases} 0 \in \dot{w}(t) + \partial f(w(t)) + \psi(t) \ \text{ a.e. } \ t \in [T_0, \theta + \delta] \\ \psi(t) \in F(t, w(t)) \ \text{ a.e. } \ t \in [T_0, \theta + \delta] \\ w(T_0) = x_0 \\ w([T_0, \theta + \delta]) \subset \text{dom } f, \end{cases}$$

and by the lower semicontinuity of f at $\overline{u}$, it is not difficult to show that for any $t \in [T_0, \theta + \delta]$, the inequality

$$\int_{T_0}^t ||\dot{w}(s)||^2 ds \le 2(f(x_0) - f(w(t))) + \int_{T_0}^t ||\psi(s)||^2 ds$$

holds true. Thus $w(.)$ is a continuation of $u(.)$ on $[\theta, \theta + \delta]$ which contradicts the maximality of $u(.)$. Then $\theta = +\infty$ and $u(.)$ is an expected global solution of the inclusion under consideration on $[T_0, +\infty[$. □

Now we present a variant of the preceding results via a new technique of discretization.

Theorem 2.4. *Let $T_0 \in \mathbf{R}^+$. Let $f : H \to \mathbf{R} \cup \{\infty\}$ be proper lsc and pln on dom f. Suppose that for some positive number α,*

(H_1) $f(x) \ge -\alpha(1 + ||x||), \forall x \in H$.

Let $F : [T_0, +\infty[\times H \rightrightarrows H$ *be a nonempty convex compact valued scalarly upper semicontinuous multifunction, which satisfies the growth type condition:*

(H_2) *there is a nonnegative function* φ *in* $L^2([T_0, +\infty[)$ *and a compact convex set* K *in* $(H, ||.||)$ *verifying* $0 \in K \subset \overline{B}_H(0, 1)$ *such that*

$$\forall (t, x) \in [T_0, +\infty[\times H, \quad F(t, x) \subset \varphi(t)(1 + ||x||)K.$$

Then, for each $x_0 \in dom\ f$*, there exists a locally absolutely continuous mapping* $u : [T_0, +\infty[\to H$ *that satisfies*

$$(\mathcal{I}_{\partial f,F}) \qquad \begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + F(t, u(t)) & a.e.\ \ t \in [T_0, +\infty[\\ u(T_0) = x_0 \\ u([T_0, +\infty[) \subset dom\ f, \end{cases}$$

and such that, for all $r, t \in [T_0, +\infty[$*,* $r \leq t$*,*

$$\int_r^t ||\dot{u}(s)||^2 ds \leq 2(f(x_0) - f(u(t))) + \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||u(s)||)^2 ds.$$

Proof. A) We first prove the existence of a local solution for $(\mathcal{I}_{\partial f,F})$. Let T be a fixed number $> T_0$. For each $n \in \mathbf{N}$, for each $k = 1, .., n + 1$, define

$$t_k^n := T_0 + (k-1)\frac{T - T_0}{n}$$

and consider for $k \in \{1, ..., n\}$, $\delta_k^n \in [t_k^n, t_{k+1}^n]$ such that

$$\varphi(\delta_k^n) \leq \inf_{t \in [t_k^n, t_{k+1}^n[} \varphi(t) + 1. \tag{2.10}$$

Then, fix any $n \in \mathbf{N}$. Put $u_1^n(t_1^n) = x_0$ and choose $v_1^n \in F(\delta_1^n, x_0)$. Then, relying on Theorem 4.1.7 in [15], denote by $u_1^n : [t_1^n, T] \to H$ the absolutely continuous solution on $[t_1^n, T]$ of the inclusion

$$\begin{cases} 0 \in \dot{y}(t) + \partial f(y(t)) + v_1^n & \text{a.e. } t \in [t_1^n, T] \\ y(t_1^n) = x_0 = u_1^n(t_1^n). \end{cases}$$

Next for each $k \in \{2, ..., n\}$, choose $v_k^n \in F(\delta_k^n, u_{k-1}^n(t_k^n))$ and let $u_k^n : [t_k^n, T] \to H$ be the absolutely continuous solution of

$$\begin{cases} 0 \in \dot{y}(t) + \partial f(y(t)) + v_k^n & \text{a.e. } t \in [t_k^n, T] \\ y(t_k^n) = u_{k-1}^n(t_k^n). \end{cases}$$

In view of Theorem 4.1.7 in [15], recall that for any $k \in \{1, ..., n\}$,

$$\int_r^t ||\dot{u}_k^n(s)||^2 ds \leq 2(f(u_k^n(t_k^n)) - f(u_k^n(t))) + (t - t_k^n)||v_k^n||^2 \tag{2.11}$$

whenever $r, t \in [t_k^n, T], r \leq t$. Now, we define $w_n : [T_0, T] \to H$ by

$$w_n(t) = \begin{cases} u_k^n(t) & \text{if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1, ..., n\} \\ u_n^n(T) & \text{if } t = T. \end{cases}$$

Such a map w_n is absolutely continuous on $[T_0, T]$. Consider the mappings $\theta_n, \Delta_n : [T_0, T] \to [T_0, T]$ such that

$$\theta_n(t) = \begin{cases} t_k^n & \text{if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1, ..., n\} \\ T & \text{if } t = T. \end{cases}$$

and

$$\Delta_n(t) = \begin{cases} \delta_k^n & \text{if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1, ..., n\} \\ \delta_n^n & \text{if } t = T. \end{cases}$$

Next define $v_n : [T_0, T] \to H$ by

$$v_n(t) = \begin{cases} v_k^n & \text{if } t \in [t_k^n, t_{k+1}^n[\text{ for some } k \in \{1, ..., n\} \\ v_n^n & \text{if } t = T. \end{cases}$$

Then, for each $n \in \mathbf{N}$, we have the following

(a) $\forall t \in [T_0, T], v_n(t) \in F(\Delta_n(t), w_n(\theta_n(t))) \subset \varphi(\Delta_n(t))(1 + ||w_n(\theta_n(t))||)K$,
(b) $\forall t \in [T_0, T], ||v_n(t)|| \leq \varphi(\Delta_n(t))(1 + ||w_n(\theta_n(t))||)$,
(c) $w_n(T_0) = x_0$,
(d) $0 \in \dot{w}_n(t) + \partial f(w_n(t)) + v_n(t)$ a.e. $t \in [T_0, T]$,

and hence

$$0 \in \dot{w}_n(t) + \partial f(w_n(t)) + F(\Delta_n(t), w_n(\theta_n(t))) \quad \text{a.e.} \quad t \in [T_0, T].$$

Further by (2.11) it is not difficult to see that for all $T_0 \leq r \leq t \leq T$,

$$\int_r^t ||\dot{w}_n(s)||^2 ds \leq 2(f(x_0) - f(w_n(t))) + \int_{T_0}^t ||v_n(s)||^2 ds \tag{2.12}$$

thus, using (H_1) and (2.10), it comes

$$\begin{aligned} \int_r^t ||\dot{w}_n(s)||^2 ds & \\ & \leq 2(f(x_0) + \alpha(1 + ||w_n(t)||)) + \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||w_n(\theta_n(s))||)^2 ds. \end{aligned} \tag{2.13}$$

Let us denote by s_0, c_0, Q_0 some positive constants associated with the pln property of f at x_0, and fix $\eta_0 \in]0, s_0[$. Then, we fix a real number $\tau \in]T_0, T[$ such that

$$\begin{aligned} (\tau - T_0)^{\frac{1}{2}} [2(f(x_0) + \alpha(1 + s_0 + ||x_0||) & \\ & + 2(1 + s_0 + ||x_0||)^2 (||\varphi||_{L^2}^2 + T - T_0))]^{\frac{1}{2}} < \eta_0. \end{aligned} \tag{2.14}$$

Then, relying on estimation (2.13) and (2.14), it can be shown that

$$\forall n \in \mathbf{N}, \ w_n([T_0, \tau]) \subset \overline{B}_H(x_0, s_0). \tag{2.15}$$

For each $n \in \mathbf{N}$, and any $t \in [T_0, \tau]$, define $z_n(t) := \int_{T_0}^t v_n(s) ds$. Then z_n is absolutely continuous on $[T_0, \tau]$. By virtue of (b) and (2.15), for $T_0 \leq r \leq t \leq \tau$, one has

$$||v_n(t)|| \leq (1 + s_0 + ||x_0||)(\varphi(t) + 1) \quad \text{and} \tag{2.16}$$

$$
\begin{aligned}
(2.17) \qquad ||z_n(t) - z_n(r)|| &\leq (1 + s_0 + ||x_0||) \int_r^t \varphi(\Delta_n(s))ds \\
&\leq (1 + s_0 + ||x_0||) \int_r^t (\varphi(s) + 1)ds
\end{aligned}
$$

so that (z_n) is equicontinuous in $\mathcal{C}([T_0, \tau], H)$.

Furthermore, since K is convex with $0 \in K$, it follows from (a) (2.10) and (2.15) that

$$\forall n \in \mathbf{N},\ \forall t \in [T_0, \tau],\ v_n(t) \in (1 + s_0 + ||x_0||)(\varphi(t) + 1)K.$$

As K is closed and convex, this yields that for all $n \geq 1$, and $t \in [T_0, \tau]$

$$z_n(t) \in [(1 + s_0 + ||x_0||) \int_{T_0}^t (\varphi(s) + 1)ds]K$$

and once more, as K is convex with $0 \in K$, we deduce that for any $t \in [T_0, \tau]$, $\{z_n(t), n \in \mathbf{N}\}$ is a subset of the strongly compact set $[(1 + s_0 + ||x_0||) \int_{T_0}^{\tau} (\varphi(s) + 1)ds]K$.

Hence Ascoli's theorem ensures that, up to a subsequence, (z_n) converges uniformly on $[T_0, \tau]$ to some continuous mapping $z(.)$. Further, (2.13) and (2.15) ensure that

$$(2.18) \qquad \sup_{n \in \mathbf{N}} ||\dot{w}_n||_{L^2([T_0,\tau];H)} < +\infty.$$

Now making use of the pln property of f at x_0, we will show that the corresponding subsequence (w_n) converges uniformly to some local solution of the differential inclusion under consideration. For any $n \in \mathbf{N}$, and any $t \in [T_0, \tau]$, define $X_n(t) := w_n(t) + z_n(t)$, which is clearly absolutely continuous. We denote by $\mathcal{N}$ the Lebesgue null subset of $[T_0, \tau]$ out of which the inclusion (d) holds for any $n \in \mathbf{N}$. Then, by (d), for any fixed, $n, p \in \mathbf{N}$ and $t \in [T_0, \tau] \setminus \mathcal{N}$, one has

$$-\dot{X}_n(t) = -\dot{w}_n(t) - v_n(t) \in \partial f(w_n(t))$$

and

$$-\dot{X}_p(t) = -\dot{w}_p(t) - v_p(t) \in \partial f(w_p(t))$$

with $\{w_n(t), w_p(t)\} \subset B(x_0, s_0)$. Therefore the pln property of f at x_0 yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}||X_n(t) - X_p(t)||^2 = \langle \dot{X}_n(t) - \dot{X}_p(t), X_n(t) - X_p(t) \rangle \\
&= \langle \dot{X}_n(t) - \dot{X}_p(t), w_n(t) - w_p(t) \rangle + \langle \dot{X}_n(t) - \dot{X}_p(t), z_n(t) - z_p(t) \rangle \\
&\leq (Q_0 + c_0^{-1}(||\dot{X}_n(t)|| + ||\dot{X}_p(t)||))||w_n(t) - w_p(t)||^2 \\
&\quad + \langle \dot{X}_n(t) - \dot{X}_p(t), z_n(t) - z_p(t) \rangle \\
&\leq \langle \dot{X}_n(t) - \dot{X}_p(t), z_n(t) - z_p(t) \rangle \\
&\quad + 2(Q_0 + c_0^{-1}(||\dot{X}_n(t)|| + ||\dot{X}_p(t)||))||z_n(t) - z_p(t)||^2 \\
&\quad + 2(Q_0 + c_0^{-1}(||\dot{X}_n(t)|| + ||\dot{X}_p(t)||))||X_n(t) - X_p(t)||^2.
\end{aligned}
$$

Thus, applying Gronwall's lemma, for all $t \in [T_0, \tau]$, one obtains

$$||X_n(t) - X_p(t)||^2 \leq \int_{T_0}^t a(s) \exp(\int_s^t b(r)dr)ds \tag{2.19}$$

where for a.e. $s \in [T_0, \tau]$,

$$a(s) = \langle \dot{X}_n(s) - \dot{X}_p(s), z_n(s) - z_p(s) \rangle + 2(Q_0 + c_0^{-1}(||\dot{X}_n(s)|| + ||\dot{X}_p(s)||))||z_n(s) - z_p(s)||^2$$

and

$$b(s) = 2(Q_0 + c_0^{-1}(||\dot{X}_n(s)|| + ||\dot{X}_p(s)||)).$$

Now deducing that, by (2.18) $(\dot{w}_n)$ is bounded in $L^2([T_0, \tau]; H)$ and, since via (b)

$$\sup_{n \in \mathbf{N}} ||\dot{z}_n||_{L^2([T_0,\tau];H)} \leq (1 + s_0 + ||x_0||) \int_{T_0}^\tau (\varphi(s) + 1)ds < +\infty,$$

we conclude that

$$S := \sup_{n \in \mathbf{N}} ||\dot{X}_n||_{L^2([T_0,\tau];H)} < +\infty.$$

Then, it follows from (2.19) that

$$\sup_{t \in [T_0,\tau]} ||X_n(t) - X_p(t)||^2 \leq 2||z_n - z_p||_\infty (S + ||z_n - z_p||_\infty (Q_0(\tau - T_0) + 2c_0^{-1} S) \exp(2(Q_0(\tau - T_0) + 2c_0^{-1} S)).$$

Hence (X_n) is a uniform Cauchy sequence in $\mathcal{C}([T_0, \tau], H)$. So (X_n) converges uniformly on $[T_0, \tau]$ to some $X \in \mathcal{C}([T_0, \tau]; H)$, and $(w_n) = (X_n - z_n)$ converges uniformly on $[T_0, \tau]$ to some continuous mapping $w(\cdot)$ from $[T_0, \tau]$ into $B_H(x_0, s_0))$ with $w(T_0) = x_0$ using (c). Moreover, $w(\cdot)$ is absolutely continuous, using the boundedness of $(\dot{w}_n)$ in $L^2([T_0, \tau]; H)$. Furthermore, in view of (2.12), for all $t \in [T_0, \tau]$ and for all $n \in \mathbf{N}$,

$$f(w_n(t)) \leq f(x_0) + \frac{1}{2}(1 + s_0 + ||x_0||)^2 \int_{T_0}^\tau (\varphi(s) + 1)^2 ds$$

which implies that $w([T_0, \tau]) \subset \text{dom } f$. We claim that

$$0 \in \dot{w}(t) + \partial f(w(t)) + F(t, w(t)) \text{ for a.e. } t \in [T_0, \tau]. \tag{2.20}$$

Recall that $-\dot{X}_n(t) \in \partial f(w_n(t))$ and $v_n(t) \in F(\Delta_n(t), w_n(\theta_n(t)))$ for all $t \in [T_0, \tau] \setminus \mathcal{N}$ where $\lim_{n\to\infty} \max\{|\Delta_n(t) - t|; |\theta_n(t) - t|\} = 0$ and

$$\sup_{n \in \mathbf{N}} ||v_n||^2_{L^2([T_0,\tau];H)} \leq (1 + s_0 + ||x_0||)^2 \int_{T_0}^\tau (\varphi(s) + 1)^2 ds < +\infty.$$

We may assume that (v_n) and $(\dot{w}_n)$ converge weakly in $L^2([T_0, \tau]); H)$ to v and $\dot{w}$ respectively. Then, the corresponding subsequence $(\dot{X}_n)$ converges weakly in $L^2([T_0, \tau]; H)$ to $v + \dot{w}$. From the inclusion

$$-\dot{w}_n(t) - v_n(t) \in \partial f(w_n(t)) \text{ for a.e. } t \in [T_0, \tau]$$

and the preceding convergences results, invoking the closure lemma in ([15], Lemma 3.1.9), we conclude that

$$-\dot{w}(t) - v(t) \in \partial f(w(t)) \text{ for a.e. } t \in [T_0, \tau]. \tag{2.21}$$

It remains to show that

$$v(t) \in F(t, w(t)) \text{ for a.e. } t \in [T_0, \tau].$$

Indeed, by construction we have

$$v_n(t) \in F(\Delta_n(t), w_n(\theta_n(t))) \text{ for a.e. } t \in [T_0, \tau].$$

As $(\Delta_n(t), w_n(\theta_n(t)))$ pointwisely converges to $(t, w(t))$ and (v_n) weakly converges in $L^2([T_0, \tau]; H)$ to v, and F is scalarly upper semicontinuous on $[T_0, \tau] \times H$, invoking the closure lemma in ([7], Theoreme VI-4), we get the required inclusion. Combining with (2.21), we conclude that w is an absolutely continuous solution of

$$0 \in \dot{w}(t) + \partial f(w(t)) + F(t, w(t)) \text{ for a.e. } t \in [T_0, \tau];\ w(T_0) = x_0$$

and is a local solution of $(\mathcal{I}_F)$.

As an estimation on the velocity, let us underline that, letting $n \to +\infty$ in (2.13) yields

$$\int_r^t ||\dot{w}(s)||^2 ds \leq 2(f(x_0) + \alpha(1 + ||w(t)||)) + \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||w(s)||)^2 ds$$

for any $r, t \in [T_0, \tau], r \leq t$. Similarly, passing to the limit when $n \to +\infty$ in (2.12), we get the estimate

$$f(w(t)) \leq f(x_0) + \frac{1}{2} \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||w(s)||)^2 ds$$

for any $t \in [T_0, \tau]$.

B) Now we prove the existence of a global solution for $(\mathcal{I}_F)$ by using some arguments given Theorem 2.3.

Denote by $u : [T_0, \theta[\to H$ with $\theta \leq +\infty$, the *maximal* locally absolutely continuous solution of the inclusion

$$\begin{cases} 0 \in \dot{u}(t) + \partial f(u(t)) + F(t, u(t)), \text{ a.e. } t \in [T_0, \theta[\\ u(T_0) = x_0 \\ u([T_0, \theta[) \subset \text{dom } f \end{cases}$$

for which

(i) $f(u(t)) \leq f(x_0) + \frac{1}{2} \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||u(s)||)^2 ds$

(ii) $\int_r^t ||\dot{u}(s)||^2 ds \leq 2(f(x_0) - f(u(t))) + \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||u(s)||)^2 ds$ for any $r, t \in [T_0, \theta[, r \leq t$.

Our aim is to show that $\theta = +\infty$. First, let us observe a few facts. Fix any $t \in [T_0, \theta[$. By virtue of (H_1) and (ii), one has

$$\int_{T_0}^t ||\dot{u}(s)||^2 ds \leq 2(f(x_0) + \alpha(1 + ||u(t)||)) + \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||u(s)||)^2 ds \tag{2.22}$$

and hence

$$||u(t) - x_0||^2 \leq 2(t - T_0)[f(x_0) + \alpha + \int_{T_0}^{t} (\varphi(s) + 1)^2 ds] \\ + 2(t - T_0)[\alpha||u(t)|| + \int_{T_0}^{t} (\varphi(s) + 1)^2 ||u(s)||^2 ds].$$

This implies that

$$||u(t)||^2 - 4\alpha(t - T_0)||u(t)|| \\ \leq 2||x_0||^2 + 4(t - T_0)[f(x_0) + \alpha + \int_{T_0}^{t} (\varphi(s) + 1)^2 ds + \int_{T_0}^{t} (\varphi(s) + 1)^2 ||u(s)||^2 ds].$$

Then we deduce that

$$||u(t)|| \leq 4\alpha(t - T_0) \\ + 2[2||x_0||^2 + 4(t - T_0)(f(x_0) + \alpha + \int_{T_0}^{t} (\varphi(s) + 1)^2 ds + \int_{T_0}^{t} (\varphi(s) + 1)^2 ||u(s)||^2 ds)]^{\frac{1}{2}}$$

and hence

$$||u(t)||^2 \leq 8(4\alpha^2(t - T_0)^2 + 2||x_0||^2 + 4(t - T_0)(f(x_0) + \alpha + \int_{T_0}^{t} (\varphi(s) + 1)^2 ds)) \\ + 32(t - T_0) \int_{T_0}^{t} (\varphi(s) + 1)^2 ||u(s)||^2 ds.$$

Thus, applying Gronwall's inequality yields

$$(2.23) \quad ||u(t)||^2 \\ \leq a(t) + 32(t - T_0) \int_{T_0}^{t} a(s)(\varphi(s) + 1)^2 \exp(32 \int_{s}^{t} (\varphi(r) + 1)^2 (r - T_0) dr) ds$$

here

$$a(t) := 8[4\alpha^2(t - T_0)^2 + 2||x_0||^2 + 4(t - T_0)(f(x_0) + \alpha + \int_{T_0}^{t} (\varphi(s) + 1)^2 ds)]$$

for each $t \in [T_0, \theta[$.

Now, to show that $\theta = +\infty$, we proceed by contradiction. Assume that $\theta < +\infty$. Then we easily deduce from preceding estimate that

$$(2.24) \qquad M_\theta := \sup_{t \in [T_0, \theta[} ||u(t)|| < +\infty.$$

Then, by (2.23) and (2.24), for any $r, t \in [T_0, \theta[$ with $r \leq t$,

$$||u(t) - u(r)|| \leq (t - r)^{\frac{1}{2}} [2(f(x_0) + \alpha(1 + M_\theta)) + (1 + M_\theta)^2 \int_{T_0}^{\theta} (\varphi(s) + 1)^2 ds]^{\frac{1}{2}}$$

which implies, by Cauchy's criterion that $\overline{u} := \lim_{t \uparrow \theta} u(t)$ exists in $(H, ||.||)$. As

$$\forall t \in [T_0, \theta[, \; f(u(t)) \leq f(x_0) + \frac{1}{2}(1 + M_\theta)^2 \int_{T_0}^{\theta} (\varphi(s) + 1)^2 ds$$

in view of (i), the lower semicontinuity of f ensures that $\overline{u} \in \text{dom } f$ and hence f is pln at $\overline{u}$. Considering θ as initial time and $\overline{u}$ as initial value, under our assumptions, the local existence step A) above guarantees that there exist $\delta > 0$ and an absolutely continuous mapping $y : [\theta, \theta + \delta] \to H$ satisfying

$$\begin{cases} 0 \in \dot{y}(t) + \partial f(y(t)) + F(t, y(t)) \text{ a.e. } t \in [\theta, \theta + \delta] \\ y(\theta) = \overline{u} \\ y([\theta, \theta + \delta]) \subset \text{dom } f \end{cases}$$

and for any $r, t \in [\theta, \theta + \delta]$, $r \leq t$,

$$\int_r^t ||\dot{y}(s)||^2 ds \leq 2(f(\overline{u}) - f(y(t)) + \int_\theta^t (\varphi(s) + 1)^2 (1 + ||y(s)||)^2 ds,$$
$$f(y(t)) \leq f(\overline{u}) + \frac{1}{2} \int_\theta^t (\varphi(s) + 1)^2 (1 + ||y(s)||)^2 ds.$$

As a result, defining $\tilde{u} : [T_0, \theta + \delta] \to H$ by

$$\tilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [T_0, \theta[\\ y(t) & \text{if } t \in [\theta, \theta + \delta] \end{cases}$$

we see that $\tilde{u}$ is absolutely continuous on $[T_0, \theta + \delta]$ and one has

$$\begin{cases} 0 \in \dot{\tilde{u}}(t) + \partial f(\tilde{u}(t)) + +F(t, \tilde{u}(t)) \text{ a.e. } t \in [T_0, \theta + \delta] \\ \tilde{u}(T_0) = x_0 \end{cases}$$

along with

$$f(\tilde{u}(t)) \leq f(x_0) + \frac{1}{2} \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||\tilde{u}(s)||)^2 ds \quad \text{and}$$
$$\int_r^t ||\dot{\tilde{u}}(s)||^2 ds \leq 2(f(x_0) - f(\tilde{u}(t))) + \int_{T_0}^t (\varphi(s) + 1)^2 (1 + ||\tilde{u}(s)||)^2 ds$$

for all $r, t \in [T_0, \theta + \delta]$, $r \leq t$. Thus $\tilde{u}(.)$ is a continuation of $u(.)$ on $[\theta, \theta + \delta]$ which contradicts the maximality of $u(.)$. Then $\theta = +\infty$ and $u(.)$ is an expected global solution of $(\mathcal{I}_{\partial f, F})$ on $[T_0, +\infty[$. $\square$

Remark. We conjecture that Theorem 2.4 holds true if we remplace the growth condition

$$F(t, x) \subset \varphi(t)(1 + ||x||)K$$

by a more general condition. Namely

$$F(t, x) \subset (1 + ||x||)\Gamma(t)$$

where $\Gamma(.) : [T_0, +\infty[\rightrightarrows H$ is a nonempty convex compact-valued L^2-integrably bounded multifonction, that is, the function $|\Gamma| : t \mapsto \max\{||x|| : x \in \Gamma(t)\}$ is $L^2([T_0, +\infty[$-integrable.

3. Applications to control and viscosity problems

Let Y (resp. Z) be two compact metric spaces. Let $\mathcal{M}^1_+(Y)$ (resp. $\mathcal{M}^1_+(Z)$) be the compact metrizable space of the set of all probability Radon measures on Y (resp. Z) endowed with the vague topology. Let $\mathcal{Y}$ (resp. $\mathcal{Z}$) be the set of all Lebesgue-measurable mappings (alias Young measures) from $[0,T]$ to $\mathcal{M}^1_+(Y)$ (resp. $\mathcal{M}^1_+(Z)$). A sequence (μ^n) (resp. (ν^n)) in $\mathcal{Y}$ (resp. $\mathcal{Z}$) *stably* converges to $\mu \in \mathcal{Y}$ (resp. $\nu \in \mathcal{Z}$), if

$$\lim_n \int_0^T \langle \mu^n_t, f_t\rangle dt = \int_0^T \langle \mu_t, f_t\rangle dt$$

for any L^1-bounded Carathéodory integrand f defined on $[0,T]\times Y$ (resp.

$$\lim_n \int_0^T \langle \nu^n_t, g_t\rangle dt = \int_0^T \langle \nu_t, g_t\rangle dt$$

for any L^1-bounded Carathéodory integrand g defined on $[0,T]\times Z$), that is $t\mapsto f_t$ and $t\mapsto g_t$ belong to $L^1([0,T];\mathcal{C}(Y))$ and $L^1([0,T];\mathcal{C}(Z))$ respectively. Recall that $\mathcal{Y}$ (resp. $\mathcal{Z}$) is a compact metrizable space for the stable convergence. For more on Young measures, we refer to ([1], [5]). As an application of the preceding results, we state first some viscosity results for an evolution inclusion governed by the subdifferential of a lispchitzean pln function where the controls are Young measures.

Suppose that $H = \mathbf{R}^d$ and let $f : \mathbf{R}^d \to \mathbf{R}$ be a Lipschitz continuous function that is pln on each closed ball centered at the origin with the same constants. Assume further that:

(H_1) $g : [0,T]\times H\times Y\times Z \to H$ is bounded, continuous, uniformly Lipschitz continuous with respect to its second variable,

(H_2) $J : [0,T]\times H\times Y\times Z \to \mathbf{R}$ is bounded and continuous.

Let V_J denote the associated value function defined on $[0,T]\times H$

$$V_J(\tau,x) := \sup_{\nu\in\mathcal{Z}} \inf_{\mu\in\mathcal{Y}} \{\int_\tau^T [\int_Z [\int_Y J(t,u_{x,\mu,\nu}(t),y,z)\,\mu_t(dy)]\,\nu_t(dz)]\,dt\},$$

where $u_{x,\mu,\nu}$ is the unique absolutely continuous solution of the inclusion

$$\begin{cases} \dot u_{x,\mu,\nu}(t) \in -\partial f(u_{x,\mu,\nu}(t)) + \int_Z[\int_Y g(t,u_{x,\mu,\nu}(t),y,z)\mu_t(dy)]\nu_t(dz) \text{ a.e. } [\tau,T]\\ u_{x,\mu,\nu}(\tau) = x \in \operatorname{dom} f.\end{cases}$$

Before going further we recall and summarize the three following results which are the key ingredients of our study.

Theorem 3.1. *Under the preceding assumptions, for each $x_0 \in dom\ f = \mathbf{R}^d$ and for each $(\mu,\nu)\in\mathcal{Y}\times\mathcal{Z}$,*

a) *there is a unique absolutely continuous solution $u_{x_0,\mu,\nu}$ of*

$$\begin{cases} \dot u_{x_0,\mu,\nu}(t) \in -\partial f(u_{x_0,\mu,\nu}(t)) + \int_Z[\int_Y g(t,u_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)]\,\nu_t(dz)\\ \text{for a.e. } t\in[0,T],\\ u_{x_0,\mu,\nu}(0) = x_0 \in dom\ f.\end{cases}$$

Furthermore, there is a constant $M > 0$ which is independent of (μ, ν) such that $||u_{x_0,\mu,\nu}(t) - u_{x_0,\mu,\nu}(s)|| \leq (t-s)^{\frac{1}{2}} M$ for all $s \leq t \in [0, T]$.

b) *If (t^n) is a sequence in $[0, T]$ converging to t^∞, (ν^n) is a sequence in $\mathcal{Z}$ converging stably to $\nu^\infty \in \mathcal{Z}$ and $u_{x_0,\mu,\nu^n}(n \in \mathbf{N} \cup \{\infty\})$ is the absolutely continuous solution of*

$$\begin{cases} \dot{u}_{x_0,\mu,\nu^n}(t) \in -\partial f(u_{x_0,\mu,\nu^n}(t)) + \int_Z [\int_Y g(t, u_{x_0,\mu,\nu^n}(t), y, z)\, \mu(dy)] \nu_t^n(dz) \\ \text{for a.e. } t \in [0, T], \\ u_{x_0,\mu,\nu^n}(0) = x_0 \end{cases}$$

then one has

$$\lim_{n \to \infty} ||u_{x_0,\mu,\nu^n}(t^n) - u_{x_0,\mu,\nu^\infty}(t^\infty)|| = 0.$$

Proof. See ([15], Theorem 5.2.1-5.2.3). Actually, a) can be deduced from Theorem 2.2 or 2.3 and the hypomononicity of ∂f. b) is proved in Theorem 5.2.3 in [15]. □

Lemma 3.1. *Let $(t_0, x_0) \in [0, T] \times dom f$. Assume that $\Lambda_1 : [0, T] \times H \times \mathcal{M}_+^1(Y) \times \mathcal{M}_+^1(Z) \to \mathbf{R}$ is continuous and $\Lambda_2 : [0, T] \times H \times \mathcal{M}_+^1(Z) \to \mathbf{R}$ is upper semicontinuous such that, for any bounded subset B of H, $\Lambda_2|_{[0,T] \times B \times \mathcal{M}_+^1(Z)}$ is bounded, and assume that $\Lambda := \Lambda_1 + \Lambda_2$ satisfies the following condition*

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0 \text{ for some } \eta > 0.$$

Further, let $V : [0, T] \times H \to \mathbf{R}$ be a continuous function such that V reaches a local maximum at (t_0, x_0). Then there exist $\overline{\mu} \in \mathcal{M}_+^1(Y)$ and $\sigma > 0$ such that

$$\sup_{\nu \in \mathcal{Z}} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t) dt < -\sigma\eta/2, \tag{3.1}$$

where $u_{x_0,\overline{\mu},\nu}$ denotes the unique absolutely continuous solution of

$$\begin{cases} \dot{u}_{x_0,\overline{\mu},\nu}(t) \in -\partial f(u_{x_0,\overline{\mu},\nu}(t)) + \int_Z [\int_Y g(t, u_{x_0,\overline{\mu},\nu}(t), y, z)\, \overline{\mu}(dy)]\, \nu_t(dz) \\ \text{for a.e. } t \in [t_0, T] \\ u_{x_0,\overline{\mu},\nu}(t_0) = x_0 \end{cases}$$

associated with the controls $(\overline{\mu}, \nu) \in \mathcal{M}_+^1(Y) \times \mathcal{Z}$, and such that

$$V(t_0, x_0) \geq V(t_0 + \sigma, u_{x_0,\overline{\mu},\nu}(t_0 + \sigma)) \tag{3.2}$$

for all $\nu \in \mathcal{Z}$.

Proof. By hypothesis we have

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0,$$

that is,

$$\min_{\mu \in \mathcal{M}_+^1(Y)} \max_{\nu \in \mathcal{M}_+^1(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)] < -\eta < 0.$$

As the function Λ_1 is continuous, so is the function

$$\mu \mapsto \max_{\nu \in \mathcal{M}_+^1(Z)} [\Lambda_1(t_0, x_0, \mu, \nu) + \Lambda_2(t_0, x_0, \nu)].$$

Hence there exists $\overline{\mu} \in \mathcal{M}^1_+(Y)$ such that

$$\max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \overline{\mu}, \nu) = \min_{\mu \in \mathcal{M}^1_+(Y)} \max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu, \nu) < -\eta < 0.$$

As the function $(t, x, \nu) \mapsto \Lambda_1(t, x, \overline{\mu}, \nu)$ is continuous and the function $(t, x, \nu) \mapsto \Lambda_2(t, x, \nu)$ is upper semicontinuous, $(t, x, \nu) \mapsto \Lambda_1(t, x, \overline{\mu}, \nu) + \Lambda_2(t, x, \nu)$ is upper semicontinuous, so is the function

$$(t, x) \mapsto \max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t, x, \overline{\mu}, \nu).$$

Hence there is $\zeta > 0$ such that

$$\max_{\nu \in \mathcal{M}^1_+(Z)} \Lambda(t, x, \overline{\mu}, \nu) < -\eta/2,$$

for $0 < t - t_0 \leq \zeta$ and $||x - x_0|| \leq \zeta$. We assert that there is $\theta > 0$ such that

$$V(t_0, x_0) \geq V(t_0 + s, u_{x_0,\overline{\mu},\nu}(t_0 + s))$$

for all $s \in]0, \theta]$ and for all $\nu \in \mathcal{Z}$. This fact needs a subtle argument due to P. Raynaud de Fitte using both the *continuity* of $(t, \nu) \mapsto u_{x_0,\overline{\mu},\nu}$ and the *compactness* of $\mathcal{Z}$. Indeed, since V has a local maximum at (t_0, x_0), for δ and $r > 0$ small enough (we can always decrease δ), we have

$$V(t_0, x_0) \geq V(t_0 + s, x)$$

for every $s \geq 0$ such that $s \leq \delta$ and for every $x \in H$ such that $\|x - x_0\| \leq r$. From the continuity of $(t, \nu) \mapsto u_{x_0,\overline{\mu},\nu}(t)$, we can find for each $\nu \in \mathcal{Z}$ an open neighborhood V_ν of ν in $\mathcal{Z}$ and $\theta_\nu \in]0, \delta]$ such that, for all $(s, \nu') \in [0, \theta_\nu[\times V_\nu$, $\|u_{x_0,\overline{\mu},\nu'}(t_0 + s) - x_0\| \leq r$. By compactness of $\mathcal{Z}$, we can find a finite family $\nu^1, \ldots, \nu^n$ such that $\mathcal{Z} = \cup_{j=1}^n V_{\nu^j}$. The assertion is then proved by taking $\theta = \min\{\theta_{\nu^j} : 1 \leq j \leq n\}$. Let us recall that

$$||u_{x_0,\overline{\mu},\nu}(t) - u_{x_0,\overline{\mu},\nu}(s)|| \leq (t - s)^{\frac{1}{2}} M$$

for all $t_0 \leq s \leq t \leq T$, where M is a positive constant independent of $(\mu, \nu) \in \mathcal{Y} \times \mathcal{Z}$. Let us choose $0 < \sigma \leq \min\{\theta, \zeta, (\frac{\zeta}{M})^2\}$, hence we get

$$||u_{x_0,\overline{\mu},\nu}(t) - u_{x_0,\overline{\mu},\nu}(t_0)|| \leq \zeta,$$

for all $t \in [t_0, t_0 + \sigma]$ and for all $\nu \in \mathcal{Z}$, so that the first estimate (3.1) follows by integration of $t \mapsto \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t)$ on $[t_0, t_0 + \sigma]$

$$\begin{aligned}\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu_t) dt &\leq \int_{t_0}^{t_0+\sigma} [\max_{\nu' \in \mathcal{M}^1_+(Z)} \Lambda(t, u_{x_0,\overline{\mu},\nu}(t), \overline{\mu}, \nu')] dt \\ &< -\sigma\eta/2 < 0,\end{aligned}$$

for all $\nu \in \mathcal{Z}$, while the second estimate (3.2) follows by the choice of σ. $\square$

Other variants of the preceding result are in ([3], [9], [4], [5], [15]. The preceding proof is borrowed from ([9], Lemma 2.3). The following is the dynamic programming theorem for the evolution problem under consideration.

Theorem 3.2 (of dynamic programming). *Let $(\tau, x) \in [0,T] \times dom\ f$ and $\sigma > 0$ such that $\tau + \sigma < T$. Then one has*

$$V_J(\tau, x) = \sup_{\nu\in\mathcal{Z}} \inf_{\mu\in\mathcal{Y}} \{ \int_\tau^{\tau+\sigma} [\int_Z [\int_Y J(t, u_{x,\mu,\nu}(t), y, z)\, \mu_t(dy)\, \nu_t(dz)]\, dt + V_J(\tau+\sigma, u_{x,\mu,\nu}(\tau+\sigma))\},$$

where

$$V_J(\tau+\sigma, u_{x,\mu,\nu}(\tau+\sigma)) = \sup_{\gamma\in\mathcal{Z}} \inf_{\beta\in\mathcal{Y}} \int_{\tau+\sigma}^T \int_Z \int_Y J(t, v_{x,\beta,\gamma}(t), y, z)\, \beta_t(dy)\, \gamma_t(dz) dt,$$

where $v_{x,\beta,\gamma}$ denotes the trajectory solution of the evolution inclusion

$$\dot v_{x,\beta,\gamma}(t) \in -\partial f(v_{x,\beta,\gamma}(t)) + \int_Z \int_Y g(t, v_{x,\beta,\gamma}(t), y, z) \beta_t(dy) \gamma_t(dz) \quad a.e.\ in\ [\tau+\sigma, T]$$

associated with the controls $(\beta, \gamma) \in \mathcal{Z} \times \mathcal{Z}$ with initial condition $v_{x,\beta,\gamma}(\tau+\sigma) = u_{x,\mu,\nu}(\tau+\sigma)$.

Theorem 3.3 (Existence of viscosity subsolutions). *Under the above assumptions, let V_J denote the associated value function defined on $[0,T] \times \mathbf{R}^d$*

$$V_J(\tau, x) := \sup_{\nu\in\mathcal{Z}} \inf_{\mu\in\mathcal{Y}} \{ \int_\tau^T [\int_Z [\int_Y J(t, u_{x,\mu,\nu}(t), y, z)\, \mu_t(dy)]\, \nu_t(dz)]\, dt\},$$

where $u_{x,\mu,\nu}$ is the unique absolutely continuous solution of the inclusion

$$\begin{cases} \dot u_{x,\mu,\nu}(t) \in -\partial f(u_{x,\mu,\nu}(t)) + \int_Z \int_Y g(t, u_{x,\mu,\nu}(t), y, z) \mu_t(dy) \nu_t(dz) \ a.e.\ in\ [\tau, T] \\ u_{x,\mu,\nu}(\tau) = x \in dom\ f. \end{cases}$$

Let H be the Hamiltonian on $[0,T] \times \mathbf{R}^d \times \mathbf{R}^d$ given by

$$H(t, x, \rho) = \inf_{\mu\in\mathcal{M}^1_+(Y)} \sup_{\nu\in\mathcal{M}^1_+(Z)} \{\langle \rho, \int_Z [\int_Y g(t, x, y, z)\, \mu(dy)]\, \nu(dz)\rangle + \int_Z [\int_Y J(t, x, y, z)\ \mu(dy)] \mu(dz)\} + \delta^*(\rho, -\partial f(x)),$$

here $\delta^*(\rho, -\partial f(x))$ denotes the support function of the upper semicontinuous convex compact valued mapping $x \rightrightarrows -\partial f(x)$. Then, V_J is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(t, x) + H(t, x, \nabla V(t, x)) = 0,$$

that is to say: for any $\varphi \in \mathcal{C}^1([0,T] \times \mathbf{R}^d)$ such that $V_J - \varphi$ reaches a local maximum at $(t_0, x_0) \in [0,T] \times \mathbf{R}^d$, one has

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla\varphi(t_0, x_0)) \geq 0.$$

Proof. Here, we adapt techniques from Castaing and al. [3], [5], [4], [9] and originally used in Evans-Souganidis [13], [12]. However this needs a careful look because we deal here with a new class of evolution inclusion involving Young measures. We assume by contradiction that there exist some $\varphi \in \mathcal{C}^1([0,T] \times \mathbf{R}^d)$ and a point $(t_0, x_0) \in [0,T] \times \operatorname{dom} f$ for which

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla\varphi(t_0, x_0)) < -\eta \text{ for some } \eta > 0.$$

By Proposition I.17 in [20], the convex compact valued mapping $x \in \mathbf{R}^d \Rightarrow \partial f(x)$ is upper semicontinuous, (∂ coinciding with the Clarke subdifferential operator because of the pln assumption on f). It follows that the function

$$(t,x) \in [0,T] \times \mathbf{R}^d \mapsto \Lambda_2(t,x) := \delta^*(\nabla\varphi(t,x), -\partial f(x))$$

is upper semicontinuous. Moreover, $\Lambda_2|_{[0,T]\times B}$ is bounded for any bounded subset B of $\mathbf{R}^d$, owing to the continuity of $\nabla\varphi(.,.)$ and the boundedness of $\bigcup_{x\in B} \partial f(x)$. On the other hand, under our assumptions, it is not difficult to see that the function $\Lambda_1 : [0,T] \times \mathbf{R}^d \times \mathcal{M}^1_+(Y) \times \mathcal{M}^1_+(Z) \to \mathbf{R}$ defined by

$$\begin{aligned}\Lambda_1(t,x,\mu,\nu) := &\int_Z [\int_Y J(t,x,y,z)\,\mu(dy)]\nu(dz)\\ &+ \langle \nabla\varphi(t,x), \int_Z [\int_Y g(t,x,y,z)\mu(dy)]\nu(dz)\rangle + \frac{\partial\varphi}{\partial t}(t,x)\end{aligned}$$

is continuous, $\mathcal{M}^1_+(Y)$ and $\mathcal{M}^1_+(Z)$ being endowed with the vague topology $\sigma(\mathcal{M}(Y), \mathcal{C}(Y))$ and $\sigma(\mathcal{M}(Z), \mathcal{C}(Z))$ respectively. Thus, we apply Lemma 3.1 to $\Lambda := \Lambda_1 + \Lambda_2$ and find $\overline{\mu} \in \mathcal{M}^1_+(Y)$ and $\sigma > 0$ *independent of* $\nu \in \mathcal{Z}$ such that

$$\begin{aligned}(3.3)\quad -\frac{\sigma\eta}{2} > \sup_{\nu\in\mathcal{Z}}\{&\int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\mu},\nu}(t))dt\\ &+ \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y J(t, u_{x_0,\overline{\mu},\nu}(t), y, z)\overline{\mu}(dy)]\nu_t(dz)]dt\\ &+ \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y \langle \nabla\varphi(t, u_{x_0,\overline{\mu},\nu}(t)), g(t, u_{x_0,\overline{\mu},\nu}(t), y, z)\rangle\overline{\mu}(dy)]\nu_t(dz)]dt\\ &+ \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\nu}(t)), -\partial f(u_{x_0,\overline{\mu},\nu}(t)))dt\}\end{aligned}$$

where $u_{x_0,\overline{\mu},\nu} : [\tau, T] \to \mathbf{R}^d$ is the absolutely continuous solution of the inclusion

$$\begin{cases}\dot{u}_{x_0,\overline{\mu},\nu}(t) \in -\partial f(u_{x_0,\overline{\mu},\nu}(t)) + \int_Z[\int_Y g(t, u_{x_0,\overline{\mu},\nu}(t), y, z)\overline{\mu}(dy)]\nu_t(dz)\\ \text{for a.e. } t \in [\tau, T]\\ u_{x_0,\overline{\mu},\nu}(\tau) = x_0\end{cases}$$

associated with the control $(\overline{\mu}, \nu) \in \mathcal{M}^1_+(Y) \times \mathcal{Z}$ and such that

$$(3.4)\quad V_J(t_0, x_0) - \varphi(t_0, x_0) \geq V_J(t_0+\sigma, u_{x_0,\overline{\mu},\nu}(t_0+\sigma)) - \varphi(t_0+\sigma, u_{x_0,\overline{\mu},\nu}(t_0+\sigma))$$

for all $\nu \in \mathcal{Z}$. Next, according to Theorem 3.2 of dynamic programming, we deduce that

$$V_J(t_0, x_0) \le \sup_{\nu\in\mathcal{Z}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t, u_{x_0,\overline{\mu},\nu}(t), z))\,\overline{\mu}(dy)]\,\nu_t(dz)]dt \\ + V_J(t_0+\sigma, u_{x_0,\overline{\mu},\nu}(t_0+\sigma))\}.$$

Now to finish the proof, we make use of an argument from ([8], Proposition 6.2). For each $n \in \mathbf{N}$, there is $\nu^n \in \mathcal{Z}$ such that

$$V_J(t_0, x_0) \le \int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\overline{\mu}(dy)]\nu_t^n(dz)]\,dt \\ + V_J(t_0+\sigma, u_{x_0,\overline{\mu},\nu^n}(t_0+\sigma)) + 1/n.$$

Therefore from (3.4) we deduce that

$$V_J(t_0+\sigma, u_{x_0,\overline{\mu},\nu^n}(t_0+\sigma)) - \varphi(t_0+\sigma, u_{x_0,\overline{\mu},\nu^n}(t_0+\sigma)) \\ \le \int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\overline{\mu}(dy)]\nu_t^n(dz)]\,dt + 1/n \\ - \varphi(t_0, x_0) + V_J(t_0+\sigma, u_{x_0,\overline{\mu},\nu^n}(t_0+\sigma)).$$

Consequently we get

$$0 \le \int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\overline{\mu}(dy)]\nu_t^n(dz)]\,dt \\ + \varphi(t_0+\sigma, u_{x_0,\overline{\mu},\nu^n}(t_0+\sigma)) - \varphi(t_0, x_0) + 1/n.$$

As φ is $\mathcal{C}^1$ and $u_{x_0,\overline{\mu},\nu^n}$ is the trajectory solution of our evolution inclusion

$$\varphi(t_0+\sigma, u_{x_0,\overline{\mu},\nu^n}(t_0+\sigma)) - \varphi(t_0, x_0) \\ \le \int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y \langle\nabla\varphi(t, u_{x_0,\overline{\mu},\nu^n}(t)), g(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\rangle\overline{\mu}(dy)]\nu_t^n(dz)]\,dt \\ + \int_{t_0}^{t_0+\sigma}\delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\nu^n}(t)), -\partial f(u_{x_0,\overline{\mu},\nu^n}(t)))\,dt \\ + \int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\mu},\nu^n}(t))\,dt.$$

For each n, we have

$$0 \le \int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\overline{\mu}(dy)]\nu_t^n(dz)]\,dt \\ + \int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y \langle\nabla\varphi(t, u_{x_0,\overline{\mu},\nu^n}(t)), g(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\rangle\overline{\mu}(dy)]\nu_t^n(dz)]\,dt \\ + \int_{t_0}^{t_0+\sigma}\delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\nu^n}(t)), -\partial f(u_{x_0,\overline{\mu},\nu^n}(t)))dt \\ + \int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\mu},\nu^n}(t))dt + 1/n. \tag{3.5}$$

As $\mathcal{Z}$ is compact metrizable for the stable topology, we may assume that (ν^n) stably converges to a Young measure $\overline{\nu} \in \mathcal{Z}$. This implies that $u_{x_0,\overline{\mu},\nu^n}$ converges uniformly to $u_{x_0,\overline{\mu},\overline{\nu}}$ that is a trajectory solution of our dynamic

$$\begin{cases}\dot{u}_{x,\overline{\mu},\overline{\nu}}(t) \in -\partial f(u_{x,\overline{\mu},\overline{\nu}}(t)) + \int_Z [\int_Y g(t, u_{x_0,\overline{\mu},\overline{\nu}}(t), y, z)\overline{\mu}(dy)]\overline{\nu}_t(dz) \\ \text{for a.e } t \in [\tau, T] \\ u_{x_0,\overline{\mu},\overline{\nu}}(\tau) = x_0 \end{cases}$$

associated with the control $(\overline{\mu}, \overline{\nu}) \in \mathcal{M}^1_+(Y) \times \mathcal{Z}$ and $\delta_{u_{x_0,\overline{\mu},\nu^n}} \otimes \nu^n$ stably converges to $\delta_{u_{x_0,\overline{\mu},\overline{\nu}}} \otimes \overline{\nu}$ (see [4], [5], [3] for details). It follows that

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y J(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\overline{\mu}(dy)]\nu^n_t(dz)]\, dt \\ = \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y J(t, u_{x_0,\overline{\mu},\overline{\nu}}(t), y, z)\overline{\mu}(dy)]\overline{\nu}_t(dz)]\, dt,$$

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y \langle \nabla\varphi(t, u_{x_0,\overline{\mu},\nu^n}(t)), g(t, u_{x_0,\overline{\mu},\nu^n}(t), y, z)\rangle \overline{\mu}(dy)]\nu^n_t(dz)]\, dt \\ = \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y \langle \nabla\varphi(t, u_{x_0,\overline{\mu},\overline{\nu}}(t)), g(t, u_{x_0,\overline{\mu},\overline{\nu}}(t), y, z)\rangle \overline{\mu}(dy)]\overline{\nu}_t(dz)]\, dt.$$

Moreover

$$\limsup_{n\to\infty} \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\nu^n}(t)), -\partial f(u_{x_0,\overline{\mu},\nu^n}(t)))\, dt \\ \le \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\overline{\nu}}(t)), -\partial f(u_{x_0,\overline{\mu},\overline{\nu}}(t)))\, dt,$$

because

$$\limsup_{n\to\infty} \delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\nu^n}(t)), -\partial f(u_{x_0,\overline{\mu},\nu^n}(t))) \\ \le \delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\overline{\nu}}(t)), -\partial f(u_{x_0,\overline{\mu},\overline{\nu}}(t)))$$

and

$$\lim_{n\to\infty} \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\mu},\nu^n}(t))\, dt = \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\mu},\overline{\nu}}(t))\, dt.$$

Consequently by passing to the limit in (3.5) when $n \to \infty$ we get

$$0 \le \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y J(t, u_{x_0,\overline{\mu},\overline{\nu}}(t), y, z)\overline{\mu}(dy)]\overline{\nu}_t(dz)]\, dt \\ + \int_{t_0}^{t_0+\sigma} [\int_Z [\int_Y \langle \nabla\varphi(t, u_{x_0,\overline{\mu},\overline{\nu}}(t)), g(t, u_{x_0,\overline{\mu},\overline{\nu}}(t), y, z)\rangle \overline{\mu}(dy)]\overline{\nu}_t(dz)]\, dt \\ + \int_{t_0}^{t_0+\sigma} \delta^*(\nabla\varphi(t, u_{x_0,\overline{\mu},\overline{\nu}}(t)), -\partial f(u_{x_0,\overline{\mu},\overline{\nu}}(t)))\, dt \\ + \int_{t_0}^{t_0+\sigma} \frac{\partial\varphi}{\partial t}(t, u_{x_0,\overline{\mu},\overline{\nu}}(t))\, dt.$$

This contradicts (3.3) and the proof is therefore complete. $\square$

Now we examine the superviscosity property of the value function V_J by adding some extra conditions on f, g, J and on the first space of Young measure controls. Namely we assume

(H_1) $\mathcal{H}$ is a compact subset of $\mathcal{Y}$ for the convergence in probability, in particular $\mathcal{H}$ is compact for the stable convergence (see e.g. [5]).

It is worth mentioning that (H_1) implies that the mapping $(\mu, \nu) \mapsto u_{x_0,\mu,\nu}$ is continuous on $\mathcal{H} \times \mathcal{Z}$ using the fiber product of Young measures [5] and the arguments of Theorem 5.1 in [5], along with Theorem 5.2.3 in [15].

(H_2) J and g are bounded and continuous with g uniformly lipschitzean on $H = \mathbf{R}^d$ (in the sequel), $(J(.,.,\mu,\nu))_{(\mu,\nu)\in\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$ (resp. $(g(.,.,\mu,\nu))_{(\mu,\nu)\in\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$), is equicontinuous on $[0, T] \times H$.

(H_3) $f : \mathbf{R}^d \to \mathbf{R}$ is Lipschitz continuous function that is pln on each closed ball centered at the origin with the same constants, and is C^1 on H so that $\partial f(x) = \{\nabla f(x)\}$ for any $x \in H$, see ([20], Prop. I-18).

Using (H_1)–(H_3) we have a variant of Lemma 3.1 which permits to state the desired superviscosity. Namely

Lemma 3.2. *Let $(t_0, x_0) \in [0, T] \times H$. Assume that $\Lambda : [0, T] \times H \times \mathcal{M}^1_+(Y) \times \mathcal{M}^1_+(Z) \to \mathbf{R}$ is continuous and the family $(\Lambda(.,.,\mu,\nu))$, $(\mu,\nu) \in \mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)$, is equicontinuous on $[0, T] \times H$ and assume that*

$$\min_{\mu\in\mathcal{M}^1_+(Y)} \max_{\nu\in\mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu, \nu) > \eta > 0 \text{ for some } \eta > 0.$$

Further, let $V : [0, T] \times H \to \mathbf{R}$ be a continuous function such that V reaches a local minimum at (t_0, x_0). Then, there exists $\sigma > 0$ such that for each $\mu \in \mathcal{H}$, we have

$$\sup_{\nu\in\mathcal{Z}} \int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\mu,\nu}(t), \mu_t, \nu_t)dt > \sigma\eta/2, \tag{3.6}$$

where $u_{x_0,\mu,\nu}$ denotes the unique absolutely continuous solution of

$$\begin{cases} \dot u_{x_0,\mu,\nu}(t) = -\nabla f(u_{x_0,\mu,\nu}(t)) + \int_Z [\int_Y g(t, u_{x_0,\mu,\nu}(t), y, z)\, \mu_t(dy)]\, \nu_t(dz) \\ \textit{for a.e. } t \in [0, T] \\ u_{x_0,\mu,\nu}(t_0) = x_0, \end{cases}$$

associated with the controls $(\mu, \nu) \in \mathcal{H} \times \mathcal{Z}$, and such that

$$V(t_0, x_0) \le V(t_0 + \sigma, u_{x_0,\mu,\nu}(t_0 + \sigma)) \tag{3.7}$$

for all $(\mu, \nu) \in \mathcal{H} \times \mathcal{Z}$.

Proof. Since V has a local minimum at (t_0, x_0), there are $\theta > 0,\ r > 0$ such that

$$V(t_0, x_0) \le V(t, x) \text{ whenever } 0 < t - t_0 \le \theta \text{ and } x \in B(x_0, r).$$

By equicontinuity of the family $(\Lambda(.,.,\mu,\nu))_{(\mu,\nu)\in\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)}$ there is ζ with $0 < \zeta < r$ independent of (μ, ν) such that for all $t \in [t_0, t_0 + \zeta]$ and x with $||x - x_0|| \le \zeta$

$$\Lambda(t_0, x_0, \mu, \nu) - \frac{\eta}{2} < \Lambda(t, x, \mu, \nu)$$

for any $(\mu, \nu) \in \mathcal{M}^1_+(Y) \times \mathcal{M}^1_+(Z)$.

Now let μ be an arbitrary element in $\mathcal{H}$. Then there exists a Lebesgue-measurable mapping $\nu^\mu : [0,T] \to \mathcal{M}^1_+(Z)$ such that

$$\Lambda(t_0, x_0, \mu_t, \nu^\mu_t) = \max_{\nu' \in \mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu_t, \nu')$$

for all $t \in [0,T]$, because the nonempty compact-valued multifunction

$$t \to \{\nu \in \mathcal{M}^1_+(Z) : \Lambda(t_0, x_0, \mu_t, \nu) = \max_{\nu' \in \mathcal{M}^1_+(Z)} \Lambda(t_0, x_0, \mu_t, \nu')\}$$

has its graph in $\mathcal{L}([0,T]) \otimes \mathcal{B}(\mathcal{M}^1_+(Z))$. Let us recall that

$$||u_{x_0,\mu,\nu}(t) - u_{x_0,\mu,\nu}(s)|| \le (t-s)^{\frac{1}{2}} M$$

for all $t_0 \le s \le t \le T$, here M is a positive constant independent of $(\mu,\nu) \in \mathcal{Y} \times \mathcal{Z}$. Take $\sigma > 0$ such that $0 < \sigma \le \min\{\theta, (\frac{\zeta}{M})^2, \zeta\}$, we get

$$||u_{x_0,\mu,\nu}(t) - u_{x_0,\mu,\nu}(t_0)|| \le \zeta,$$

for all $t \in [t_0, t_0+\sigma]$ and for all $\nu \in \mathcal{Z}$. By integrating,

$$\int_{t_0}^{t_0+\sigma} \Lambda(t, u_{x_0,\mu,\nu^\mu}(t), \mu_t, \nu^\mu_t)dt \ge \int_{t_0}^{t_0+\sigma} [\Lambda(t_0, x_0, \mu_t, \nu^\mu_t) - \frac{\eta}{2}]dt > \int_{t_0}^{t_0+\sigma} \frac{\eta}{2} dt = \frac{\sigma\eta}{2}.$$

while (3.6) follows from the choice of σ. $\square$

Theorem 3.4 (Existence of viscosity supersolutions). *Under (H_1)–(H_3), let V_J denote the associated value function defined on $[0,T] \times H$*

$$V_J(\tau, x) := \sup_{\nu \in \mathcal{Z}} \inf_{\mu \in \mathcal{H}} \{ \int_\tau^T [\int_Z [\int_Y J(t, u_{x,\mu,\nu}(t), y, z)\, \mu_t(dy)]\, \nu_t(dz)]\, dt\},$$

where $u_{x,\mu,\nu}$ is the unique absolutely continuous solution of

$$\begin{cases} \dot u_{x,\mu,\nu}(t) = -\nabla f(u_{x,\mu,\nu}(t)) + \int_Z [\int_Y g(t, u_{x,\mu,\nu}(t), y, z) \mu_t(dy)] \nu_t(dz) \\ a.e\ in\ [\tau, T] \\ u_{x,\mu,\nu}(\tau) = x. \end{cases}$$

Let H be the Hamiltonian on $[0,T] \times \mathbf{R}^d \times \mathbf{R}^d$ given by

$$H(t,x,\rho) = \inf_{\mu \in \mathcal{M}^1_+(Y)} \sup_{\nu \in \mathcal{M}^1_+(Z)} \{\langle \rho, \int_Z [\int_Y g(t,x,y,z)\, \mu(dy)]\, \nu(dz)\rangle$$
$$+ \int_Z [\int_Y J(t,x,y,z)\ \mu(dy)] \nu(dz)\} + \langle \rho, -\nabla f(x)\rangle.$$

Then, V_J is a viscosity supersolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(t,x) + H(t, x, \nabla V(t,x)) = 0,$$

that is to say : for any $\varphi \in \mathcal{C}^1([0,T] \times \mathbf{R}^d)$ such that $V_J - \varphi$ reaches a local minimum at $(t_0, x_0) \in [0,T] \times \mathbf{R}^d$, one has

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + H(t_0, x_0, \nabla\varphi(t_0, x_0)) \le 0.$$

Proof. It is similar to the one of Theorem 3.2 with appropriate modifications. Assume by contradiction that there exist $\varphi \in \mathcal{C}^1_E([0,T]\times E)$ and a point $(t_0,x_0)\in[0,T]\times\mathbf{R}^d$ for which

$$\frac{\partial\varphi}{\partial t}(t_0,x_0)+H(t_0,x_0,\nabla\varphi(t_0,x_0))>\eta \tag{3.8}$$

for some $\eta>0$. Since $V_J-\varphi$ has a local minimum at (t_0,x_0), applying Lemma 3.2 to $V_J-\varphi$ and the integrand Λ defined by on $[0,T]\times\mathbf{R}^d\times\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)$ by

$$\begin{aligned}\Lambda(t,x,\mu,\nu)=\int_Z\int_Y J(t,x,y,z)\mu(dy)\nu(dz)+\frac{\partial\varphi}{\partial t}(t,x)+\langle\nabla\varphi(t,x),-\nabla f(x)\rangle\\ +\langle\nabla\varphi(t,x),\int_Z[\int_Y g(t,x,y,z)\mu(dy)]\nu(dz)\rangle\end{aligned}$$

for all $(t,x,\mu,\nu)\in[0,T]\times\mathbf{R}^d\times\mathcal{M}^1_+(Y)\times\mathcal{M}^1_+(Z)$ provides $\sigma>0$ such that

$$\begin{aligned}\sup_{\nu\in\mathcal{Z}}\min_{\mu\in\mathcal{H}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t,u_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)]\nu_t(dz)]\,dt\\ +\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y\langle\nabla\varphi(t,u_{x_0,\mu,\nu}(t)),g(t,u_{x_0,\mu,\nu}(t),y,z)\rangle\mu_t(dy)]\nu_t(dz)]\,dt\\ +\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\mu,\nu}(t))dt+\int_{t_0}^{t_0+\sigma}\langle\nabla\varphi(t,u_{x_0,\mu,\nu}(t)),-\nabla f(u_{x_0,\mu,\nu}(t))\rangle dt\}\geq\frac{\sigma\eta}{2}\end{aligned} \tag{3.9}$$

where $u_{x_0,\mu,\nu}$ is the trajectory solution associated with the control $(\mu,\nu)\in\mathcal{H}\times\mathcal{Z}$ of

$$\begin{cases}\dot u_{x_0,\mu,\nu}(t)=-\nabla f(u_{x_0,\mu,\nu}(t))+\int_Z[\int_Y g(t,u_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)]\nu_t(dz)\\ u_{x_0,\mu,\nu}(t_0)=x_0\end{cases}$$

and such that

$$V_J(t_0,x_0)-\varphi(t_0,x_0)\leq V_J(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))-\varphi(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma)) \tag{3.10}$$

for all $(\mu,\nu)\in\mathcal{H}\times\mathcal{Z}$.
From (3.10) and Theorem 3.2 of dynamic programming we have

$$\begin{aligned}\sup_{\nu\in\mathcal{Z}}\min_{\mu\in\mathcal{H}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t,u_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)]\nu_t(dz)]\,dt\\ +V_J(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))\}+\varphi(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))-\varphi(t_0,x_0)\\ -V_J(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))\leq 0.\end{aligned} \tag{3.11}$$

Let us choose $\overline{\mu}\in\mathcal{H}$ such that

$$\begin{aligned}\sup_{\nu\in\mathcal{Z}}\min_{\mu\in\mathcal{H}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t,u_{x_0,\mu,\nu}(t),y,z)\mu_t(dy)]\nu_t(dz)]\,dt\\ +V_J(t_0+\sigma,u_{x_0,\mu,\nu}(t_0+\sigma))\}\\ =\sup_{\nu\in\mathcal{Z}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t,u_{x_0,\overline{\mu},\nu}(t),y,z)\overline{\mu}_t(dy)]\nu_t(dz)]\,dt\\ +V_J(t_0+\sigma,u_{x_0,\overline{\mu},\nu}(t_0+\sigma))\}\end{aligned} \tag{3.12}$$

Coming back to (3.10) and (3.12) we deduce

(3.13)
$$\sup_{\nu\in\mathcal{Z}}\{\int_{t_0}^{t_0+\sigma}\int_Z\int_Y J(t,u_{x_0,\overline{\mu},\nu}(t),y,z)\overline{\mu}_t(dy)\nu_t(dz)dt+V_J(t_0+\sigma,u_{x_0,\overline{\mu},\nu}(t_0+\sigma))\}$$
$$+\sup_{\nu\in\mathcal{Z}}\{\varphi(t_0+\sigma,u_{x_0,\overline{\mu},\nu}(t_0+\sigma))-\varphi(t_0,x_0)-V_J(t_0+\sigma,u_{x_0,\overline{\mu},\nu}(t_0+\sigma))\}\le 0.$$

Hence we deduce

$$0\ge\sup_{\nu\in\mathcal{Z}}\{\int_{t_0}^{t_0+\sigma}\int_Z\int_Y J(t,u_{x_0,\overline{\mu},\nu}(t),y,z)\overline{\mu}_t(dy)\nu_t(dz)dt+\varphi(t_0+\sigma,u_{x_0,\overline{\mu},\nu}(t_0+\sigma))-\varphi(t_0,x_0)\}. \tag{3.14}$$

As φ is $\mathcal{C}^1$ and $u_{x_0,\overline{\mu},\nu}$ is the trajectory solution of our dynamic

$$\begin{aligned}&\varphi(t_0+\sigma,u_{x_0,\overline{\mu},\nu}(t_0+\sigma))-\varphi(t_0,x_0)\\&=\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y\langle\nabla\varphi(t,u_{x_0,\overline{\mu},\nu}(t)),g(t,u_{x_0,\overline{\mu},\nu}(t),y,z)\rangle\overline{\mu}_t(dy)]\nu_t(dz)]\,dt\\&+\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\overline{\mu},\nu}(t))\,dt+\int_{t_0}^{t_0+\sigma}\langle\nabla\varphi(t,u_{x_0,\overline{\mu},\nu}(t)),-\nabla f(u_{x_0,\overline{\mu},\nu}(t))\rangle dt.\end{aligned} \tag{3.15}$$

By substituting (3.15) in (3.14) we get

$$\begin{aligned}&\sup_{\nu\in\mathcal{Z}}\{\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y J(t,u_{x_0,\overline{\mu},\nu}(t),y,z)\overline{\mu}_t(dy)]\nu_t(dz)]\,dt\\&+\int_{t_0}^{t_0+\sigma}[\int_Z[\int_Y\langle\nabla\varphi(t,u_{x_0,\overline{\mu},\nu}(t)),g(t,u_{x_0,\overline{\mu},\nu}(t),y,z)\rangle]\overline{\mu}_t(dy)]\nu_t(dz)]\,dt\\&+\int_{t_0}^{t_0+\sigma}\frac{\partial\varphi}{\partial t}(t,u_{x_0,\overline{\mu},\nu}(t))dt+\int_{t_0}^{t_0+\sigma}\langle\nabla\varphi(t,u_{x_0,\overline{\mu},\nu}(t)),-\nabla f(u_{x_0,\overline{\mu},\nu}(t))\rangle dt\}\le 0.\end{aligned} \tag{3.16}$$

Comparing (3.16) and (3.9) we get a contradiction. □

The following is a direct application of Theorem 2.2 to a bang-bang type result in control problems.

Theorem 3.5. *Let $[0,T], 0<T$. Let H be a separable Hilbert space, and let $f: H\to\mathbf{R}\cup\{\infty\}$ be a proper lsc function with closed domain dom f. Suppose that f is bounded and pln on dom f. Suppose further that for some real number $\alpha>0$,*

- (H_1) *$f(x)\ge-\alpha(1+||x||), \forall x\in H$.*
- (H_2) *f is inf-ball compact around each point of dom f, i.e, $\forall x\in$ dom f, there exists $r>0$ such that, $\forall\lambda>0$, the set $\{f\le\lambda\}\cap\overline{B}_H(x,r)$ is compact in $(H,||.||)$.*

Let $K:=\overline{B}_H(0,1)$ be the closed unit ball in H, and ext(K) the set of extreme points of K. Let $x_0\in$ dom f. Then the solutions set $\mathcal{S}_{x_0}(K)$ of the inclusion

$$(\mathcal{I}_{\partial f,K})\qquad\begin{cases}\dot u_{x_0}(t)\in-\partial f(u_{x_0}(t))+K\\ u_{x_0}(0)=x_0\in dom\ f\end{cases}$$

is compact with respect to the topology of uniform convergence and the solutions set $\mathcal{S}_{x_0}(ext(K))$ *of the inclusion*

$$(\mathcal{I}_{\partial f, ext(K)}) \qquad \begin{cases} \dot{u}_{x_0}(t) \in -\partial f(u_{x_0}(t)) + ext(K) \\ u_{x_0}(0) = x_0 \in dom\ f \end{cases}$$

is dense in the compact set $\mathcal{S}_{x_0}(K)$*.*

Proof. Let S_K (resp. $S_{ext(K)}$) denote the set of all measurable selections of K (resp. $ext(K)$). Then S_K is convex weakly compact for the topology $\sigma(L^\infty([0,T];H), L^1([0,T];H))$ and $S_{ext(K)}$ is dense in S_K for this topology (see e.g. [2]) by virtue of Ljapunov theorem. Further by Theorem 2.2, $\mathcal{S}_{x_0}(ext(K))$ and $\mathcal{S}_{x_0}(K)$ are nonempty. Making use of the arguments of Theorem 2.1 and the closure property for the operator subdifferential of l.s.c pln function (cf. Proposition 4.1.8 in [15]), it is easy to see that $\mathcal{S}_{x_0}(K)$ is compact for the uniform convergence, namely the mapping $h \mapsto u_h$ where u_h is the unique absolutely continuous solution of the inclusion

$$\begin{cases} \dot{u}_{x_0}(t) \in -\partial f(u_{x_0}(t)) + h(t) \\ u_{x_0}(0) = x_0 \in \text{dom } f \end{cases}$$

associated with the control $h \in S_K$, is continuous on the convex weakly compact set S_K in $L^2([0,T];H)$. Then the result follows by density. □

4. A new class of functional evolution inclusions

To end this paper, we present an application of Theorem 2.2 to a new class of functional evolution inclusions (FEI). Let $r > 0$ be a finite delay, $\mathcal{C}_0 = \mathcal{C}([-r,0], H)$ be the Banach space of all continuous H-valued functions defined on $[-r,0]$ equipped with the norm of uniform convergence and $F : [0,T] \times \mathcal{C}([-r,0],H) \rightrightarrows H$ be a separately scalarly measurable and separately scalarly upper semicontinuous convex weakly compact valued multifunction. For any $t \in [0,T]$, let $\tau(t) : \mathcal{C}([-r,t],H) \to \mathcal{C}_0$ defined by $(\tau(t)u)(s) = u(t+s), \forall s \in [-r,0]$ and $\forall u \in \mathcal{C}([-r,t],H)$. Let φ be a given element of $\mathcal{C}_0$ with $\varphi(0) \in \text{dom } f$. We are concerned with the existence of solutions to the FEI of the form

$$\begin{cases} \dot{u}(t) \in -\partial f(u(t)) + F(t, \tau(t)u), \quad \text{a.e } \ t \in [0,T] \\ u(s) = \varphi(s),\ \forall s \in [-r,0];\ u(t) \in \text{dom } f,\ \forall t \in [0,T]. \end{cases}$$

By solution we mean a function $u : [-r,T] \to H$ such that its restriction on $[-r,0]$ is equal to φ and its restriction to $[0,T]$ is absolutely continuous and satisfies the above inclusion.

Theorem 4.1. *Let* $[0,T], 0 < T$*. Let* H *be a separable Hilbert space. Let* $f : H \to \mathbf{R} \cup \{\infty\}$ *is a proper lsc function with closed domain* $dom\ f$*. Suppose that* f *is bounded and pln on* $dom\ f$*. Suppose further that for some real number* $\alpha > 0$*,*

- (H_1) $f(x) \geq -\alpha(1 + ||x||), \forall x \in H$*.*
- (H_2) f *is inf-ball compact around each point of* $dom\ f$*, i.e,* $\forall x \in dom\ f$*, there exists* $r > 0$ *such that,* $\forall \lambda > 0$*, the set* $\{f \leq \lambda\} \cap \overline{B}_H(x,r)$ *is compact in* $(H, ||.||)$*.*

Let $F : [0,T] \times \mathcal{C}([-r,0],H) \rightrightarrows H$ be a separately scalarly measurable and separately scalarly upper semicontinuous convex weakly compact valued multifunction satisfying $F(t,u) \subset \gamma(t)\overline{B}_H(0,1)$ for all $(t,u) \in [0,T] \times \mathcal{C}_H([-r,0])$ for some $\gamma \in L^2([0,T])$. Let $\varphi \in \mathcal{C}_0$ with $\varphi(0) \in dom\ f$.

Then the solutions set $\mathcal{S}_\varphi$ of the FEI

$$\begin{cases} \dot{u}(t) \in -\partial f(u(t)) + F(t, \tau(t)u), \quad a.e\ \ t \in [0,T] \\ u(s) = \varphi(s),\ \forall s \in [-r,0];\ u(t) \in dom\ f,\ \forall t \in [0,T] \end{cases}$$

is nonempty and compact in the Banach space $\mathcal{C}([-r,T],H)$.

Proof. The proof is long, making use of the estimation of the velocity of solutions given in Theorem 2.2 and the discretization techniques developed in ([6], Theorem 2.1). For shortness we omit the details. □

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C. Castaing
Département de Mathématiques, Université Montpellier II, 34095 Montpellier Cedex 5, France.
E-mail address: `castaing.charles@numericable.fr`

S. Marcellin
Département de Mathématiques, Université Antilles Guyane, 97110 Pointe à Pitre, Guadeloupe.
E-mail address: `sylvie.marcellin@univ-ag.fr`